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Fixed-point theorem in classes of function with values in a dq-metric space

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Abstract. We prove a fixed point result for nonlinear operators, acting on some classes of functions with values in a dq-metric space, and show some applications of it. The result has been motivated by some issues arising in Ulam stability. We use a restricted form of a contraction condition.

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1. Introduction

The name of Ulam has been somehow connected with various definitions of stability (see, e.g., [1,12,16,24]), but roughly speaking, the following one describes our considerations in this paper (A^B denotes the family of all functions mapping a nonempty set B into a nonempty set A, $\mathbb{R}$ stands for the set of all real numbers and $\mathbb{R}_+ := [0, \infty)$).

Definition 1. Let (Y, d) be a metric space, E be a nonempty set, $\mathcal{D}_0 \subset \mathcal{D} \subset Y^E$ and $\mathcal{E} \subset {\mathbb{R}_+}^E$ be nonempty, $\mathcal{T} : \mathcal{D} \to Y^E$ and $\mathcal{S} : \mathcal{E} \to {\mathbb{R}_+}^E$. We say that the equation

$$\mathcal{T}(\psi)(t) = \psi(t), \qquad t \in E,$$

is $\mathcal{S}$-stable in $\mathcal{D}_0$ provided, for any $\psi \in \mathcal{D}_0$ and $\delta \in \mathcal{E}$ with

$$d(\mathcal{T}(\psi)(t), \psi(t)) \le \delta(t), \qquad t \in E,$$

there is a solution $\phi \in \mathcal{D}$ of the equation, such that

$$d(\phi(t), \psi(t)) \le \mathcal{S}\delta(t), \qquad t \in E.$$

There are some close connections between Ulam stability and fixed-point theory (see, e.g., [6]). In particular, the subsequent theorem has been presented in [7, Theorem 2] and it has been shown there how to deduce some quite general Ulam stability results from it (see also [6,9,14]). To formulate it, we need the following hypothesis concerning operators $\Lambda : {\mathbb{R}_+}^E \to {\mathbb{R}_+}^E$ (E is a nonempty set):

($\mathcal{C}$) If $(\delta_n)_{n\in\mathbb{N}}$ is a sequence in ${\mathbb{R}_+}^E$ with $\lim_{n\to\infty} \delta_n(t) = 0$ for $t \in E$, then

$$\lim_{n\to\infty} \Lambda\delta_n(t) = 0, \qquad t \in E.$$

Let us yet recall that $\Lambda : {\mathbb{R}_+}^E \to {\mathbb{R}_+}^E$ is non-decreasing provided

$$\Lambda\xi(t) \le \Lambda\eta(t), \qquad t \in E,$$

for every $\xi, \eta \in {\mathbb{R}_+}^E$ with $\xi(t) \le \eta(t)$ for every $t \in E$.

Theorem 2. *Assume that (Y, d) is a complete metric space, E is a nonempty set, $\Lambda : {\mathbb{R}_+}^E \to {\mathbb{R}_+}^E$ is non-decreasing and satisfies hypothesis ($\mathcal{C}$), and $\mathcal{T} : Y^E \to Y^E$ is such that*

$$d\big((\mathcal{T}\xi)(t), (\mathcal{T}\mu)(t)\big) \le \Lambda\big(d(\xi, \mu)\big)(t), \qquad \xi, \mu \in Y^E, t \in E, \tag{1}$$

and functions $\varepsilon : E \to \mathbb{R}_+$ and $\varphi : E \to Y$ fulfil

$$d\big((\mathcal{T}\varphi)(t), \varphi(t)\big) \le \varepsilon(t), \qquad t \in E,$$

and

$$\sigma(t) := \sum_{n\in\mathbb{N}_0} (\Lambda^n\varepsilon)(t) < \infty, \qquad t \in E.$$

Then, for every $t \in E$, the limit

$$\lim_{n\to\infty} (\mathcal{T}^n\varphi)(t) =: \psi(t)$$

exists and the function $\psi \in Y^E$, defined in this way, is a fixed point of $\mathcal{T}$ with

$$d\big(\varphi(t), \psi(t)\big) \le \sigma(t), \qquad t \in E.$$

In the next section, we present a similar fixed-point theorem for dislocated quasi-metric spaces that generalizes Theorem 2 and several similar outcomes in [5,7,8]. In particular, we apply a restricted version of a weaker form of condition (1) (see Remark 3).

Let us recall that a dislocated quasi-metric (*dq-metric*, for short), in a nonempty set Y, is a function $d \colon Y \times Y \to [0, +\infty)$ that satisfies the following two conditions:

(A1) if $d(x, y) = d(y, x) = 0$, then $x = y$,
(A2) $d(x, y) \leqslant d(x, z) + d(z, y)$

for all $x, y, z \in Y$. The notion of a dq-metric space is a natural generalization of the usual definitions of metric, quasi-metric, partial metric, and metric-like spaces and plays crucial roles in computer science and cryptography (see, e.g., [2,4,11,13,15,20–22,25,26]).

Remark 1. Let $a, b \in (0, \infty)$, $n, k \in \mathbb{N}$ (positive integers), $\alpha : \mathbb{R} \to \mathbb{R}_+$ and $\alpha^{-1}(\{0\}) = \{0\}$. Then, it is easy to check that the function $d\colon \mathbb{R}\times\mathbb{R} \to [0, \infty)$, given by any of the following six formulas, is a dq-metric:

$$
\begin{aligned}
d(x, y) &= \alpha(x), \qquad x, y \in \mathbb{R},\\
d(x, y) &= \max\{a|x|^k, b|y|^n\}, \qquad x, y \in \mathbb{R},\\
d(x, y) &= a|x|^k + b|y|^n, \qquad x, y \in \mathbb{R},\\
d(x, y) &= \sqrt{a|x|^k + b|y|^n}, \qquad x, y \in \mathbb{R},\\
d(x, y) &= \sqrt[n]{\max\{x - y, 0\}}, \qquad x, y \in \mathbb{R},\\
d(x, y) &= \max\{x - [y], 0\}, \qquad x, y \in \mathbb{R},
\end{aligned}
$$

where $[y]$ denotes the integer part of y, i.e., $[y]{:=}\max\{n \in \mathbb{Z} : n \le y\}$ and $\mathbb{Z}$ stands for the set of integers. For some further examples we refer to, e.g., [2,4,13,22] and the references therein.

Let d be a dq-metric in a nonempty set Y. We say that $x \in Y$ is a *limit* of a sequence $(x_n)_{n=1}^{\infty}$ in Y provided

$$\lim_{n\to\infty} \max\{d(x_n, x), d(x, x_n)\} = 0;$$

then we write $x_n \to x$ or $x = \lim_{n\to\infty} x_n$; in view of (A2), it is easy to note that such a limit must be unique. Next, we say that a sequence $(x_n)_{n=1}^{\infty}$ in Y is Cauchy if

$$\lim_{N\to\infty} \sup_{m,n\geqslant N} d(x_n, x_m) = 0;$$

d is complete if every Cauchy sequence in Y has a limit in Y.

Remark 2. Usually, in a dq-metric space, the Cauchy sequence is defined in a somewhat different way; e.g., in a metric-like space (Y, d), a sequence $(x_n)_{n=1}^{\infty}$ is said to be Cauchy if the limit $\lim_{N\to\infty} \sup_{m,n\geqslant N} d(x_n, x_m)$ exists and is finite (see [3]). However, such definitions are too weak and would exclude from our considerations the metric and quasi-metric spaces. The same concerns the notion of completeness.

Our definition of a limit of a sequence is stronger than the usual, but this seems to be necessary in the proof of the main result; moreover, it actually corresponds to our definition of the Cauchy sequence and makes such limit unique (which is not the case in general) and, therefore, more useful.

2. The main result

In what follows, we always assume that (Y, d) is a complete dq-metric space, i.e., d is a complete dq-metric in a nonempty set Y. Moreover, E denotes a nonempty set and $d : Y^E \times Y^E \to {\mathbb{R}_+}^E$ is defined by

$$d(\xi, \mu)(t){:=}d(\xi(t), \mu(t)), \qquad \xi, \mu \in Y^E, t \in E.$$

Analogously, as in the classical metric spaces, if $(\chi_n)_{n\in\mathbb{N}}$ is a sequence of elements of Y^E, then a function $\chi \in Y^E$ is a pointwise limit of $(\chi_n)_{n\in\mathbb{N}}$ provided

$$\lim_{n\to\infty} \max\left\{d(\chi,\chi_n)(t), d(\chi_n,\chi)(t)\right\} = 0, \qquad t \in E;$$

$\chi \in Y^E$ is a uniform limit of $(\chi_n)_{n\in\mathbb{N}}$ provided

$$\lim_{n\to\infty} \sup_{t\in E} \max\left\{d(\chi,\chi_n)(t), d(\chi_n,\chi)(t)\right\} = 0.$$

A nonempty subset $\mathcal{F}$ of Y^E is called p-closed (u-closed, respectively) if every $\chi \in Y^E$, which is a pointwise (uniform, resp.) limit of a sequence $(\chi_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{F}$, belongs to $\mathcal{F}$.

Furthermore, given $f, g \in \mathbb{R}^E$, we write $f \le g$ if $f(t) \le g(t)$ for $t \in E$. Let $\emptyset \neq \mathcal{C} \subset Y^E$, $\Lambda\colon {\mathbb{R}_+}^E \to {\mathbb{R}_+}^E$, and $\omega \in {\mathbb{R}_+}^E$. We say that $\mathcal{T}\colon \mathcal{C} \to Y^E$ is (ω,Λ)—contractive provided

$$d(\mathcal{T}\xi, \mathcal{T}\mu) \le \Lambda\delta$$

for any $\xi, \mu \in \mathcal{C}$ and $\delta \in {\mathbb{R}_+}^E$ with

$$\delta \le \omega, \qquad d(\xi,\mu) \le \delta.$$

Given a set $A \neq \emptyset$ and $f \in A^A$, we define $f^n \in A^A$ (for $n \in \mathbb{N}_0$) by

$$f^0(x) = x, \qquad f^{n+1}(x) = f(f^n(x)), \qquad x \in A, n \in \mathbb{N}_0.$$

Finally, to simplify some formulas, we denote by Λ_0 the identity operator on ${\mathbb{R}_+}^E$, i.e., $\Lambda_0\delta = \delta$ for each $\delta \in {\mathbb{R}_+}^E$.

Now, we are in a position to present the fixed-point theorem, which is the main result of this paper.

Theorem 3. *Let $\mathcal{C} \subset Y^E$ be nonempty, $\Lambda_n\colon {\mathbb{R}_+}^E \to {\mathbb{R}_+}^E$ for $n \in \mathbb{N}$, and $\mathcal{T}\colon \mathcal{C} \to \mathcal{C}$. Assume that there exist functions $\varepsilon_1, \varepsilon_2 \in {\mathbb{R}_+}^E$ and $\varphi \in \mathcal{C}$, such that*

$$\varepsilon_j^*(t) := \sum_{i=0}^{\infty} \Lambda_i \varepsilon_j(t) < \infty, \qquad t \in E, j = 1, 2, \tag{2}$$

$$d(\mathcal{T}\varphi, \varphi) \le \varepsilon_1, \qquad d(\varphi, \mathcal{T}\varphi) \le \varepsilon_2, \tag{3}$$

$$\liminf_{n\to\infty} \Lambda_1\Big(\sum_{i=n}^{\infty} \Lambda_i\varepsilon_j\Big)(t) = 0, \qquad t \in E, j = 1, 2, \tag{4}$$

and write $\varepsilon^*(t) := \max\{\varepsilon_1(t), \varepsilon_2(t)\}$ for $t \in E$. Let $\mathcal{T}^n$ be $(\varepsilon^*, \Lambda_n)$—contractive for $n \in \mathbb{N}$ and one of the following two hypotheses be valid.

(i) *$\mathcal{C}$ is p-closed.*
(ii) *$\mathcal{C}$ is u-closed and the sequence $\left(\sum_{i=0}^{n} \Lambda_i\varepsilon_j\right)_{n\in\mathbb{N}}$ tends uniformly to ε_j^* on E for $j = 1, 2$.*

Then, for each $t \in E$, there exists the limit

$$\psi(t) := \lim_{n\to\infty} \mathcal{T}^n\varphi(t) \tag{5}$$

and the function $\psi \in \mathcal{C}$*, defined in this way, is a fixed point of* $\mathcal{T}$ *with*

$$d(\mathcal{T}^n\varphi, \psi) \le \sum_{i=n}^{\infty} \Lambda_i \varepsilon_1, \qquad d(\psi, \mathcal{T}^n\varphi) \le \sum_{i=n}^{\infty} \Lambda_i \varepsilon_2, \qquad n \in \mathbb{N}_0. \tag{6}$$

Moreover, the following two statements are valid:

(a) *for every sequence* $(k_n)_{n\in\mathbb{N}}$ *of positive integers with* $\lim_{n\to\infty} k_n = \infty$, ψ *is the unique fixed point of* $\mathcal{T}$*, such that*

$$d(\mathcal{T}^{k_n}\varphi, \psi) \le \sum_{i=k_n}^{\infty} \Lambda_i \varepsilon_j, \qquad d(\psi, \mathcal{T}^{k_n}\varphi) \le \sum_{i=k_n}^{\infty} \Lambda_i \varepsilon_l, \qquad n \in \mathbb{N},$$

with some $j, l \in \{1, 2\}$*;*

(b) *if*

$$\liminf_{n\to\infty} \Lambda_n \varepsilon_j^*(t) = 0, \qquad j = 1, 2, t \in E, \tag{7}$$

then ψ *is the unique fixed point of* $\mathcal{T}$ *with*

$$d(\varphi, \psi) \le \varepsilon_1^*, \qquad d(\psi, \varphi) \le \varepsilon_2^*,$$

and for every $j, l \in \{1, 2\}$

$$\psi(t) = \lim_{n\to\infty} \mathcal{T}^{k_n}\xi(t), \qquad \xi \in \mathcal{C}, d(\xi, \psi) \le \varepsilon_j^*, d(\psi, \xi) \le \varepsilon_l^*, t \in E, \tag{8}$$

for every sequence $(k_n)_{n\in\mathbb{N}}$ *of positive integers with* $\lim_{n\to\infty} \Lambda_{k_n} \varepsilon_m^*(t) = 0$ *for* $t \in E$ *and* $m \in \{j, l\}$*.*

Proof. Clearly, (3) implies that, for any $k, l \in \mathbb{N}$ and $n \in \mathbb{N}_0$

$$\begin{aligned} d(\mathcal{T}^{n+k}\varphi, \mathcal{T}^n\varphi) &\le \sum_{i=0}^{k-1} d(\mathcal{T}^{n+i+1}\varphi, \mathcal{T}^{n+i}\varphi) \\ &\le \sum_{i=n}^{n+k-1} \Lambda_i \varepsilon_1 \le \sum_{i=n}^{\infty} \Lambda_i \varepsilon_1, \end{aligned} \tag{9}$$

$$\begin{aligned} d(\mathcal{T}^n\varphi, \mathcal{T}^{n+l}\varphi) &\le \sum_{i=0}^{l-1} d(\mathcal{T}^{n+i}\varphi, \mathcal{T}^{n+i+1}\varphi) \\ &\le \sum_{i=n}^{n+l-1} \Lambda_i \varepsilon_2 \le \sum_{i=n}^{\infty} \Lambda_i \varepsilon_2, \end{aligned} \tag{10}$$

whence

$$\begin{aligned} d(\mathcal{T}^{n+k}\varphi, \mathcal{T}^{n+l}\varphi) &\le d(\mathcal{T}^{n+k}\varphi, \mathcal{T}^n\varphi) + d(\mathcal{T}^n\varphi, \mathcal{T}^{n+l}\varphi) \\ &\le \sum_{i=n}^{\infty} \Lambda_i \varepsilon_1 + \sum_{i=n}^{\infty} \Lambda_i \varepsilon_2. \end{aligned}$$

Therefore, by (2), $(\mathcal{T}^n\varphi(t))_{n\in\mathbb{N}}$ is a Cauchy sequence in Y for each $t \in E$. Since Y is complete, this sequence is convergent. Consequently, (5) defines a function $\psi \in \mathcal{C}$.

Letting $k \to \infty$ in (9) and $l \to \infty$ in (10), on account of (5), we get

$$d(\mathcal{T}^n\varphi, \psi) \le \sum_{i=n}^{\infty} \Lambda_i \varepsilon_1, \qquad d(\psi, \mathcal{T}^n\varphi) \le \sum_{i=n}^{\infty} \Lambda_i \varepsilon_2, \qquad n \in \mathbb{N}_0, \tag{11}$$

which is (6). Next, using (11), we get

$$d(\mathcal{T}^{n+1}\varphi, \mathcal{T}\psi) \le \Lambda_1\Big(\sum_{i=n}^{\infty} \Lambda_i \varepsilon_1\Big), \qquad d(\mathcal{T}\psi, \mathcal{T}^{n+1}\varphi) \le \Lambda_1\Big(\sum_{i=n}^{\infty} \Lambda_i \varepsilon_2\Big)$$

for $n \in \mathbb{N}_0$. Hence, for each $n \in \mathbb{N}_0$

$$d(\psi, \mathcal{T}\psi) \le d(\psi, \mathcal{T}^{n+1}\varphi) + d(\mathcal{T}^{n+1}\varphi, \mathcal{T}\psi) \le d(\psi, \mathcal{T}^{n+1}\varphi) + \Lambda_1\Big(\sum_{i=n}^{\infty} \Lambda_i \varepsilon_1\Big),$$

$$d(\mathcal{T}\psi, \psi) \le \Lambda_1\Big(\sum_{i=n}^{\infty} \Lambda_i \varepsilon_2\Big) + d(\mathcal{T}^{n+1}\varphi, \psi),$$

which with $n \to \infty$ yields $d(\psi, \mathcal{T}\psi) = 0$ and $d(\mathcal{T}\psi, \psi) = 0$ [in view of (4)], and consequently $\mathcal{T}\psi = \psi$.

Let $(k_n)_{n\in\mathbb{N}}$ be a sequence of positive integers with $\lim_{n\to\infty} k_n = \infty$ and $\xi \in Y^E$ be a fixed point of $\mathcal{T}$ with

$$d(\mathcal{T}^{k_n}\varphi(x), \xi(x)) \le \sum_{i=k_n}^{\infty} \Lambda_i \varepsilon_j(t), \qquad d(\xi(x), \mathcal{T}^{k_n}\varphi(x)) \le \sum_{i=k_n}^{\infty} \Lambda_i \varepsilon_l(t),$$
$$n \in \mathbb{N}, t \in E,$$

with some $j, l \in \{1, 2\}$. Then, by (6)

$$\begin{aligned} d(\xi(t), \psi(t)) \le\ & d(\xi(t), \mathcal{T}^{k_n}\varphi(t)) + d(\mathcal{T}^{k_n}\varphi(t), \psi(t)) \\ \le\ & \sum_{i=k_n}^{\infty} \Lambda_i \varepsilon_l(t) + \sum_{i=k_n}^{\infty} \Lambda_i \varepsilon_1(t), \qquad n \in \mathbb{N}_0, t \in E, \end{aligned}$$

$$\begin{aligned} d(\psi(t), \xi(t)) \le\ & d(\psi(t), \mathcal{T}^{k_n}\varphi(t)) + d(\mathcal{T}^{k_n}\varphi(t), \xi(t)) \\ \le\ & \sum_{i=k_n}^{\infty} \Lambda_i \varepsilon_2(t) + \sum_{i=k_n}^{\infty} \Lambda_i \varepsilon_j(t), \qquad n \in \mathbb{N}_0, t \in E, \end{aligned}$$

whence letting $n \to \infty$, we get $\xi = \psi$.

It remains to prove statement (b). Therefore, assume that (7) holds and $\xi \in Y^E$ is a fixed point of $\mathcal{T}$ with

$$d(\varphi, \xi) \le \varepsilon_1^*, \qquad d(\xi, \varphi) \le \varepsilon_2^*.$$

Then, for any $n \in \mathbb{N}_0$, we have

$$\begin{aligned} d(\psi, \xi) &\le d(\psi, \mathcal{T}^n\varphi) + d(\mathcal{T}^n\varphi, \mathcal{T}^n\xi) \\ &\le d(\psi, \mathcal{T}^n\varphi) + \Lambda_n \varepsilon_1^*, \end{aligned}$$

$$\begin{aligned} d(\xi, \psi) &\le d(\mathcal{T}^n\xi, \mathcal{T}^n\varphi) + d(\mathcal{T}^n\varphi, \psi) \\ &\le \Lambda_n \varepsilon_2^* + d(\psi, \mathcal{T}^n\varphi), \end{aligned}$$

whence letting $n \to \infty$, we can easily see that $\xi = \psi$.

Now, let $j, l \in \{1, 2\}$ and $(k_n)_{n\in\mathbb{N}}$ be a sequence of positive integers with

$$\lim_{n\to\infty} \Lambda_{k_n}\varepsilon_m^*(t) = 0, \qquad t \in E, m \in \{j, l\}.$$

Let $\xi \in \mathcal{C}$ be a function such that $d(\xi, \psi) \le \varepsilon_j^*$ and $d(\psi, \xi) \le \varepsilon_l^*$. Then

$$d(\mathcal{T}^{k_n}\xi, \psi) = d(\mathcal{T}^{k_n}\xi, \mathcal{T}^{k_n}\psi) \le \Lambda_{k_n}\varepsilon_j^*, \qquad n \in \mathbb{N},$$
$$d(\psi, \mathcal{T}^{k_n}\xi) = d(\mathcal{T}^{k_n}\psi, \mathcal{T}^{k_n}\xi) \le \Lambda_{k_n}\varepsilon_l^*, \qquad n \in \mathbb{N}.$$

Letting $n \to \infty$, we get (8). □

Theorem 3 implies at once the following.

Theorem 4. *Let $\mathcal{C} \subset Y^E$ be nonempty, $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ and $\Lambda : {\mathbb{R}_+}^E \to {\mathbb{R}_+}^E$. Assume that there exist functions $\varepsilon_1, \varepsilon_2 \in {\mathbb{R}_+}^E$ and $\varphi \in \mathcal{C}$, such that*

$$\varepsilon_j^*(x) := \sum_{i=0}^{\infty} \Lambda^i \varepsilon_j(t) < \infty, \qquad t \in E, j = 1, 2,$$

$$d(\mathcal{T}\varphi, \varphi) \le \varepsilon_1, \qquad d(\varphi, \mathcal{T}\varphi) \le \varepsilon_2, \tag{12}$$

$$\liminf_{n\to\infty} \Lambda\Big(\sum_{i=n}^{\infty} \Lambda^i \varepsilon_j\Big)(t) = 0, \qquad t \in E, j = 1, 2, \tag{13}$$

and $\mathcal{T}$ is (ε^*, Λ)-contractive, where $\varepsilon^*(t) := \max\{\varepsilon_1(t), \varepsilon_2(t)\}$ for $t \in E$. Next, let one of the following two hypotheses hold.

(i) *$\mathcal{C}$ is p-closed.*
(ii) *$\mathcal{C}$ is u-closed and the sequence $\big(\sum_{i=0}^n \Lambda^i \varepsilon_j\big)_{n\in\mathbb{N}}$ tends uniformly to ε_j^* on E for $j = 1, 2$.*

Then, for each $t \in E$, there exists the limit

$$\psi(t) := \lim_{n\to\infty} \mathcal{T}^n\varphi(t)$$

and the function $\psi \in \mathcal{C}$, defined in this way, is a fixed point of $\mathcal{T}$ with

$$d(\mathcal{T}^n\varphi, \psi) \le \sum_{i=n}^{\infty} \Lambda^i \varepsilon_1, \qquad d(\psi, \mathcal{T}^n\varphi) \le \sum_{i=n}^{\infty} \Lambda^i \varepsilon_2, \qquad n \in \mathbb{N}_0.$$

Moreover, the following two statements are valid:

(a) *For every sequence $(k_n)_{n\in\mathbb{N}}$ of positive integers with $\lim_{n\to\infty} k_n = \infty$, ψ is the unique fixed point of $\mathcal{T}$ with*

$$d(\mathcal{T}^{k_n}\varphi, \psi) \le \sum_{i=k_n}^{\infty} \Lambda^i \varepsilon_1, \qquad d(\psi, \mathcal{T}^{k_n}\varphi) \le \sum_{i=k_n}^{\infty} \Lambda^i \varepsilon_2, \qquad n \in \mathbb{N}.$$

(b) *If*

$$\liminf_{n\to\infty} \Lambda^n \varepsilon_j^*(t) = 0, \qquad t \in E, j = 1, 2, \tag{14}$$

then ψ is the unique fixed point of $\mathcal{T}$ with

$$d(\varphi, \psi) \le \varepsilon_1^*, \qquad d(\psi, \varphi) \le \varepsilon_2^*.$$

and for every $j, l \in \{1,2\}$,

$$\psi(t) = \lim_{n\to\infty} \mathcal{T}^{k_n}\xi(t), \qquad \xi \in \mathcal{C}, d(\xi,\psi) \le \varepsilon_j^*, d(\psi,\xi) \le \varepsilon_l^*, t \in E,$$

for every sequence $(k_n)_{n\in\mathbb{N}}$ *of positive integers with* $\lim_{n\to\infty} \Lambda_{k_n}\varepsilon_m^*(t) = 0$ *for* $t \in E$ *and* $m \in \{j,l\}$.

Proof. It is enough to notice that $\mathcal{T}^n$ is $(\varepsilon^*, \Lambda^n)$—contractive for each $n \in \mathbb{N}$ and use Theorem 3. □

Remark 3. There arises a natural question whether, in some situations, assumption (2) can be weaker than (12) with $\Lambda{:=}\Lambda_1$. Below, we provide a somewhat trivial example that this is the case.

Let $Y = \mathbb{R}^3$ be endowed with the euclidean norm, $c \in \mathbb{R}$ and $E = \mathbb{R}$. Define the operator $\mathcal{T}\colon Y^{\mathbb{R}} \to Y^{\mathbb{R}}$ by

$$\mathcal{T}\phi(x) = (0, \phi_1(x), \phi_2(x) + c), \qquad x \in \mathbb{R},$$

for every $\phi = (\phi_1, \phi_2, \phi_3) \in Y^{\mathbb{R}}$. Then

$$\begin{aligned}\|\mathcal{T}\phi(x) - \mathcal{T}\mu(x)\| = {} & \|(0, \phi_1(x) - \mu_1(x), \phi_2(x) - \mu_2(x))\| \\ \le {} & \|\phi(x) - \mu(x)\|, \qquad x \in \mathbb{R},\end{aligned}$$

for every $\phi = (\phi_1, \phi_2, \phi_3), \mu = (\mu_1, \mu_2, \mu_3) \in Y^{\mathbb{R}}$. This shows that Λ_1 and Λ_2 exist, because it is enough to take any Λ_1 and Λ_2 with $\Lambda_i\delta \ge \delta$ for $\delta \in {\mathbb{R}_+}^{\mathbb{R}}$ and $i = 1, 2$. Next

$$\mathcal{T}^n\phi(x) = (0, 0, nc), \qquad x \in \mathbb{R}, n \in \mathbb{N}, n \ge 3,$$

for every $\phi = (\phi_1, \phi_2, \phi_3) \in Y^{\mathbb{R}}$ and we can take $\Lambda_n\delta(x) = 0$ for $\delta \in {\mathbb{R}_+}^{\mathbb{R}}$, $x \in \mathbb{R}$ and $n \in \mathbb{N}$, $n \ge 3$. Clearly, in such a case, (2) holds for any $\varepsilon_1, \varepsilon_2 \in \mathbb{R}_+^{\mathbb{R}}$.

We show that, for any such Λ_1, we must have

$$\Lambda_1\delta \ge \delta, \qquad \delta \in {\mathbb{R}_+}^{\mathbb{R}}. \tag{15}$$

Therefore, take arbitrary $\delta \in {\mathbb{R}_+}^{\mathbb{R}}$ and define $\phi, \psi \in Y^{\mathbb{R}}$ by

$$\phi(x) = (\delta(x), 0, 0), \qquad \psi(x) = (0, 0, 0), \qquad x \in \mathbb{R}.$$

Then

$$\mathcal{T}\phi(x) = (0, \delta(x), c), \qquad \mathcal{T}\psi(x) = (0, 0, c), \qquad x \in \mathbb{R},$$
$$\|\phi(x) - \psi(x)\| = \|(\delta(x), 0, 0)\| = \delta(x), \qquad x \in \mathbb{R},$$

and

$$\|\mathcal{T}\phi(x) - \mathcal{T}\psi(x)\| = \|(0, \delta(x), 0)\| = \delta(x) \le \Lambda_1\delta(x), \qquad x \in \mathbb{R}.$$

This shows that (15) holds, whence, by induction, we obtain that for each $n \in \mathbb{N}$

$${\Lambda_1}^n\delta \ge \delta,$$

and therefore

$$\sum_{i=0}^{\infty} {\Lambda_1}^i\delta(x) = \infty, \qquad x \in \mathbb{R}, \delta(x) \ne 0.$$

3. Some comments

We need yet the following hypothesis concerning operators $\Lambda\colon {\mathbb{R}_+}^E \to {\mathbb{R}_+}^E$.

($\mathcal{C}$) If $(\delta_n)_{n\in\mathbb{N}}$ is a sequence of elements of ${\mathbb{R}_+}^E$ with

$$\lim_{n\to\infty} \delta_n(t) = 0, \qquad t \in E,$$

then

$$\liminf_{n\to\infty} \Lambda\delta_n(t) = 0, \qquad t \in E.$$

Remark 4. Note that if Λ_1 fulfils hypothesis ($\mathcal{C}$), then (4) results at once from (2). Analogously, (12) yields (13) if Λ fulfils ($\mathcal{C}$).

Remark 5. Let $j \in \mathbb{N}$ and $\mathbb{K}$ be either the set of reals $\mathbb{R}$ or the set of complex numbers $\mathbb{C}$. Fix $f_i\colon E \to E$ and $L_i\colon E \to \mathbb{K}$ for $i = 1,\dots,j$. Then, the operator $\mathcal{T}\colon \mathbb{K}^E \to \mathbb{K}^E$, given by

$$\mathcal{T}\phi(t) := \sum_{i=1}^{j} L_i(t)\phi(f_i(t)), \qquad \phi \in \mathbb{K}^E, t \in E,$$

is (ω,Λ) contractive, with any $\omega \in \mathbb{R}_+^E$ and $\Lambda\colon {\mathbb{R}_+}^E \to {\mathbb{R}_+}^E$ defined by the formula

$$\Lambda\delta(t) := \sum_{i=1}^{j} |L_i(t)|\delta(f_i(t)), \qquad \delta \in {\mathbb{R}_+}^E, t \in E.$$

Moreover, ($\mathcal{C}$) holds.

Next, for any function $\varepsilon_0\colon E \to \mathbb{R}_+$ with ε_0^* given by [see (12)]

$$\varepsilon_0^*(x) := \sum_{i=0}^{\infty} \Lambda^i \varepsilon_0(t) < \infty, \qquad t \in E, j = 1,2,$$

we have

$$\begin{aligned}\Lambda\varepsilon_0^*(t) = & \sum_{i=1}^{j} |L_i(t)| \sum_{k=0}^{\infty} (\Lambda^k\varepsilon_0)(f_i(t)) = \sum_{k=0}^{\infty}\sum_{i=1}^{j} |L_i(t)|(\Lambda^k\varepsilon_0)(f_i(t)) \\ = & \sum_{k=1}^{\infty} (\Lambda^k\varepsilon_0)(t), \qquad t \in E,\end{aligned}$$

and analogously, by induction, we get

$$\Lambda^n\varepsilon_0^*(t) = \sum_{k=n}^{\infty} (\Lambda^k\varepsilon_0)(t), \qquad t \in E, n \in \mathbb{N}_0.$$

This means that (12) yields (14). Therefore, [9, Theorem 1] can be derived from Theorem 4.

Remark 6. Let $F : E \times \mathbb{R}_+ \to \mathbb{R}_+$ be subadditive and non-decreasing with respect to the second variable (i.e., $F(x, a + b) \le F(x, a) + F(x, b)$ and $F(x, a) \le F(x, c)$ for $a, b, c \in \mathbb{R}_+$ with $a \le c$ and $x \in E$). Let $f : E \to E$ be given and $\Lambda\colon {\mathbb{R}_+}^E \to {\mathbb{R}_+}^E$ be defined by

$$\Lambda\varepsilon(x) = F(x, \varepsilon(f(x))), \qquad x \in E, n \in \mathbb{N}_0, \varepsilon \in {\mathbb{R}_+}^E.$$

We show that for such Λ, condition (12) yields (13) and (14).

Therefore, assume that (12) holds for some suitable ε_j with $j = 1, 2$. Fix $x \in E$ and define a function $F_0 : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$F_0(a) = F(x, a), \qquad a \in \mathbb{R}_+.$$

Since F_0 is non-decreasing and $\Lambda^n \varepsilon_1(f(x)) \geq 0$ for each $n \in \mathbb{N}_0$, we have

$$\Lambda^{n+1}\varepsilon_1(x) = F_0\big(\Lambda^n \varepsilon_1(f(x))\big) \geq F_0(0).$$

Hence, by (12), we get $F_0(0) = 0$.

Fix $j \in \{1, 2\}$. Next, we prove that F_0 is continuous at 0 or there exists $l_0 \in \mathbb{N}$ with

$$\Lambda^n \varepsilon_j(f(x)) = 0, \qquad n \in \mathbb{N}, n > l_0.$$

To this end suppose that F_0 is not continuous at 0 and there is a strictly increasing sequence $\big(k_n\big)_{n\in\mathbb{N}}$ of positive integers, such that $\Lambda^{k_n}\varepsilon_j(f(x)) \neq 0$ for $n \in \mathbb{N}$. Since F_0 is non-decreasing and $F(0) = 0$, there exists $d > 0$ with $F_0(c) > d$ for every $c > 0$, whence

$$\Lambda^{k_n+1}\varepsilon_j(x) = F_0\big(\Lambda^{k_n}\varepsilon_j(f(x))\big) \geq d, \qquad n \in \mathbb{N},$$

which contradicts to (12).

Thus, we have proved that

$$\lim_{j\to\infty} F_0\Big(\sum_{n=j}^{\infty} \Lambda^n \varepsilon_j(f(x))\Big) = 0, \qquad j = 1, 2.$$

Furthermore, by subadditivity of F_0, for every $k, l \in \mathbb{N}_0$, $l > k$, we get

$$F_0\Big(\sum_{n=k}^{\infty} \Lambda^n \varepsilon_j(f(x))\Big) \leq \sum_{n=k}^{l} \Lambda^{n+1}\varepsilon_j(x) + F_0\Big(\sum_{n=l+1}^{\infty} \Lambda^n \varepsilon_j(f(x))\Big)$$

whence letting $l \to \infty$, we obtain

$$\Lambda\Big(\sum_{n=k}^{\infty} \Lambda^n \varepsilon_j(x)\Big) = F_0\Big(\sum_{n=k}^{\infty} \Lambda^n \varepsilon_j(f(x))\Big) \leq \sum_{n=k+1}^{\infty} \Lambda^n \varepsilon_j(x)$$

and consequently, by induction (with $k = 0$)

$$\Lambda^l\Big(\sum_{n=0}^{\infty} \Lambda^n \varepsilon_j(x)\Big) \leq \sum_{n=l}^{\infty} \Lambda^n \varepsilon_j(x), \qquad l \in \mathbb{N}.$$

Clearly, using those inequalities, we can easily deduce (13) and (14) from (12).

Now, consider a very special situation when the set E has only one element, $E = \{s\}$. Then, actually, each $\mathcal{C} \subset Y^E$ can be considered as a subset of Y of the form $C{:=}\{\phi(s) : \phi \in \mathcal{C}\}$.

Given $e \in \mathbb{R}_+$, $\lambda\colon \mathbb{R}_+ \to \mathbb{R}_+$ and $C \subset Y$, analogously as before, we say that $T\colon C \to C$ is (e, λ)—contractive provided

$$d(Ty, Tz) \leq \lambda(\delta),$$

for every $y, z \in Y$ and $\delta \in \mathbb{R}_+$, such that $d(y, z) \leq \delta \leq e$.

Next, for $\lambda_1\colon \mathbb{R}_+ \to \mathbb{R}_+$, hypothesis $(\mathcal{C})$ takes the following form:

$(\mathcal{C}_0)$ If $(\delta_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}_+$ with

$$\lim_{n\to\infty} \delta_n = 0,$$

then

$$\liminf_{n\to\infty} \lambda_1(\delta_n) = 0.$$

Theorem 3, with $y_0 = \varphi(s)$ and $z_0 = \psi(s)$, takes the following form (we write $\lambda_0(\varepsilon){:=}\varepsilon$ for each $\varepsilon \in \mathbb{R}_+$).

Theorem 5. *Let $T : Y \to Y$, $\lambda_n\colon \mathbb{R}_+ \to \mathbb{R}_+$ for $n \in \mathbb{N}$, and λ_1 satisfy hypothesis $(\mathcal{C}_0)$. Suppose that there exist $y_0 \in Y$ and $\varepsilon_1, \varepsilon_2 \in \mathbb{R}_+$, such that*

$$d(T(y_0), y_0) \le \varepsilon_1, \qquad d(y_0, T(y_0)) \le \varepsilon_2,$$
$$\varepsilon_j^*{:=}\sum_{i=0}^{\infty} \lambda_i(\varepsilon_j) < \infty, \qquad j = 1, 2, \tag{16}$$

and T^n is $(\varepsilon^*, \lambda_n)$—contractive for $n \in \mathbb{N}$ with $\varepsilon^*{:=}\max\{\varepsilon_1^*, \varepsilon_2^*\}$. Then, the limit

$$z_0{:=}\lim_{n\to\infty} T^n(y_0)$$

exists and z_0 is a unique fixed point of T with

$$d(T^n(y_0), z_0) \le \sum_{i=n}^{\infty} \lambda_i(\varepsilon_1), \qquad d(z_0, T^n(y_0)) \le \sum_{i=n}^{\infty} \lambda_i(\varepsilon_2), \qquad n \in \mathbb{N}_0.$$

Moreover, the following two statements are valid:

(a) *for every sequence $(k_n)_{n\in\mathbb{N}}$ of positive integers with $\lim_{n\to\infty} k_n = \infty$, z_0 is the unique fixed point of T with*

$$d(T^{k_n}(y_0), z_0) \le \sum_{i=k_n}^{\infty} \lambda^i(\varepsilon_1), \qquad d(z_0, T^{k_n}(y_0)) \le \sum_{i=k_n}^{\infty} \lambda^i(\varepsilon_2), \qquad n \in \mathbb{N};$$

(b) *if*

$$\liminf_{n\to\infty} \lambda_n(\varepsilon_j^*) = 0, \qquad j = 1, 2,$$

then z_0 is the unique fixed point of T, such that

$$d(y_0, z_0) \le \varepsilon_1^*, \qquad d(z_0, y_0) \le \varepsilon_2^*.$$

Clearly, if there is $\lambda \in \mathbb{R}_+$, such that $\lambda_n(a) = \lambda^n a$ for $a \in \mathbb{R}_+$ and $n \in \mathbb{N}$, then Theorem 5 becomes a natural modification of the Banach Contraction Principle (with a local contraction condition) and (16) means that $\lambda < 1$.

4. Ulam stability

Now, we show how we can derive some simple Ulam stability outcomes from the results of the previous section. To this end, given $e > 0$ or $e = \infty$, we need the subsequent hypothesis.

(H1) $j \in \mathbb{N}$, $L_i : E \to \mathbb{R}_+$ for $i = 1, \dots, j$, $\Phi : E \times Y^j \to Y$, and

$$d(\Phi(t, w_1, ..., w_j), \Phi(t, z_1, ..., z_j)) \le \sum_{k=1}^{j} L_k(t) d(w_k, z_k)$$

for any $t \in E$ and $(w_1, ..., w_j), (z_1, ..., z_j) \in Y^j$, such that $d(z_i, w_i) \le e$ for $i = 1, \dots, j$.

The following corollary also can be easily deduced from Theorem 2.

Corollary 6. *Assume that* $\varepsilon_1, \varepsilon_2 : E \to \mathbb{R}_+$, *hypothesis* (H1) *is valid with* $e := \sup\{\varepsilon_j^*(t) : t \in E, j = 1, 2\}$, *where*

$$\varepsilon_j^*(t) := \sum_{i=0}^{\infty} \Lambda^i \varepsilon_j(t) < \infty, \qquad t \in E, j = 1, 2,$$

and $\Lambda : \mathbb{R}_+^E \to \mathbb{R}_+^E$ *is given by*

$$\Lambda\delta(t) = \sum_{k=1}^{j} L_k(t)\delta(f_k(t)), \qquad \delta \in \mathbb{R}_+^E, t \in E,$$

with some $f_1, \dots, f_j : E \to E$, *and* $\varphi : E \to Y$ *is such that*

$$d(\Phi(t, \varphi(f_1(t)), ..., \varphi(f_j(t))), \varphi(t)) \le \varepsilon_1(t), \qquad t \in E, \tag{17}$$

$$d(\varphi(t), \Phi(t, \varphi(f_1(t)), ..., \varphi(f_j(t)))) \le \varepsilon_2(t), \qquad t \in E. \tag{18}$$

Then, the limit

$$\psi(t) := \lim_{n\to\infty} \mathcal{T}^n \varphi(t) \tag{19}$$

exists for each $t \in E$, *with* $\mathcal{T}$ *given by*

$$\mathcal{T}\varphi(t) := \Phi(t, \varphi(f_1(t)), ..., \varphi(f_j(t))), \qquad \varphi \in Y^E, t \in E,$$

and the function $\psi : E \to Y$, *defined by (19), is the unique solution of the functional equation:*

$$\Phi(t, \psi(f_1(t)), ..., \psi(f_j(t))) = \psi(t), \qquad t \in E, \tag{20}$$

such that

$$d(\varphi(t), \psi(t)) \le \varepsilon_1^*(t), \qquad d(\psi(t), \varphi(t)) \le \varepsilon_2^*(t), \qquad t \in E. \tag{21}$$

Proof. Let us note that inequalities (17) and (18) imply (3). Next

$$\liminf_{n\to\infty} \Lambda^n \varepsilon_j^*(t) = 0, \qquad t \in E, j = 1, 2,$$

$$\liminf_{n\to\infty} \Lambda\Big(\sum_{i=n}^{\infty} \Lambda^i \varepsilon_j\Big)(t) = 0, \qquad t \in E, j = 1, 2,$$

in view of Remarks 4 and 5. Therefore, by Theorem 4, the function ψ defined by (19) is the unique fixed point of $\mathcal{T}$ (that is a solution of (20)) satisfying (21). □

Stability of functional equations of form (20) (or related to it) has been already studied by several authors, and for further information, we refer to the survey papers [1,8]. A particular case of (20) is the linear functional equation of the form

$$\phi(t):=\sum_{i=1}^{j} L_i(t)\phi(f_i(t)), \qquad \varphi \in Y^E,\ t \in E,$$

under the assumptions as in Remark 5; some recent results concerning stability of less general cases of it can be found in [10,18,19,23].

As an example of applications of Corollary 6 consider stability of the difference equation:

$$\psi(i) = \Phi(i, \psi(i+1)), \qquad i \in \mathbb{N}, \tag{22}$$

where $\Phi : \mathbb{N} \times Y \to Y$ is given and $\psi : \mathbb{N} \to Y$ is unknown. Clearly, (22) is a very simple particular case of (20), with $E = \mathbb{N}$, $j = 1$ and $f_1(i) = i + 1$ for $i \in X$.

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive reals, such that

$$\sum_{k=1}^{\infty} \prod_{l=0}^{k-1} a_{i+l} < \infty, \qquad i \in \mathbb{N}. \tag{23}$$

For instance, we can take $\rho \in (0, 1)$ and write

$$a_{2n} = \frac{1}{\rho}, \qquad a_{2n-1} = \rho^2, \qquad n \in \mathbb{N}.$$

Then

$$\prod_{l=0}^{2k} a_{i+l} = \rho \prod_{l=0}^{2k-2} a_{i+l}, \qquad \prod_{l=0}^{2k+1} a_{i+l} = \rho \prod_{l=0}^{2k-1} a_{i+l}, \qquad k \in \mathbb{N},$$

whence (23) is valid and

$$\sum_{k=1}^{\infty} \prod_{l=0}^{k-1} a_{i+l} = \frac{a_i(1 + a_{i+1})}{1-\rho}, \qquad i \in \mathbb{N}.$$

Let operator $\Lambda : \mathbb{R}_+^{\mathbb{N}} \to \mathbb{R}_+^{\mathbb{N}}$ be defined by

$$\Lambda\delta(i) = a_i\delta(i + 1), \qquad \delta \in \mathbb{R}_+^{\mathbb{N}}, i \in \mathbb{N}.$$

Note that

$$\Lambda^k\delta(i) = \delta(i + k) \prod_{l=0}^{k-1} a_{i+l}, \qquad k \in \mathbb{N}, \delta \in \mathbb{R}_+^{\mathbb{N}},$$

whence

$$\sum_{k=1}^{n} \Lambda^k\delta(i) = \sum_{k=1}^{n} \delta(i + k) \prod_{l=0}^{k-1} a_{i+l}, \qquad n \in \mathbb{N}, \delta \in \mathbb{R}_+^{\mathbb{N}}. \tag{24}$$

Take $\gamma > 0$ and let $\phi : \mathbb{N} \to Y$ fulfil inequalities (17) and (18) with some $\varepsilon_1, \varepsilon_2 : \mathbb{N} \to [0, \gamma]$, that is

$$d(\Phi(i, \phi(i + 1)), \phi(i)) \le \varepsilon_1(i), \qquad d(\phi(i), \Phi(i, \phi(i + 1))) \le \varepsilon_2(i), \qquad i \in \mathbb{N}.$$

Then, (24) implies that, for each $j \in \{1, 2\}$

$$\varepsilon_j^*(i) := \sum_{k=0}^{\infty} \Lambda^k \varepsilon_j(i) \leq \gamma\Big(1 + \sum_{k=1}^{\infty} \prod_{l=0}^{k-1} a_{l+i}\Big) < \infty, \qquad i \in \mathbb{N}.$$

Next, if

$$d(\Phi(i, z), \Phi(i, w)) \leq a_i d(z, w), \qquad w, z \in Y, i \in \mathbb{N}, d(z, w) \leq e,$$

where $e := \sup\{\varepsilon_j^*(i) : i \in \mathbb{N}, j = 1, 2\}$, then Φ is as in (H1) with $j = 1$, and the assumptions of Corollary 6 are satisfied with

$$L_1(i) = a_i, \qquad f_1(i) = i + 1, \qquad i \in \mathbb{N}.$$

Hence, the limit

$$\psi(i) := \lim_{n\to\infty} \mathcal{T}^n \phi(i) \tag{25}$$

exists for each $i \in \mathbb{N}$, with

$$\mathcal{T}\xi(i) := \Phi(i, \xi(i + 1)), \qquad \xi \in Y^E, \ i \in \mathbb{N},$$

and $\psi : E \to Y$, given by (25), is the unique solution of (22), such that

$$d(\phi(i), \psi(i)) \leq \varepsilon_1^*(i), \qquad d(\psi(i), \phi(i)) \leq \varepsilon_2^*(i), \qquad i \in \mathbb{N}.$$

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