Fixed point homomorphisms for parameterized maps

Roman Srzednicki

To Professor Kazimierz Gęba on his 80th birthday

Abstract. Let X be an ANR (absolute neighborhood retract), Λ a k -dimensional topological manifold with topological orientation η , and $f: D \rightarrow X$ a locally compact map, where D is an open subset of $X \times \Lambda$. We define $\text{Fix}(f)$ as the set of points $(x, \lambda) \in D$ such that $x = f(x, \lambda)$. For an open pair (U, V) in $X \times \Lambda$ such that $\text{Fix}(f) \cap U \setminus V$ is compact we construct a homomorphism $\Sigma_{(f, U, V)}: H^k(U, V) \rightarrow R$ in the singular cohomologies H^* over a ring-with-unit R , in such a way that the properties of Solvability, Excision and Naturality, Homotopy Invariance, Additivity, Multiplicativity, Normalization, Orientation Invariance, Commutativity, Contraction, Topological Invariance, and Ring Naturality hold. In the case of a C^∞ -manifold Λ , these properties uniquely determine Σ . By passing to the direct limit of $\Sigma_{(f, U, V)}$ with respect to the pairs (U, V) such that $K = \text{Fix}(f) \cap U \setminus V$, we define a homomorphism $\sigma_{(f, K)}: \check{H}^k(\text{Fix}(f), \text{Fix}(f) \setminus K) \rightarrow R$ in the Čech cohomologies. Properties of Σ and σ are equivalent each to the other. We indicate how the homomorphisms generalize the fixed point index.

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1. Introduction

Let X and Λ be topological spaces. We consider a continuous map

$$f: D \rightarrow X,$$

where D is a subset of $X \times \Lambda$. Define the set of parameterized fixed points as

$$\text{Fix}(f) := \{(x, \lambda) \in D: f(x, \lambda) = x\}.$$

Unless otherwise stated, in the present paper we assume that X is an ANR (absolute neighborhood retract), Λ is a (Hausdorff) topological manifold of

dimension k , oriented over a commutative ring-with-unit R , D is open in $X \times \Lambda$, and f is locally compact. Our aim is to define a counterpart of the fixed point index for the map f . Let (U, V) be a pair of open subsets of $X \times \Lambda$ such that the set $\text{Fix}(f) \cap U \setminus V$ is compact. With f , (U, V) , and a given topological orientation η of Λ over R we associate a homomorphism

$$\Sigma_{(\eta, f, U, V)}: H^k(U, V) \rightarrow R,$$

where H^* denotes the singular cohomology functor over R , such that the properties of Solvability, Excision and Naturality, Homotopy Invariance, Additivity, Multiplicativity, Normalization, Orientation Invariance, Commutativity, Contraction, Topological Invariance, and Ring Naturality are satisfied (see Theorem 2.1). Actually, if we restrict ourselves to the case of C^∞ -manifolds, the properties uniquely determine the homomorphism (cf. Theorem 2.2). Since η is usually fixed, frequently we write $\Sigma_{(f, U, V)}$ instead of $\Sigma_{(\eta, f, U, V)}$ and we use other abbreviations, clear from the context.

In the case of one-point set $\Lambda = \text{pt}$, by the identification $X = X \times \text{pt}$, the set $\text{Fix}(f)$ is equal to the set of fixed points of f . The homomorphism Σ generalizes the fixed point index in the following way. Let $R = \mathbb{Z}$ and assume that the set of fixed points of a map $f: U \rightarrow X$ is compact. Then the number $\Sigma_{(f, U)}(1_U)$, where 1_U is the unit cohomology class in $H^0(U)$, is equal to the fixed point index of f in U (see also Proposition 9.1). Moreover, if $X = \mathbb{R}^n$, f is smooth, and x is a regular point of $\text{id}_{\mathbb{R}^n} - f$, we generalize the fact that the fixed point index of f in a neighborhood of x is equal to the sign of the determinant of $\text{id}_{\mathbb{R}^n} - d_x f$ (see Proposition 11.1).

It would be convenient to have a numeric invariant for the set of parameterized fixed points rather than a homomorphism. However, in opposition to the case $\Lambda = \text{pt}$, where there is $1 \in H^0(X)$ for every nonempty X , in the general case, there is no such a distinguished nontrivial class. Nevertheless, a given class $u \in H^k(X)$ provides a numerical invariant which has similar properties to the fixed point index: with open U such that $\text{Fix}(f) \cap U$ is compact associate $\Sigma_{(f, U)}(u|_U) \in R$. Numerical invariants obtained in this way for $\Lambda = \mathbb{R}$ and some specific classes over $\mathbb{Z}_p$ with p prime, lead to a generalization of the Fuller index (cf. [Fu]) which was given in [Sr1] in the finite-dimensional case. We return to this topic in a forthcoming paper.

Motivated by another notation related to topological invariants (like the Fuller index and the Conley index), for a compact set K contained in $\text{Fix}(f)$ we define a homomorphism

$$\sigma_{(f, K)} := \sigma_{(\eta, f, K)}: \check{H}^k(\text{Fix}(f), \text{Fix}(f) \setminus K) \rightarrow R$$

in the Čech cohomologies as the direct limit of the homomorphisms $\Sigma_{(f, U, V)}$, where U is a neighborhood of $\text{Fix}(f)$ and $K = \text{Fix}(f) \setminus V$. The homomorphism σ inherits the properties of Σ (see Theorems 2.3 and 2.4). (It is more convenient to formulate Commutativity for σ than for Σ .) In fact, Σ and σ are in some sense equivalent (cf. Remark 2.1), hence Theorems 2.3 and 2.4 do not require separate proofs.

The paper is organized as follows. In Section 2 we state Theorems 2.1–2.4, which are the main results here. The remaining part of the paper is devoted to the proofs of Theorem 2.1 (Sections 3–8) and Theorem 2.2 (Sections 9–12), although some results presented there might be of separate interest. In particular, in Section 3 we define Σ in the case X is a finite-dimensional vector space, in Section 4 we state an abstract lemma on Commutativity property as a consequence of other properties and apply it in the proof of the required properties of Σ in the finite-dimensional setting, in Section 5 we state some general results related to the notion of compactness, in Section 6 we extend the definition of Σ to the case of normed spaces and prove some of its properties—proofs of the remaining properties are given in Section 7, and in Section 8 we construct Σ for ANRs and we finish the proof of Theorem 2.1. Section 9 establishes a connection of Σ and σ to the fixed point index theory (Proposition 9.1), in Section 10 we consider mutual relations between homology and cohomology generators and orientations of vector spaces, in Section 11 we establish Proposition 11.1 on determination of σ in the smooth case, and finally, in Section 12 we finish the proof of Theorem 2.2.

We use the following notation and terminology. By

$$j: X \times \Lambda \rightarrow X, \quad p: X \times \Lambda \rightarrow \Lambda$$

we denote the projections. I denotes the closed interval $[0, 1]$. By $\|\cdot\|$ we denote the norm of a normed space X and by $B(x, \epsilon)$ we denote the closed ball $\{y \in X: \|x - y\| \leq \epsilon\}$. A map between topological spaces is called *compact* provided it is continuous and the closure of its image is compact. It is called *locally compact* provided its restriction to some neighborhood of each point of its domain is compact. Actually, a locally compact map is compact in some neighborhood of each compact subset of its domain. In order to shorten notation, we call a collection of maps $f_t: X \rightarrow Y$ (where $t \in I$) a *homotopy* provided

$$f: X \times I \ni (x, t) \rightarrow f_t(x) \in X$$

is continuous (i.e., the map f is a homotopy in the usual meaning). The homotopy is compact (resp., locally compact) provided f is compact (resp., locally compact). The notation concerning pairs of sets is standard (cf. [D1]); in particular a set A is treated as the pair $(A, \emptyset)$, for maps g and h , $g(A, B)$ and $h^{-1}(A, B)$ denote the pairs $(g(A), g(B))$ and $(h^{-1}(A), h^{-1}(B))$, respectively, and

$$(A, B) \times (A', B') := (A \times A', A \times B' \cup B \times A').$$

Unless otherwise stated, H and H^* denote the singular homology and, respectively, the singular cohomology functors with coefficients in R . We treat the direct sum $\phi \oplus \psi$ and the tensor product $\phi \otimes \psi$ of homomorphisms $\phi: M \rightarrow R$ and $\psi: N \rightarrow R$ of modules over R as the maps $(x, y) \rightarrow \phi(x) + \psi(y)$ and, respectively, $(x, y) \rightarrow \phi(x)\psi(y)$. We regard the Čech cohomologies $\check{H}^*$ as the direct limit of the singular ones; more exactly, for a pair (A, B) of locally compact subspaces of an ANR space X ,

$$\check{H}^*(A, B) := \text{dir lim } H^*(U, V),$$

where the limit is taken over the inverse system of all open neighborhoods (U, V) of (A, B) and the corresponding inclusions. In consequence, there are natural maps

$$\begin{aligned}\nu: H^*(U, V) &\rightarrow \check{H}^*(A, B), \\ \mu: \check{H}^*(A, B) &\rightarrow H^*(A, B);\end{aligned}$$

μ is an isomorphism if both A and B are ANRs (therefore we identify $\check{H}^*(A, B)$ and $H^*(A, B)$ in that case). By f_* , f^* , and $\check{f}$ we denote the homomorphism induced by f in singular homologies, singular cohomologies, and Čech cohomologies, respectively. All nondescribed arrows in the diagrams are induced by inclusions. The image of a cohomology class u under a homomorphism induced by an inclusion is called a *restriction* of u . By $\times$, $\langle \cdot, \cdot \rangle$, $\smile$, $\frown$, $\backslash$ we denote, respectively, both the homology and cohomology cross products, the scalar product, the cup product, the cap product, and the cohomology slant product defined as in [M] and [Sp] (or [D1], but with different sign conventions than given there). By a topological orientation η of Λ over R we mean a concordant family of homology classes $\eta_L \in H_k(\Lambda, \Lambda \setminus L)$, where L is a compact subset of Λ , such that η_λ is a generator of $H_k(\Lambda, \Lambda \setminus \lambda) \cong R$ for every $\lambda \in \Lambda$. (Recall that if Λ is oriented over $\mathbb{Z}$, then it is oriented over an arbitrary R and, in general, Λ is always oriented over $\mathbb{Z}_2$.) We denote also by η the induced orientation on each open subset of Λ . For a one-point manifold pt we assume that the orientation is given by the (trivial) 0-dimensional singular simplex. If Λ' is another manifold with an orientation η' over R , by $\eta \times \eta'$ we denote the orientation of $\Lambda \times \Lambda'$ (as well as of each of its open subset) over R determined by

$$(\eta \times \eta')_{L \times L'} := \eta_L \times \eta'_{L'}$$

for all compact $L \subset \Lambda$ and $L' \subset \Lambda'$. If $\alpha: \Lambda \rightarrow \Xi$ is a homeomorphism, by $\alpha_*(\eta)$ we denote the induced orientation on Ξ , i.e., the orientation determined by $\alpha_*(\eta_L) \in H_k(\Xi, \Xi \setminus \alpha(L))$.

This paper is a revised and extended version of a part of the unpublished preprint [Sr2].

2. The homomorphisms and their properties

The main result of the current paper is the following.

Theorem 2.1. *For a locally compact map $f: D \rightarrow X$, where $D \subset X \times \Lambda$ is open, X is an ANR, and Λ is a k -dimensional topological manifold with an orientation η , and an open pair (U, V) in $X \times \Lambda$ such that $\text{Fix}(f) \cap U \setminus V$ is compact, there exists a homomorphism*

$$\Sigma_{(f, U, V)} := \Sigma_{(\eta, f, U, V)}: H^k(U, V) \rightarrow R$$

which has the following properties.

- (I) Solvability. *If $\Sigma_{(f, U, V)} \neq 0$, then $\text{Fix}(f) \cap U \setminus V \neq \emptyset$.*

(II) Excision and Naturality.

$$\Sigma_{(f|_U, U, V)} = \Sigma_{(f, U, V)}$$

and if (U', V') is open in $X \times \Lambda$, $(U, V) \subset (U', V')$,

$$\text{Fix}(f) \cap U' \setminus V' \subset \text{Fix}(f) \cap U \setminus V,$$

then $\text{Fix}(f) \cap U' \setminus V'$ is also compact and the diagram

$$\begin{array}{ccc} H^k(U', V') & & \\ \downarrow & \searrow \Sigma_{(f, U', V')} & \\ & R & \\ \uparrow \Sigma_{(f, U, V)} & & \\ H^k(U, V) & & \end{array}$$

commutes.

(III) Homotopy Invariance. If $f_t: U \rightarrow X$ is a locally compact homotopy and $\bigcup_{t \in I} \text{Fix}(f_t) \cap U \setminus V$ is compact, then

$$\Sigma_{(f_0, U, V)} = \Sigma_{(f_1, U, V)}.$$

(IV) Additivity. Assume $(U, V) = (U_0 \cup U_1, V_0 \cup V_1)$ and $U_0 \cap U_1 = \emptyset$. Then the diagram

$$\begin{array}{ccc} H^k(U, V) & & \\ \downarrow \cong & \searrow \Sigma_{(f, U, V)} & \\ & R & \\ \uparrow \Sigma_{(f, U_0, V_0)} \oplus \Sigma_{(f, U_1, V_1)} & & \\ H^k(U_0, V_0) \oplus H^k(U_1, V_1) & & \end{array}$$

commutes.

(V) Multiplicativity. Let Λ' be a manifold of dimension k' and let η' be its orientation over R . Assume that X' is an ANR, (U', V') is an open pair in $X' \times \Lambda'$, and $f': U' \rightarrow X'$ is locally compact. Let

$$\pi: X \times X' \times \Lambda \times \Lambda' \rightarrow X \times \Lambda \times X' \times \Lambda'$$

permute the coordinates. Then the diagram

$$\begin{array}{ccc} H^k(U, V) \otimes H^k(U', V') & & \\ \downarrow \times & \searrow \Sigma_{(\eta, f, U, V)} \otimes \Sigma_{(\eta', f', U', V')} & \\ H^{k+k'}((U, V) \times (U', V')) & & R \\ \downarrow \pi^* \cong & \nearrow \Sigma_{(\eta \times \eta', f \times f' \circ \pi, \pi^{-1}((U, V) \times (U', V')))} & \\ H^{k+k'}(\pi^{-1}((U, V) \times (U', V'))) & & \end{array}$$

commutes.

- (VI) Normalization. Let $x_0 \in X$ and let $c: X \times \Lambda \rightarrow X$ be the constant map $(x, \lambda) \rightarrow x_0$, hence $\text{Fix}(c) = x_0 \times \Lambda$. If L is a compact subset of Λ and $v \in H^k(\Lambda, \Lambda \setminus L)$, then

$$\Sigma_{(c, X \times (\Lambda, \Lambda \setminus L))}(1_X \times v) = \langle v, \eta_L \rangle.$$

- (VII) Orientation Invariance. If Λ' is a k -dimensional manifold with an orientation η' , $\alpha: \Lambda \rightarrow \Lambda'$ is a continuous injection (hence a homeomorphism onto $\alpha(\Lambda)$ which is open in Λ' by Domain Invariance Theorem), and the induced orientation $\alpha_*(\eta)$ coincides with η' on $\alpha(\Lambda)$, then the diagram

$$\begin{array}{ccc} H^k(\text{id}_X \times \alpha(U, V)), & & \\ \downarrow (\text{id}_X \times \alpha)^* \cong & \searrow \Sigma_{(\eta', f \circ (\text{id}_X \times \alpha)^{-1}, \text{id}_X \times \alpha(U, V))} & \\ & R & \\ & \nearrow \Sigma_{(\eta, f, U, V)} & \\ H^k(U, V) & & \end{array}$$

commutes.

- (VIII) Commutativity. Let X' be another ANR, let D be an open subset of $X \times \Lambda$, let D' be open in $X' \times \Lambda$, and let $g: D \rightarrow X'$ and $g': D' \rightarrow X$ be continuous. Assume that one of the maps g or g' is locally compact. Define $G: D \rightarrow X' \times \Lambda$ by $G(x, \lambda) := (g(x, \lambda), \lambda)$ and $G': D' \rightarrow X \times \Lambda$ by $G'(x', \lambda) := (g'(x', \lambda), \lambda)$. Then

- (a) G and G' induce mutually inverse homeomorphisms

$$G: \text{Fix}(g' \circ G) \rightleftarrows \text{Fix}(g \circ G'): G',$$

- (b) for an open pair (U', V') in $X' \times \Lambda$, $U' \subset D'$, such that

$$\text{Fix}(g \circ G') \cap U' \setminus V'$$

is compact and an open pair (U, V) in $X \times \Lambda$ such that

$$G(U, V) \subset (U', V'),$$

$$\text{Fix}(g \circ G') \cap G(U) \setminus G(V) = \text{Fix}(g \circ G') \cap U' \setminus V',$$

the diagram

$$\begin{array}{ccc} H^k(U', V') & & \\ \downarrow G^* & \searrow \Sigma_{(g \circ G', U', V')} & \\ & R & \\ & \nearrow \Sigma_{(g' \circ G, U, V)} & \\ H^k(U, V) & & \end{array}$$

commutes.

- (IX) Contraction. If Y is an ANR contained in X , $i: Y \hookrightarrow X$ is the inclusion, (U, V) is an open pair in $X \times \Lambda$, and $g: U \rightarrow Y$ is locally compact, then

the diagram

$$\begin{array}{ccc}
 H^k(U, V) & & \\
 \downarrow & \searrow \Sigma_{(i \circ g, U, V)} & \\
 & R & \\
 & \nearrow \Sigma_{(g|_{Y \times \Lambda}, U \cap Y, V \cap Y)} & \\
 H^k(U \cap Y, V \cap Y) & &
 \end{array}$$

commutes.

- (X) Topological Invariance. If $h: X \rightarrow X'$ is a homeomorphism, then the diagram

$$\begin{array}{ccc}
 H^k(h \times \text{id}_\Lambda(U, V)), & & \\
 \downarrow (h \times \text{id}_\Lambda)^* \cong & \searrow \Sigma_{(f \circ (h \times \text{id}_\Lambda)^{-1}, h \times \text{id}_\Lambda(U, V))} & \\
 & R & \\
 & \nearrow \sigma_{(f, U, V)} & \\
 H^k(U, V) & &
 \end{array}$$

commutes.

- (XI) Ring Naturality. Let $\rho: R \rightarrow R'$ be a homomorphism of rings-with-unit and let ρ_* denote both the natural map between the homologies and between the cohomologies with coefficients in R and R' induced by it. Then the diagram

$$\begin{array}{ccc}
 H^k(U, V; R) & \xrightarrow{\Sigma_{(\eta, f, U, V)}} & R \\
 \downarrow \rho_* & & \downarrow \rho \\
 H^k(U, V; R') & \xrightarrow{\Sigma_{(\rho_*(\eta), f, U, V)}} & R'
 \end{array}$$

commutes, where the orientation $\rho_*(\eta)$ is given by the homology classes of the form $\rho_*(\eta_L) \in H_k(\Lambda, \Lambda \setminus L; R')$.

Actually, the properties of Contraction and Topological Invariance are direct consequences of Commutativity. Moreover, we have the following theorem.

Theorem 2.2 (Uniqueness). *If the considered manifolds Λ are C^∞ -differentiable, then the properties (I)–(XI) uniquely determine the homomorphism Σ .*

In Sections 3–8, in several steps we provide a construction of the homomorphism Σ satisfying Theorem 2.1. The proof of Theorem 2.2 is postponed to Section 12.

Let Σ be given by Theorem 2.1. Assume that K is a compact subset of $\text{Fix}(f)$. The set of open pairs $(U, V) \supset (\text{Fix}(f), \text{Fix}(f) \setminus K)$, and the inclusions

among them, is an inverse system, hence by (II), $\Sigma_{(f,U,V)}$ form a direct system of homomorphisms. Define

$$\sigma_{(f,K)} := \sigma_{(\eta,f,K)} := \operatorname{dir} \lim \Sigma_{(\eta,f,U,V)} : \check{H}^k(\operatorname{Fix}(f), \operatorname{Fix}(f) \setminus K) \rightarrow R.$$

Theorem 2.3. *The homomorphism σ has the following properties.*

(I $^\sigma$) Excision. *If U is open in $X \times \Lambda$ and $K \subset U$, then the diagram*

$$\begin{array}{ccc} \check{H}^k(\operatorname{Fix}(f), \operatorname{Fix}(f) \setminus K) & & \\ \cong \downarrow & \searrow \sigma_{(f,K)} & \\ & & R \\ & \nearrow \sigma_{(f|_U, K)} & \\ \check{H}^k(\operatorname{Fix}(f|_U), \operatorname{Fix}(f|_U) \setminus K) & & \end{array}$$

commutes.

(II $^\sigma$) Naturality. *If $K \subset K' \subset \operatorname{Fix}(f)$, then the diagram*

$$\begin{array}{ccc} \check{H}^k(\operatorname{Fix}(f), \operatorname{Fix}(f) \setminus K) & & \\ \downarrow & \searrow \sigma_{(f,K)} & \\ & & R \\ & \nearrow \sigma_{(f,K')} & \\ \check{H}^k(\operatorname{Fix}(f), \operatorname{Fix}(f) \setminus K') & & \end{array}$$

commutes.

(III $^\sigma$) Homotopy Invariance. *Let $f_t: D \rightarrow X$ be a locally compact homotopy and let*

$$F := \bigcup_t \operatorname{Fix}(f_t).$$

Assume that K is a compact subset of F . Then $K_t := K \cap \operatorname{Fix}(f_t)$ is compact and the diagram

$$\begin{array}{ccccc} & & \check{H}^k(\operatorname{Fix}(f_0), \operatorname{Fix}(f_0) \setminus K_0) & & \\ & \nearrow & & \searrow \sigma_{(f_0, K_0)} & \\ \check{H}^k(F, F \setminus K) & & & & R \\ & \searrow & & \nearrow \sigma_{(f_1, K_1)} & \\ & & \check{H}^k(\operatorname{Fix}(f_1), \operatorname{Fix}(f_1) \setminus K_1) & & \end{array}$$

commutes.

(IV $^\sigma$) Additivity. If K_0 and K_1 are compact disjoint subsets of $\text{Fix}(f)$, then the diagram

$$\begin{array}{ccc}
 \check{H}^k(\text{Fix}(f), \text{Fix}(f) \setminus (K_0 \cup K_1)) & & \\
 \downarrow \cong & \searrow \sigma_{(f, K_0 \cup K_1)} & \\
 & & R \\
 & \nearrow \sigma_{(f, K_0)} \oplus \sigma_{(f, K_1)} & \\
 \check{H}^k(\text{Fix}(f), \text{Fix}(f) \setminus K_0) \oplus \check{H}^k(\text{Fix}(f), \text{Fix}(f) \setminus K_1) & &
 \end{array}$$

commutes.

(V $^\sigma$) Multiplicativity. Under the notation of (V), if K' is compact in $\text{Fix}(f')$, then the diagram

$$\begin{array}{ccc}
 \check{H}^k(\text{Fix}(f), \text{Fix}(f) \setminus K) \otimes \check{H}^k(\text{Fix}(f'), \text{Fix}(f') \setminus K') & & \\
 \downarrow \times & \searrow \sigma_{(\eta, f, K)} \otimes \sigma_{(\eta', f', K')} & \\
 \check{H}^{k+k'}(\text{Fix}(f) \times \text{Fix}(f'), \text{Fix}(f) \times \text{Fix}(f') \setminus K \times K') & & R \\
 \downarrow \tilde{\pi} \cong & \nearrow \sigma_{(\eta \times \eta', f \times f' \circ \pi, \pi^{-1}(K \times K'))} & \\
 \check{H}^{k+k'}(\pi^{-1}(\text{Fix}(f) \times \text{Fix}(f')), \text{Fix}(f) \times \text{Fix}(f') \setminus K \times K') & &
 \end{array}$$

commutes.

(VI $^\sigma$) Normalization. Under the notation of (VI),

$$\sigma_{(c, x_0 \times L)}(1_{x_0} \times \nu(v)) = \langle v, \eta_{\lambda_0} \rangle$$

for each class $v \in H^k(\Lambda, \Lambda \setminus \lambda_0)$, where $\nu: H^k(\Lambda, \Lambda \setminus L) \rightarrow \check{H}^k(L)$ is the natural map.

(VII $^\sigma$) Orientation Invariance. Under the notation of (VII),

$$\sigma_{(\eta, f, K)} = \sigma_{(\eta', f \circ (\text{id}_X \times \alpha)^{-1}, \text{id}_X \times \alpha(K))} \circ \text{id}_X \check{\times} \alpha.$$

(VIII $^\sigma$) Commutativity. Under the notation of (VII) (which implies, in particular, (a) in (VII)),

(b $^\sigma$) if K is compact in $\text{Fix}(g' \circ G)$ and $K' := G(K)$, then the diagram

$$\begin{array}{ccc}
 \check{H}^k(\text{Fix}(g \circ G'), \text{Fix}(g \circ G') \setminus K') & & \\
 \downarrow \cong & \searrow \sigma_{(g \circ G', K')} & \\
 & R & \\
 & \nearrow \sigma_{(g' \circ G, K)} & \\
 \check{H}^k(\text{Fix}(g' \circ G), \text{Fix}(g' \circ G) \setminus K) & &
 \end{array}$$

commutes.

(IX $^\sigma$) Contraction. If Y is an ANR contained in X , $i: Y \hookrightarrow X$ is the inclusion, D is open in $X \times \Lambda$, and $g: D \rightarrow Y$ is locally compact, then

$$\sigma_{(i \circ g, K)} = \sigma_{(g|_{Y \times \Lambda}, K)}.$$

(X $^\sigma$) Topological Invariance. If $h: X \rightarrow X'$ is a homeomorphism, then

$$\sigma_{(f, K)} = \sigma_{(f \circ (h \times \text{id}_\Lambda)^{-1}, h \times \text{id}_\Lambda(K))} \circ h \check{\times} \text{id}_\Lambda.$$

(XI $^\sigma$) Ring Naturality. Under the notation of (XI), the diagram

$$\begin{array}{ccc}
 \check{H}^k(\text{Fix}(f), \text{Fix}(f) \setminus K; R) & \xrightarrow{\sigma_{(\eta, f, K)}} & R \\
 \downarrow \rho_* & & \downarrow \rho \\
 \check{H}^k(\text{Fix}(f), \text{Fix}(f) \setminus K; R') & \xrightarrow{\sigma_{(\rho_*(\eta), f, K)}} & R'
 \end{array}$$

commutes.

The counterpart of Solvability for σ is redundant: if $K = \emptyset$, then the cohomologies of the pair $(\text{Fix}(f), \text{Fix}(f) \setminus K)$ are equal to 0. The properties of σ follow the corresponding properties of Σ by passing to the limit. However, we do not treat Theorem 2.3 as a corollary of Theorem 2.1 since at some stage of the construction of Σ , our proof of the property (VIII $^\sigma$) predeceases the proof of (VIII) (see Step 3 in the proof of Lemma 4.1).

Remark 2.1. If $K = \text{Fix}(f) \cap U \setminus V$ is compact, then

$$\Sigma_{(f, U, V)} = \sigma_{(f|_U, K)} \circ \nu: H^k(U, V) \rightarrow R,$$

hence each of the properties (II)–(XI) is equivalent to the corresponding properties among (I $^\sigma$)–(XI $^\sigma$).

By Remark 2.1, Theorem 2.2 has the following equivalent interpretation for σ .

Theorem 2.4 (Uniqueness). If the considered manifolds Λ are C^∞ -differentiable, then the properties (I $^\sigma$)–(XI $^\sigma$) uniquely determine the homomorphism σ .

3. Construction of Σ in finite-dimensional vector spaces

Let (U, V) be an open pair in $X \times \Lambda$ and let $K := \text{Fix}(f) \cap U \setminus V$ be compact. Our aim is to construct $\Sigma_{(f,U,V)}$. Unless otherwise stated, in what follows we will assume that U is contained in the domain D of f , since in the general case the composition

$$\Sigma_{(f,U,V)}: H^k(U, V) \rightarrow H^k(U \cap D, V \cap D) \xrightarrow{\Sigma_{(f,U \cap D, V \cap D)}} R$$

will satisfy all requirements. It follows, in particular, that

$$\Sigma_{(f,U,V)} = \Sigma_{(f|_U, U, V)}.$$

Assume first $X = \mathbb{R}^n$. Let o^n be a generator of $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus 0)$ and let $s^n \in H^n(\mathbb{R}^n, \mathbb{R}^n \setminus 0)$ be the dual generator to o^n , i.e., $\langle s^n, o^n \rangle = 1$. The generator o^n determines the orientation of $\mathbb{R}^n$ denoted by o . Define

$$\begin{aligned} \Sigma_{(f,U,V)}: H^k(U, V) &\xrightarrow{\smile(o \times \eta)_K} H_n(U, U \setminus \text{Fix}(f)) \\ &\xrightarrow{(j-f)_*} H_n(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \xrightarrow{\langle s^n, \cdot \rangle} R. \end{aligned}$$

Actually, if the products are defined as in [D1], the above composition of homomorphisms should be multiplied by $(-1)^{nk}$.

Remark 3.1. In the considered case $\Sigma_{(f,U,V)}$ can be alternatively defined in the spirit of the fixed point transfer from [D2], as the composition

$$\begin{aligned} \Sigma': H^k(U, V) &\xrightarrow{s^n \times} H^{n+k}((\mathbb{R}^n, \mathbb{R}^n \setminus 0) \times (U, V)) \\ &\xrightarrow{(j-f, \text{id}_U)^*} H^{n+k}(U, U \setminus K) \xrightarrow{\langle \cdot, (o \times \eta)_K \rangle} R. \end{aligned}$$

Indeed, by formulas on products given in [D1] (taking into account the fact that the definitions of products in [D1] differ by a sign from the definitions in [M] and [Sp]) one has

$$\begin{aligned} \Sigma'(u) &= \langle s^n \times u, (j-f, \text{id}_U)_*((o \times \eta)_K) \rangle \\ &= \langle s^n, u \setminus (j-f, \text{id}_U)_*((o \times \eta)_K) \rangle \\ &= \langle s^n, (j-f)_*(u \smile (o \times \eta)_K) \rangle \\ &= \Sigma_{(f,U,V)}(u). \end{aligned}$$

We prefer the present definition of $\Sigma_{(f,U,V)}$ rather than Σ' since it seems to have a simpler geometric meaning and directly generalizes a standard homology approach to the fixed point index (like in [D1]).

It is easy to see that the above-defined $\Sigma_{(f,U,V)}$ does not depend on the choice of o^n . Using a linear isomorphism we extend that definition to the case of an n -dimensional vector space X . Properties of homologies and cohomologies, and their products, imply immediately the following lemma.

Lemma 3.1. *If X and X' are finite-dimensional vector spaces, then the homomorphism Σ satisfies properties (I)–(VII), (X), and (XI).*

4. Commutativity property for locally compact maps

We begin with an observation that (a) in (VIII) does not require any assumption on compactness of g or g' , and it is easy to verify. Therefore (b) is the essential part of Commutativity. The following lemma reduces the proof of (b) for locally compact maps to verification of other properties. Let $\mathcal{E}$ denote a subclass of the class of all normed spaces, closed with respect to the cartesian products.

Lemma 4.1. *Assume Σ that is defined for all spaces in the class $\mathcal{E}$. If properties (II), (III), (V), (VI), and (VII) are satisfied in $\mathcal{E}$, then also (VIII) holds in $\mathcal{E}$ provided g and g' are locally compact.*

Proof. We follow an idea from [D1, Subsection VII.5.9] or [G, Section 8]. Set

$$K' := \text{Fix}(g \circ G') \cap U' \setminus V'$$

and $K := G^{-1}(K')$. Denote by q and q' the projections

$$q: X \times X' \times \Lambda \rightarrow X \times \Lambda, \quad q': X \times X' \times \Lambda \rightarrow X' \times \Lambda,$$

define sets

$$\tilde{U} := \{(x, x', \lambda): (x, \lambda) \in U, (x', \lambda) \in U'\},$$

$$\tilde{V} := \{(x, x', \lambda): (x, \lambda) \in V, (x', \lambda) \in V'\},$$

and a map

$$\tilde{G}: \tilde{U} \ni (x, x', \lambda) \rightarrow (g'(x, \lambda), g(x, \lambda)) \in X \times X'.$$

It follows that

$$\text{Fix}(\tilde{G}) = \{(x, x', \lambda): (x, \lambda) \in \text{Fix}(g' \circ G)\}$$

and $q: \text{Fix}(\tilde{G}) \rightarrow \text{Fix}(g' \circ G)$ and $q': \text{Fix}(\tilde{G}) \rightarrow \text{Fix}(g \circ G')$ are homeomorphisms, hence $\text{Fix}(\tilde{G}) \cap \tilde{U} \setminus \tilde{V}$ is compact. By assumptions, $\tilde{G}$ is a locally compact map and

$$\tilde{K} := q'^{-1}(K')$$

is compact.

Step 1. *The diagram*

$$\begin{array}{ccc} & H^k(U, V) & \\ q^* \swarrow & & \searrow \Sigma_{(g' \circ G, U, V)} \\ H^k(\tilde{U}, \tilde{V}) & \xrightarrow{\Sigma_{(\tilde{G}, \tilde{U}, \tilde{V})}} & R \end{array}$$

commutes.

In order to prove the claim, define the following subsets of $X \times X' \times \Lambda$:

$$W := \{(x, x', \lambda): (x, \lambda) \in U\},$$

$$Z := \{(x, x', \lambda): (x, \lambda) \in V\}$$

and the locally compact homotopies:

$$\begin{aligned} a_t: \tilde{U} \ni (x, x', \lambda) &\rightarrow ((1-t)g' \circ G(x, \lambda) + tg'(x', \lambda), g(x, \lambda)) \in X \times X', \\ b_t: W \ni (x, x', \lambda) &\rightarrow (g' \circ G(x, \lambda), tg(x, \lambda)) \in X \times X'. \end{aligned}$$

In particular, $a_1 = \tilde{G}$. One has

$$\begin{aligned} \text{Fix}(a_t) &= \{(x, x', \lambda): x = g'(x', \lambda), x' = g(x, \lambda)\}, \\ \text{Fix}(b_t) &= \{(x, x', \lambda): x = g' \circ G(x, \lambda), x' = tg(x, \lambda)\}. \end{aligned}$$

It follows by (III) that

$$\Sigma_{(a_0, \tilde{U}, \tilde{V})} = \Sigma_{(\tilde{G}, \tilde{U}, \tilde{V})}, \quad \Sigma_{(b_0, W, Z)} = \Sigma_{(b_1, W, Z)}.$$

Since $a_0(x, x', \lambda) = b_1(x, x', \lambda)$, $(\tilde{U}, \tilde{V}) \subset (W, Z)$,

$$\begin{aligned} \text{Fix}(b_1) \cap W \setminus Z &= \{(x, x', \lambda): (x, \lambda) \in U \setminus V, x = g'(x', \lambda), x' = g(x, \lambda)\} \\ &\subset \text{Fix}(b_1) \cap \tilde{U} \setminus \tilde{V} \end{aligned}$$

(because $G(U) \subset U'$ by assumption), the diagram

$$\begin{array}{ccc} H^k(W, Z) & & \\ \downarrow & \searrow \Sigma_{(b_0, W, Z)} = \Sigma_{(b_1, W, Z)} & \\ & R & \\ & \nearrow \Sigma_{(b_1, \tilde{U}, \tilde{V})} = \Sigma_{(\tilde{G}, \tilde{U}, \tilde{V})} & \\ H^k(\tilde{U}, \tilde{V}) & & \end{array}$$

commutes by (II). Let P denote a one-point space (hence a 0-dimensional connected manifold) and let an orientation θ of P be determined by the cycle equal to the (unique) 0-dimensional singular simplex on P . Let p and α be the projections

$$p: X \times \Lambda \times X' \times P \rightarrow X \times \Lambda, \quad \alpha: \Lambda \times P \rightarrow \Lambda.$$

α is a homeomorphism such that

$$\alpha_*(\eta \times \theta) = \eta.$$

In the diagram

$$\begin{array}{ccccc} & & H^k(U, V) & & \\ & \swarrow p^* & \downarrow \text{id} \otimes 1 & \searrow \Sigma_{(g' \circ G, U, V)} & \\ H^k((U, V) \times X' \times P) & \xleftarrow{\times} & H^k(U, V) \otimes H^0(X' \times P) & \xrightarrow{\Sigma \otimes \Sigma} & R \\ \downarrow \pi^* \cong & & \nearrow \Sigma & & \uparrow \Sigma_{(b_0, W, Z)} \\ H^k(W \times P, Z \times P) & \xrightarrow[(\cong)]{(\text{id}_{X \times X'} \times \alpha^{-1})^*} & & & H^k(W, Z) \end{array}$$

Σ denotes $\Sigma_{(\eta \times \theta, b_0 \circ (\text{id} \times \alpha), W \times P, Z \times P)}$ and $\Sigma \otimes \Sigma$ denotes

$$\Sigma_{(g' \circ G, U, V)} \otimes \Sigma_{(\theta, 0, X' \times P)}.$$

The diagram commutes. Indeed, the upper left triangle commutes by the units property of the cohomology cross product, the upper right triangle commutes since X' is path connected and therefore the homomorphism

$$\Sigma_{(\theta, 0, X' \times P)}: H^0(X' \times P) \rightarrow R$$

sends the generator $1_{(X' \times P)}$ to 1 by (VI), the lower triangle above the diagonal commutes by (V), and the lower triangle below the diagonal commutes by (VII). Since

$$q = p \circ \pi \circ (\text{id}_{X \times X'} \times \alpha^{-1}),$$

by the commutativity of the above two diagrams, we get

$$\Sigma_{(\tilde{G}, \tilde{U}, \tilde{V})} \circ q^* = \Sigma_{(g' \circ G, U, V)},$$

hence the claim is proved.

Step 2. *The diagram*

$$\begin{array}{ccc}
 & \check{H}^k(\text{Fix}(\gamma), \text{Fix}(\gamma) \setminus K) & \\
 \swarrow \tilde{q} \cong & & \searrow \sigma_{(\gamma, K)} \\
 \check{H}^k(\text{Fix}(\tilde{G}), \text{Fix}(\tilde{G}) \setminus \tilde{K}) & \xrightarrow{\sigma_{(\tilde{G}, \tilde{K})}} & R \\
 \nwarrow \cong \tilde{q}' & & \nearrow \sigma_{(\gamma', K')} \\
 & \check{H}^k(\text{Fix}(\gamma'), \text{Fix}(\gamma') \setminus K') &
 \end{array}$$

commutes, where $\gamma := g' \circ G|_U$ and $\gamma' := g \circ G'|_{U'}$.

Indeed, the upper triangle commutes by Step 1 and the limit passage. Define

$$U'' := G'^{-1}(U) \cap U',$$

$$V'' := G'^{-1}(V) \cap V',$$

$$\tilde{U}' := \{(x, x', \lambda): (x, \lambda) \in U, (x', \lambda) \in U''\},$$

$$\tilde{V}' := \{(x, x', \lambda): (x, \lambda) \in V, (x', \lambda) \in V''\}.$$

Since G' restricted to the set of fixed points is a homeomorphism (i.e., (a) in (VIII) holds),

$$\text{Fix}(g \circ G') \cap G'^{-1}(U) = G'(\text{Fix}(g' \circ G) \cap U) = \text{Fix}(g \circ G') \cap U',$$

$$\text{Fix}(g \circ G') \cap G'^{-1}(V) = G'(\text{Fix}(g' \circ G) \cap V) = \text{Fix}(g \circ G') \cap V',$$

hence

$$\text{Fix}(g \circ G') \cap U'' \setminus V'' = \text{Fix}(g \circ G') \cap U' \setminus V'.$$

By Step 1,

$$\Sigma_{(\tilde{G}, \tilde{U}', \tilde{V}')} \circ q'^* = \Sigma_{(g \circ G', U'', V'')}.$$

Since the diagram

$$\begin{array}{ccc} H^k(U', V') & \longrightarrow & H^k(U'', V'') \\ \downarrow q'^* & & \downarrow q'^* \\ H^k(\tilde{U}, \tilde{V}) & \longrightarrow & H^k(\tilde{U}', \tilde{V}') \end{array}$$

commutes, by (II) we get

$$\Sigma_{(\tilde{G}, \tilde{U}, \tilde{V})} \circ q'^* = \Sigma_{(g \circ G', U', V')},$$

hence by passing to the limit we get the commutativity of the lower triangle and Step 2 is proved.

Step 3. *The conclusion of the lemma holds.*

Indeed, since

$$G = q' \circ q^{-1}: \text{Fix}(g' \circ G) \rightarrow \text{Fix}(g \circ G'),$$

by Step 2, by the commutativity of the diagram

$$\begin{array}{ccc} H^k(U', V') & \xrightarrow{\nu} & \check{H}^k(\text{Fix}(\gamma'), \text{Fix}(\gamma') \setminus K') \\ \downarrow G^* & & \downarrow \check{G} \\ H^k(U, V) & \xrightarrow{\nu} & \check{H}^k(\text{Fix}(\gamma), \text{Fix}(\gamma) \setminus K), \end{array}$$

and by Remark 2.1 the result follows. $\square$

As a consequence of Lemma 3.1 and Lemma 4.1 applied to finite-dimensional vector spaces (because all continuous maps defined on their open subsets are locally compact) we get the following lemma.

Lemma 4.2. *If X , X' , and Y are finite-dimensional vector spaces, then the homomorphism Σ satisfies properties (I)–(XI).*

5. Compactness in normed spaces

In this section we assume that X is a normed space and Λ is a first countable Hausdorff space. We provide sufficient conditions of the compactness of the set $\text{Fix}(f) \cap U \setminus V$. They modify well-known criteria for unparameterized maps.

Lemma 5.1. *Let A be a closed subset of $X \times \Lambda$ such that $\overline{p(A)}$ is compact. If $f: A \rightarrow X$ is a compact map, then*

(a) $j - f$ is closed,

- (b) $\text{Fix}(f)$ is compact,
- (c) if $B \subset A$ is closed and $B \cap \text{Fix}(f) = \emptyset$, then

$$\inf\{\|x - f(x, \lambda)\| : (x, \lambda) \in B\} > 0.$$

Proof. In order to prove (a) assume that B is a closed subset of A , $(x_n, \lambda_n) \in B$, and

$$x_n - f(x_n, \lambda_n) \rightarrow y \in X.$$

Without loss of generality, we can assume that there exist $z \in X$ and $\lambda \in \Lambda$ such that

$$f(x_n, \lambda_n) \rightarrow z, \quad \lambda_n \rightarrow \lambda.$$

Then $(x_n, \lambda_n) \rightarrow (y+z, \lambda) \in B$, since B is closed. It follows that $f(y+z, \lambda) = z$, hence $y \in (j-f)(B)$. The conclusion (b) follows directly from (a), and (c) is a straightforward consequence of (a) and (b). $\square$

Lemma 5.2. *Let A be as in Lemma 5.1. If $f_t: A \rightarrow X$ is a compact homotopy, then $\bigcup_{t \in I} \text{Fix}(f_t)$ is compact.*

Proof. Let $F(\cdot, t) := f_t$. By Lemma 5.1(b), the set

$$M := \{(x, \lambda, t) \in X \times \Lambda \times I : F(x, \lambda, t) = x\}$$

is compact. Hence $\bigcup_t \text{Fix}(f_t) = q(M)$ is compact, where q denotes the projection $X \times \Lambda \times I \rightarrow X \times \Lambda$. $\square$

Let (U, V) be an open pair in $X \times \Lambda$. We apply the above lemmas to sets related to the homomorphism Σ .

Lemma 5.3. *Assume that $\overline{p(U)}$ is compact.*

- (a) *If $f: \overline{U} \rightarrow X$ is compact and*

$$\text{Fix}(f) \cap \overline{U \setminus V} \setminus U = \emptyset,$$

then $\text{Fix}(f) \cap U \setminus V$ is compact.

- (b) *If $f_t: \overline{U} \rightarrow X$ is a compact homotopy and*

$$\text{Fix}(f_t) \cap \overline{U \setminus V} \setminus U = \emptyset \quad \forall t \in I,$$

then $\bigcup_t \text{Fix}(f_t) \cap U \setminus V$ is compact.

Proof. Since $U \setminus V$ is closed in U ,

$$\overline{U \setminus V} \cap U = U \setminus V.$$

It follows by assumptions that

$$\text{Fix}(f) \cap U \setminus V = \text{Fix}(f) \cap \overline{U \setminus V}.$$

The right-hand set is compact by Lemma 5.1(b), hence (a) is proved. Similarly,

$$\bigcup_t \text{Fix}(f_t) \cap U \setminus V = \bigcup_t \text{Fix}(f_t) \cap \overline{U \setminus V},$$

hence (b) follows by Lemma 5.2. $\square$

We end this section by a simple observation on an inverse of Lemma 5.3.

Lemma 5.4. Assume $K := \text{Fix}(f) \cap U \setminus V$. Let U' be such that $K \subset U'$ and $\overline{U'} \subset U$. If $V' := U' \cap V$, then

$$\text{Fix}(f) \cap \overline{U' \setminus V'} \setminus U' = \emptyset.$$

Proof. Since $\overline{U' \setminus V'} \subset U \setminus V$ and $K \subset U'$, the result follows. $\square$

6. Construction of Σ in normed spaces

Throughout this section X denotes a normed space. At the beginning, assume that $\overline{U} \subset D$, $p(\overline{U})$ is compact, $f|_{\overline{U}}$ is a compact map, and $\text{Fix}(f) \cap \overline{U} \setminus \overline{V} \setminus U$ is empty. By Lemma 5.3(a), $\text{Fix}(f) \cap U \setminus V$ is compact and by Lemma 5.1(c),

$$\zeta := \inf \left\{ \|x - f(x, \lambda)\| : (x, \lambda) \in \overline{U} \setminus \overline{V} \setminus U \right\} > 0,$$

Let $g: \overline{U} \rightarrow X$ be a finite-dimensional ϵ -approximation of $f|_{\overline{U}}$ (by Schauder Approximation Theorem, see [GD] or [G, (4.1)]), where $0 < \epsilon < \zeta$. By Lemma 5.3(a), the set $\text{Fix}(g) \cap U \setminus V$ is compact. Let Y be a finite-dimensional subspace of X which contains the image of g and let $g_Y^U: U \cap Y \rightarrow Y$ be the restriction of g .

Lemma 6.1. *The composition*

$$\Sigma'_{g,Y}: H^k(U, V) \longrightarrow H^k(U \cap Y, V \cap Y) \xrightarrow{\Sigma_{(g_Y^U, U \cap Y, V \cap Y)}} R$$

is independent of the choice of g and Y .

Proof. The independence of the choice of Y follows by property (IX) stated in Lemma 4.2. Let $g': \overline{U} \rightarrow X$ be another ϵ -approximation of $f|_{\overline{U}}$ with the image contained in a finite-dimensional subspace Y' of X . We can assume $Y = Y'$. Define a homotopy

$$g_t: \overline{U} \ni (x, \lambda) \rightarrow (1-t)g(x, \lambda) + tg'(x, \lambda) \in X.$$

For every $(x, \lambda) \in \overline{U}$ and $t \in I$, $g_t(x, \lambda)$ is contained in the closed ball $B(f(x, \lambda), \epsilon)$. That ball does not contain x provided $(x, \lambda) \in \overline{U} \setminus \overline{V} \setminus U$, hence by Lemma 5.3(b), the set

$$\bigcup_{t \in I} \text{Fix}(g_t) \subset Y$$

is compact. Let

$$h_t := (g_t)_Y^U: U \cap Y \rightarrow Y$$

be the restriction of g_t . Since (III) holds for the homotopy h_t by Lemma 3.1,

$$\Sigma_{(h_0, U \cap Y, V \cap Y)} = \Sigma_{(h_1, U \cap Y, V \cap Y)}$$

which implies that $\Sigma'_{g,Y} = \Sigma'_{g',Y}$, hence the result is proved. $\square$

By Lemma 6.1, we define

$$\Sigma_{(f,U,V)} := \Sigma'_{g,Y}.$$

Assume now that (U, V) is an arbitrary open pair in X such that $K := \text{Fix}(f) \cap U \setminus V$ is compact. One can find U' , an open neighborhood of K ,

such that $\overline{U'} \subset U$, $f|_{\overline{U'}}$ is a compact map, and $\overline{p(U')}$ is compact (because Λ is a locally compact space). Set $V' := U' \cap V$. By Lemma 5.4, $\text{Fix}(f) \cap \overline{U'} \setminus \overline{V'} \setminus U'$ is empty. We define

$$\Sigma_{(f,U,V)}: H^k(U, V) \longrightarrow H^k(U', V') \xrightarrow{\Sigma_{(f,U',V')}} R.$$

One can check that (II) in the finite-dimensional case implies that the above definition is independent of the choice of U' .

Lemma 6.2. *If X and X' are normed spaces, then $\Sigma_{(f,U,V)}$ satisfies properties (I)–(VII), (X), and (XI).*

Proof. We apply Lemma 3.1. The properties (II), (IV)–(VII), (X), and (XI) follow directly from the corresponding properties in the finite-dimensional case.

Proof of (I). Let $\Sigma_{(f,U,V)} \neq 0$. We can assume that $\overline{U} \subset D$, $\overline{p(U)}$ is compact, and $f|_{\overline{U}}$ is a compact map. Let g_n be a finite-dimensional $1/n$ -approximation of f in $\overline{U'}$. Let

$$x_n = g_n(x_n, \lambda_n)$$

for some $(x_n, \lambda_n) \in U$, for sufficiently large n (by the finite-dimensional case of (I)). We can assume $\lambda_n \rightarrow \lambda$. By the compactness of f , we can assume $f(x_n, \lambda_n) \rightarrow y$. Since $\|f(x_n, \lambda_n) - x_n\| < 1/n$, $x_n \rightarrow y$, and thus

$$f(y, \lambda) = y.$$

Proof of (III). Choose an open set U' such that

$$\bigcup_{t \in I} \text{Fix}(f_t) \cap U \setminus V \subset U',$$

$\overline{U'} \subset U$, $\overline{p(U')}$ is compact, and $f_t|_{\overline{U'}}$ is a compact homotopy. It follows that for each $\epsilon > 0$ there exists a finite-dimensional homotopy $g_t: \overline{U'} \rightarrow X$ which is an ϵ -approximation of f_t . Set $V' := U' \setminus V$. If ϵ is small enough, by Lemmas 5.4 and 5.3(b), and by the finite-dimensional case of (III), it follows that

$$\Sigma_{(g_0,U',V')} = \Sigma_{(g_1,U',V')},$$

hence the result follows. $\square$

7. Commutativity property in normed spaces

We extend Lemma 4.1 to the following general case.

Lemma 7.1. *If X and X' are normed spaces, then property (VIII) holds.*

Proof. By Lemmas 4.1 and 6.2, property (VIII) holds provided both g and g' are locally compact. Since (VIII) holds if and only if (VIII^σ) holds, and the role of g and g' is symmetric in (VIII^σ) by (a) in (VIII), in order to finish the proof it suffices to assume that only g' is locally compact. We adapt an argument from [G, Section 8].

As in the proof of Lemma 4.1, denote

$$\begin{aligned} K &:= \text{Fix}(g' \circ G) \cap U \setminus V, \\ K' &:= \text{Fix}(g \circ G') \cap U' \setminus V'. \end{aligned}$$

By assumptions, K and $K' = G(K)$ are compact and contained in $D \cap U$ and $D' \cap U'$, respectively. Let E be an open subset of D such that $K \subset E$ and $\overline{E} \subset D$. Let W' be an open subset of X' such that $K' \subset W'$, $\overline{W'} \subset D' \cap U' \cap G'^{-1}(E)$, $g'|_{\overline{W'}}$ is a compact map, and $\overline{p'(W')}$ is compact (where p' denotes the projection $X' \times \Lambda \rightarrow \Lambda$). Thus $\overline{G'(W')}$ is compact and contained in D . Let W be an open subset of X such that $K \subset W \subset E$, $\overline{W} \subset U$, $\overline{p(W)}$ is compact, and $G(\overline{W}) \subset W'$. It follows that $\overline{W'}$ is contained in the domain of $g \circ G'$ and $\overline{W}$ is contained in the domain of $g' \circ G$. Moreover, the maps $g' \circ G|_{\overline{W}}$ and $g \circ G'|_{\overline{W'}}$ are compact. Set $Z := W \cap V$ and $Z' := W' \cap V'$. It follows that

$$\begin{aligned} K &= \text{Fix}(g' \circ G) \cap W \setminus Z, \\ K' &= \text{Fix}(g \circ G') \cap W' \setminus Z', \end{aligned}$$

hence, by (II), it suffices to prove that the diagram

$$\begin{array}{ccc} H^k(W', Z') & & \\ \downarrow G^* & \searrow \Sigma_{(g \circ G', W', Z')} & \\ & R & \\ & \nearrow \Sigma_{(g' \circ G, W, Z)} & \\ H^k(W, Z) & & \end{array}$$

commutes. Denote the norms of X and X' by $\|\cdot\|$ and $\|\cdot\|'$, respectively. By Lemmas 5.4 and 5.1(c),

$$\begin{aligned} \zeta &:= \inf \left\{ \|x - g' \circ G(x, \lambda)\| : (x, \lambda) \in \overline{W' \setminus Z} \setminus W \right\} > 0, \\ \zeta' &:= \inf \left\{ \|x' - g \circ G'(x', \lambda)\|' : (x', \lambda) \in \overline{W' \setminus Z'} \setminus W' \right\} > 0. \end{aligned}$$

For $(x, \lambda) \in \overline{G'(W')}$, let $\delta_{(x, \lambda)} > 0$ and a closed neighborhood $\Delta_{(x, \lambda)}$ of λ be such that

$$\begin{aligned} B(x, \delta_{(x, \lambda)}) \times \Delta_{(x, \lambda)} &\subset D, \\ \|g(y_1, \lambda_1) - g(y_2, \lambda_2)\|' &< \zeta' \quad \forall (y_1, \lambda_1), (y_2, \lambda_2) \in B(x, \delta_{(x, \lambda)}) \times \Delta_{(x, \lambda)}. \end{aligned}$$

By the compactness of $\overline{G'(W')}$, there exist $(x_1, \lambda_1), \dots, (x_r, \lambda_r)$ such that

$$\overline{G'(W')} \subset \bigcup_{i=1}^r B(x_i, \delta_{(x_i, \lambda_i)}/2) \times \Delta_{(x_i, \lambda_i)}.$$

Set

$$\begin{aligned} S &:= \bigcup_{i=1}^r B(x_i, \delta_{(x_i, \lambda_i)}) \times \Delta_{(x_i, \lambda_i)}, \\ \delta_0 &:= \min\{\delta_1, \dots, \delta_r\}, \\ \epsilon &:= \min\{\delta_0/2, \zeta/2\}. \end{aligned}$$

Since the balls are closed, S is a closed subset of X . Moreover, $S \subset D$. Let $h: \overline{W'} \rightarrow X$ be a finite-dimensional ϵ -approximation of $g'|_{\overline{W'}}$. Define

$$\begin{aligned} h_t: \overline{W'} \ni (x', \lambda) &\rightarrow ((1-t)g'(x', \lambda) + th(x', \lambda)) \in X, \\ H_t(x', \lambda) &:= (h_t(x', \lambda), \lambda), \\ H &:= H_1. \end{aligned}$$

It follows that $H_t(x', \lambda) \in S$ for each $(x', \lambda) \in \overline{W'}$. Since S is closed and contained in the domain of g , the closure of the image of $\overline{W'} \times I$ under the map $(x', t) \rightarrow g \circ H_t$ is compact, hence the homotopy $g \circ H_t$ is compact. Moreover, if $(x', \lambda) \in \overline{W'}$ and $G'(x', \lambda) \in B(x_i, \delta_{(x_i, \lambda_i)}/2) \times \Delta_{(x_i, \lambda_i)}$, then

$$H_t(x', \lambda) \in B(x_j, \delta_{(x_i, \lambda_i)}) \times \Delta_{(x_i, \lambda_i)},$$

hence

$$\|g \circ G'(x', \lambda) - g \circ H_t(x', \lambda)\|' < \zeta'.$$

It follows that

$$x' \neq g \circ H_t(x', \lambda)$$

for each $(x', \lambda) \in \overline{W'} \setminus Z' \setminus W'$, hence by Lemma 5.3(b) and (III),

$$\Sigma_{(g \circ G', W', Z')} = \Sigma_{(g \circ H, W', Z')}.$$

Let Y be a finite-dimensional subspace of X which contains $h(\overline{W'})$. Define

$$g^Y := g|_{D \cap Y \times \Lambda}$$

and $h^Y: W' \rightarrow Y$ as the restriction of h . Set

$$G^Y(x, \lambda) := (g^Y(x, \lambda), \lambda), \quad H^Y(x', \lambda) := (h^Y(x', \lambda), \lambda).$$

It follows that

$$g^Y \circ H^Y = g \circ H|_{W'}.$$

Since both g^Y and h^Y are locally compact, by the Commutativity property, already proved in that case, the right-hand side triangle in the diagram

$$\begin{array}{ccccc}
 & H^k(W', Z') & & & \\
 G^* \swarrow & \downarrow (G^Y)^* & \searrow \Sigma_{(g \circ G', W', Z')} = \Sigma_{(g^Y \circ H^Y, W', Z')} & & \\
 H^k(W, Z) & & & R & \\
 \searrow & & \nearrow \Sigma_{(h^Y \circ G^Y, W \cap Y, Z \cap Y)} & & \\
 & H^k(W \cap Y, Z \cap Y) & & &
 \end{array}$$

commutes. Since the left-hand side triangle also commutes and $h \circ G$ is a finite-dimensional ϵ -approximation of $g' \circ G$ on $\overline{W}$ with $\epsilon < \zeta$, its image is contained in Y , and

$$(h \circ G)_Y^W = h^Y \circ G^Y|_{W \cap Y},$$

by the definition of $\Sigma_{(g' \circ G, W, Z)}$ the result follows. $\square$

As a consequence of Lemma 7.1 we get the Contraction property for normed spaces, hence, by Lemma 6.2, we have the following lemma.

Lemma 7.2. *If X , X' , and Y are normed spaces, then the homomorphism Σ satisfies properties (I)–(XI).*

8. Construction of Σ for ANRs

If X is an open subset of a normed space E , we define

$$\Sigma_{(f, U, V)} := \Sigma_{(i \circ f, U, V)},$$

where $i: X \hookrightarrow E$ is the inclusion. Lemma 7.2 directly generalizes to the following one.

Lemma 8.1. *If X , X' , and Y are open subsets of normed spaces, then the homomorphism Σ satisfies properties (I)–(XI).*

Assume that X is an ANR. By [G, (9.3)], there exist an open subset Y of a normed space and continuous mappings $r: Y \rightarrow X$ and $s: X \rightarrow Y$ such that $r \circ s = \text{id}_X$. Define

$$\Sigma_{(f, U, V)} := \Sigma_{(s \circ f \circ (r \times \text{id}_\Lambda), (r \times \text{id}_\Lambda)^{-1}(U, V))} \circ (r \times \text{id}_\Lambda)^*.$$

By a similar argument as in [G, Section 10], one can check that the Commutativity property stated in Lemma 8.1 implies the independence of that definition from the choice of Y , r , and s . Finally, Theorem 2.1 follows by Lemma 8.1.

9. Relation of the homomorphisms to the fixed point index

In this section we assume $R = \mathbb{Z}$. Let U be an open subset of an ANR X and let $g: U \rightarrow X$ be a locally compact map with a compact set of fixed points K , hence

$$\text{Fix}(g \circ j) = K \times \Lambda.$$

Assume that Λ is a k -dimensional manifold with an orientation η over $\mathbb{Z}$. Let $\lambda \in \Lambda$, let $\eta_\lambda \in H_k(\Lambda, \Lambda \setminus \lambda)$ be the induced generator, and let $s_\lambda \in H^k(\Lambda, \Lambda \setminus \lambda)$ be the dual generator, i.e., $\langle s_\lambda, \eta_\lambda \rangle = 1$.

Proposition 9.1. *If Σ satisfies properties (I)–(VI) and (VIII) (equivalently, σ satisfies (I $^\sigma$)–(VI $^\sigma$) and (VIII $^\sigma$)), then the number*

$$\Sigma_{(\eta, g \circ j, U \times (\Lambda, \Lambda \setminus \lambda))}(1_U \times s_\lambda) = \sigma_{(\eta, g \circ j, K \times \lambda)}(1_K \times s_\lambda) \in \mathbb{Z}$$

is equal to the fixed point index of g .

Proof. The fixed point index is uniquely determined by the properties of Excision, Homotopy Invariance, Additivity, Multiplicativity, Normalization, and Commutativity (cf. [D1, Exercise VII.5.17.5] and [G, Section 13]). For the considered invariant of g , these properties follow directly from the corresponding properties of Σ . $\square$

10. Orientations and (co-)homology generators

In order to avoid confusions with topological terminology, the term “vector orientation” refers to an orientation of a vector space (over $\mathbb{R}$) in the linear algebra meaning. Let W be an n -dimensional vector space; $n \geq 1$. By $\text{Or}(W)$ we denote the set of vector orientations of W . If $o \in \text{Or}(W)$, by $-o$ we denote the other element of $\text{Or}(W)$. For a basis $w = (w_1, \dots, w_n)$ of W we denote by $o(w)$ the vector orientation determined by w . Let Y be an m -dimensional vector space, $m \geq 1$, with a basis $y = (y_1, \dots, y_m)$. Assume first that W and Y are subspaces of a vector space Z and $Z = W + Y$ is direct. Then

$$(w, y) := (w_1, \dots, w_n, y_1, \dots, y_m)$$

is a basis of Z ; define

$$o(w) \wedge o(y) := o(w, y) \in \text{Or}(Z).$$

Assume now that W and Y are arbitrary; for the trivial space 0 define bases and orientations

$$\begin{aligned} (w, 0) &:= ((w_1, 0), \dots, (w_n, 0)), \\ o(w), 0 &:= o(w, 0) \in \text{Or}(W \times 0), \\ (0, y) &:= ((0, y_1), \dots, (0, y_m)), \\ (0, o(y)) &:= o(0, y) \in \text{Or}(0 \times Y). \end{aligned}$$

The sum $W \times 0 + 0 \times Y = W \times Y$ is direct; define

$$\begin{aligned} w \times y &:= (w, 0) \wedge (0, y), \\ o(w) \times o(y) &:= o(w \times y) \in \text{Or}(W \times Y). \end{aligned}$$

One has

$$(-o) \wedge o' = o \wedge (-o) = -o \wedge o, \quad (-o) \times o' = o \times (-o') = -o \times o'.$$

Let $\psi: W \rightarrow W'$ be a linear isomorphism. It transforms a basis w into the basis $\psi(w)$ of Z , hence it induces a bijection

$$\psi_*: \text{Or}(W) \rightarrow \text{Or}(W')$$

given by $\psi_*(o(w)) := o(\psi(w))$.

Now we set up some notation concerning homology and cohomology. Assume $R = \mathbb{Z}$. Fix o^1 , a generator of $H_1(\mathbb{R}, \mathbb{R} \setminus 0)$. Let $s^1 \in H^1(\mathbb{R}, \mathbb{R} \setminus 0)$ satisfy

$$\langle s^1, o^1 \rangle = 1.$$

Define

$$\begin{aligned} o^n &:= o^1 \times \cdots \times o^1 \in H_n(\mathbb{R}^n, \mathbb{R}^n \setminus 0), \\ s^n &:= s^1 \times \cdots \times s^1 \in H^n(\mathbb{R}^n, \mathbb{R}^n \setminus 0). \end{aligned}$$

It follows, in particular, that $\langle s^n, o^n \rangle = 1$ and

$$s^n \times s^m = s^{n+m}.$$

Denote by $\text{Gcoh}(W)$ the set of generators of $H^n(W, W \setminus 0)$ and by $\text{Ghom}(W)$ the set of generators of $H_n(W, W \setminus 0)$.

We define the bijections

$$\Omega: \text{Gcoh}(W) \rightarrow \text{Or}(W), \quad \bar{\Omega}: \text{Ghom}(W) \rightarrow \text{Or}(W)$$

in the following way. Let w be a basis of W and let ϕ_w be the isomorphism $\mathbb{R}^n \rightarrow W$ which transforms the canonical basis of $\mathbb{R}^n$ into w . Define

$$\begin{aligned} \Omega^{-1}(o(w)) &:= (\phi_w^{-1})^*(s^n), \\ \bar{\Omega}^{-1}(o(w)) &:= (\phi_w)_*(o^n). \end{aligned}$$

Since $o(w) = o(w')$ if and only if $(\phi_w)^* = (\phi_{w'})^*$ and $o(w) = -o(w')$ if and only if $(\phi_w)^* = -(\phi_{w'})^*$, and analogous relations in homologies hold, Ω and $\bar{\Omega}$ are correctly defined.

Lemma 10.1. $\Omega(s) = \bar{\Omega}(o)$ if and only if $\langle s, o \rangle = 1$.

Below we concentrate on properties of Ω .

Lemma 10.2. *The diagram of bijections*

$$\begin{array}{ccc} \text{Gcoh}(W') & \xrightarrow{\Omega} & \text{Or}(W') \\ \psi^* \downarrow & & \uparrow \psi_* \\ \text{Gcoh}(W) & \xrightarrow{\Omega} & \text{Or}(W) \end{array}$$

commutes.

Proof. Let w be a basis of W . Then

$$\begin{aligned} \Omega^{-1}(\psi_*(o(w))) &= \Omega^{-1}(o(\psi(w))) = (\phi_{\psi(w)}^{-1})^*(s^n) = ((\psi\phi_w)^{-1})^*(s^n) \\ &= (\psi^{-1})^*(\phi_w^{-1})^*(s^n) = (\psi^*)^{-1}\Omega^{-1}(o(w)). \end{aligned}$$

□

Lemma 10.3. *Let $s \in \text{Gcoh}(W)$. Then*

$$\Omega(s \times 1_0) = (\Omega(s), 0), \quad \Omega(1_0 \times s) = (0, \Omega(s)).$$

Proof. Let $p: W \times 0 \rightarrow W$ be the projection. Since $p^*(s) = s \times 1_0$, the first equation follows by Lemma 10.2. $\square$

Lemma 10.4. *If $s \in \text{Gcoh}(W)$ and $t \in \text{Gcoh}(Y)$, then*

$$\Omega(s \times t) = \Omega(s) \times \Omega(t).$$

Proof. Let w and y be the bases of W and Y , respectively, and let $\Omega(s) = o(w)$ and $\Omega(t) = o(y)$. The cartesian product $\phi_w \times \phi_y$ sends the canonical basis of $\mathbb{R}^{n+m}$ to $w \times y$, hence

$$\begin{aligned} \Omega^{-1}(\Omega(s) \times \Omega(t)) &= \Omega^{-1}(o(w) \times o(y)) \\ &= \Omega^{-1}(o(w \times y)) \\ &= ((\phi_w \times \phi_y)^{-1})^*(s^{n+m}) \\ &= (\phi_w^{-1})^* \times (\phi_y^{-1})^*(s^n \times s^m) \\ &= (\phi_w^{-1})^*(s^n) \times (\phi_y^{-1})^*(s^m) \\ &= \Omega^{-1}(o(w)) \times \Omega^{-1}(o(y)) = s \times t. \end{aligned} \quad \square$$

Assume that $Z = W + Y$ is a direct sum of subspaces W and Y . Let

$$\iota: W \hookrightarrow Z, \quad \kappa: Y \hookrightarrow Z$$

be the inclusions; hence

$$\iota \oplus \kappa: W \times Y \rightarrow Z$$

is an isomorphism which sends the basis $w \times y$ to (w, y) . Thus

$$(\iota \oplus \kappa)_*(o(w) \times o(y)) = o(w) \wedge o(y).$$

Moreover,

$$\iota^*: H^n(Z, Z \setminus Y) \rightarrow H^n(W, W \setminus 0), \quad \kappa^*: H^m(Z, Z \setminus W) \rightarrow H^m(Y, Y \setminus 0)$$

are isomorphisms because the projection $Z \rightarrow W$, parallel to Y , and the projection $Z \rightarrow Y$, parallel to W , are deformation retractions.

Lemma 10.5. *If $s \in \text{Gcoh}(W)$ and $t \in \text{Gcoh}(Y)$, then*

$$\Omega((\iota^*)^{-1}(s) \smile (\kappa^*)^{-1}(t)) = \Omega(s) \wedge \Omega(t).$$

Proof. Let $p: W \times Y \rightarrow W$ and $q: W \times Y \rightarrow Y$ be the projections. Since

$$p \circ (\iota \oplus \kappa)^{-1} \circ \iota = \text{id}_W, \quad q \circ (\iota \oplus \kappa)^{-1} \circ \kappa = \text{id}_Y,$$

ι^* and $((\iota \oplus \kappa)^*)^{-1}p^*$ are mutually inverse isomorphisms between $H^n(W, W \setminus 0)$ and $H^n(Z, Z \setminus Y)$, and κ^* and $((\iota \oplus \kappa)^*)^{-1}q^*$ are mutually inverse isomorphisms between $H^m(Y, Y \setminus 0)$ and $H^m(Z, Z \setminus W)$. Thus, by the formula connecting the cup product and the cohomology cross product, and by Lemmas 10.2 and 10.4, we have

$$\begin{aligned} \Omega((\iota^*)^{-1}(s) \smile (\kappa^*)^{-1}(t)) &= \Omega(((\iota \oplus \kappa)^{-1})^*(s \times t)) \\ &= (\iota \oplus \kappa)_* \Omega(s \times t) \\ &= (\iota \oplus \kappa)_*(\Omega(s) \times \Omega(t)) \\ &= \Omega(s) \wedge \Omega(t). \end{aligned} \quad \square$$

As a corollary of Lemma 10.5 we have the following lemma.

Lemma 10.6. *Let W , Y , Z , t , and κ be the same as in Lemma 10.5. Assume that Y' is another subspace of Z such that $W + Y' = Z$ is a direct sum. Let*

$$\kappa'^*: H^n(Z, Z \setminus W) \rightarrow H^n(Y', Y' \setminus 0)$$

be the isomorphism induced by the inclusion $\kappa': Y' \hookrightarrow Z$. If ω is a vector orientation of W , then

$$\omega \wedge \Omega(\kappa'^*(\kappa^*)^{-1}(t)) = \omega \wedge \Omega(t).$$

Proof. Let $\omega = \Omega(s)$ for some generator s of $H^n(W, W \setminus 0)$. Double application of Lemma 10.5 provides

$$\begin{aligned} \Omega(s) \wedge \Omega(\kappa'^*(\kappa^*)^{-1}(t)) &= \Omega((\iota^*)^{-1}(s) \smile (\kappa'^*)^{-1}\kappa'^*(\kappa^*)^{-1}(t)) \\ &= \Omega(s) \wedge \Omega(t). \end{aligned} \quad \square$$

11. Determining Σ in the regular-value case

In this section we assume that $X = \mathbb{R}^n$ and $\Lambda = \mathbb{R}^k$, f is of C^∞ class, and 0 is a regular value of $j - f$ (hence $\text{Fix}(f)$ is a manifold and we do not distinguish between singular and Čech cohomologies of pairs of its open subsets). Let o be a (topological) orientation of $\mathbb{R}^k$. Our purpose is to determine $\Sigma_{(o,f,U,V)}$ under the assumption that Σ is an arbitrary homomorphism satisfying properties (I)–(XI). We restrict ourselves to the case of a single-point set K , i.e.,

$$K = \text{Fix}(f) \cap U \setminus V = \{x\}.$$

In the next section we describe how to reduce the case of an arbitrary compact K to that case by an application of Mayer–Vietoris sequences. Moreover, using suitable embeddings and applying properties (II), (VII), and (X), it is easy to pass to the case where X and Λ are arbitrary C^∞ -manifolds.

At first we consider the linear case and we assume $R = \mathbb{Z}$. Let $f: \mathbb{R}^n \times \mathbb{R}^k \rightarrow \mathbb{R}^n$ be a linear map and let $j: \mathbb{R}^n \times \mathbb{R}^k \rightarrow \mathbb{R}^n$ be the projection. Assume that $j - f$ is an epimorphism. Set

$$Y := \ker(j - f) = \text{Fix}(f).$$

By assumption, Y is k -dimensional. Let W be a subspace of $\mathbb{R}^n \times \mathbb{R}^k$ such that

$$W + Y = \mathbb{R}^n \times \mathbb{R}^k$$

is a direct sum. The orientation o induces o_0 , a generator of $H_k(\mathbb{R}^k, \mathbb{R}^k \setminus 0)$. Let $t \in H^k(Y, Y \setminus 0)$ be a generator. Choose vector orientations ω of W and ξ of $\mathbb{R}^n$ which satisfy

$$\omega \wedge \Omega(t) = \xi \times \bar{\Omega}(o_0).$$

In that case we call the orientations ω and ξ *concordant*.

Lemma 11.1. *Assume that σ satisfies properties (I $^\sigma$)–(VI $^\sigma$) and (VIII $^\sigma$). If $(w_1, \dots, w_n)$ is a basis of W which represents ω , then*

$$\sigma_{(o,f,0)}(t) = \begin{cases} 1 & \text{if } ((j-f)(w_1), \dots, (j-f)(w_n)) \text{ represents } \xi, \\ -1 & \text{otherwise.} \end{cases}$$

Proof. Our proof will be divided into four steps. First of them is a simple fact from linear algebra.

Step 1. *There exists a subspace Y_1 such that*

$$\begin{aligned} W + Y_1 &= \mathbb{R}^n \times \mathbb{R}^k \text{ is a direct sum,} \\ Y_1 \cap (\mathbb{R}^n \times 0) &= 0. \end{aligned}$$

Indeed, let $S = Y \cap (\mathbb{R}^n \times 0)$ and let T be a subspace such that $S + T = Y$ is a direct sum. Let $x \in S$ and assume $x \neq 0$. Then there exists $y \in W$, $y \notin \mathbb{R}^n \times 0$ (since otherwise $W + Y = W + T$ and $\dim(W + T) < n + k$). It follows that $x' = y + x \notin \mathbb{R}^n \times 0$. Let S' be such that $S = S' + \mathbb{R}x$ is direct. If $Y' := T + S' + \mathbb{R}x'$, then $W + Y' = W + Y$ and $\dim(Y' \cap \mathbb{R}^n \times 0) = \dim S - 1$, hence by repeating that argument we get the result.

Step 2. *One can assume $Y \cap (\mathbb{R}^n \times 0) = 0$.*

Let Y_1 be a subspace satisfying the conclusion of Step 1. Let $(w_1, \dots, w_n)$ be a basis of W and let $(v_1, \dots, v_k)$ be a basis of Y_1 . The map $j - f$ is represented by the matrix $[A, B]$ (where A is an $(n \times n)$ -matrix and B is an $(n \times k)$ -matrix) in the basis $(w_1, \dots, w_n, v_1, \dots, v_k)$ of $\mathbb{R}^{n+k}$ and the canonical basis in $\mathbb{R}^n$. Let a_t be a linear homotopy represented by the matrix $[A, (1 - t)B]$, $t \in I$. Set $f_t := j - a_t$, hence $Y_1 = \text{Fix}(f_1)$. Let $\kappa: Y \hookrightarrow \mathbb{R}^{n+k}$ and $\kappa_1: Y_1 \hookrightarrow \mathbb{R}^{n+k}$ be the inclusions. Set

$$t_1 := \kappa_1^*(\kappa^*)^{-1}(t) \in H^k(Y_1, Y_1 \setminus 0).$$

Since A is nonsingular,

$$\text{Fix}(f_t) \cap W = 0,$$

and (III $^\sigma$) implies that

$$\sigma_{(o,f,0)}(t) = \sigma_{(o,f_1,0)}(t_1).$$

On the other hand, Lemma 10.6 guarantees

$$\omega \wedge \Omega(t_1) = \xi \times \bar{\Omega}(o_0).$$

Step 3. *One can assume $W = \mathbb{R}^n \times 0$.*

Indeed, by Step 2 assume $Y \cap \mathbb{R}^n \times 0 = 0$. Let the basis $(w_1, \dots, w_n)$ represent ω and let $y = (y_1, \dots, y_k)$ be a basis of Y representing $\Omega(t)$. Let u_i , $i = 1, \dots, n$, be the projection of w_i onto $\mathbb{R}^n \times 0$ parallel to Y ; thus $u := (u_1, \dots, u_n)$ is a basis of $\mathbb{R}^n \times 0$ and

$$w_i = u_i + z_i$$

for some $z_i \in Y$, hence

$$(j - f)(u_i) = (j - f)(w_i).$$

For $t \in I$ set $u_i^t := u_i + tz_i$; therefore $(u_1^t, \dots, u_n^t, y_1, \dots, y_k)$ is a basis of $\mathbb{R}^n \times \mathbb{R}^k$ and it follows that

$$o(u) \wedge \Omega(t) = \xi \times \overline{\Omega}(o_0).$$

Step 4. *The conclusion of the lemma holds.*

By Step 3, we can assume $W = \mathbb{R}^n \times 0$. Then, by repeating the argument in Step 2 with Y_1 equal to $0 \times \mathbb{R}^k$, we can assume $Y = 0 \times \mathbb{R}^k$. Actually, in that case $f = g \circ j$ for some linear map $g: \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $\text{id}_{\mathbb{R}^n} - g$ is an automorphism. Moreover, $t = 1_0 \times s_0$ for some $s_0 \in H^k(\mathbb{R}^k, \mathbb{R}^k \setminus 0)$, hence by Proposition 9.1,

$$\sigma_{(o,f,0)}(t) = \langle s_0, o_0 \rangle I(g),$$

where I denotes the fixed point index. Let $z = (z_1, \dots, z_n)$ be a basis of $\mathbb{R}^n$ such that $(z, 0)$ represents ω . Since

$$(j - f)(z_i, 0) = z_i - g(z_i)$$

and since in our case

$$I(g) = \text{sgn det}(\text{id}_{\mathbb{R}^n} - g),$$

we get

$$\sigma_{(o,f,0)}(t) = \begin{cases} \langle s_0, o_0 \rangle & \text{if } ((j - f)(z_1, 0), \dots, (j - f)(z_n, 0)) \text{ represents } o(z), \\ -\langle s_0, o_0 \rangle & \text{otherwise.} \end{cases}$$

By Lemma 10.1,

$$\overline{\Omega}(o_0) = \langle s_0, o_0 \rangle \Omega(s_0),$$

hence the concordance of ω and ξ reads as

$$\omega \wedge \Omega(t) = \langle s_0, o_0 \rangle \xi \times \Omega(s_0).$$

Since $\Omega(t) = (0, \Omega(s_0))$, by Lemma 10.3,

$$\omega = \langle s_0, o_0 \rangle (\xi, 0),$$

or, equivalently, $\xi = \langle s_0, o_0 \rangle o(z)$, hence the result follows. $\square$

Now we assume that R is an arbitrary ring-with-unit and we pass to the nonlinear case. By assumption, $j - d_x f$ is an epimorphism, $\text{Fix}(f)$ is a C^∞ -submanifold of $\mathbb{R}^n \times \mathbb{R}^k$, and

$$Y := \ker(j - d_x f)$$

is equal to the tangent space of $\text{Fix}(f)$ at x . Again, let W be a complement to Y in $\mathbb{R}^n \times \mathbb{R}^k$, which means that the intersection of $\text{Fix}(f)$ and $W + x$ at x is transversal. By (II), we can restrict ourselves to the case

$$(U, V) = (\mathbb{R}^n \times \mathbb{R}^k, \mathbb{R}^n \times \mathbb{R}^k \setminus (W + x)).$$

It follows that the inclusions

$$\begin{aligned} i &: (\text{Fix}(f), \text{Fix}(f) \setminus x) \hookrightarrow (U, V), \\ i' &: (Y + x, Y + x \setminus x) \hookrightarrow (U, V) \end{aligned}$$

induce isomorphisms $i_{\mathbb{Z}}^*$ and $i'_{\mathbb{Z}}^*$ in cohomologies with coefficients in $\mathbb{Z}$. Thus $u \in H^k(U, V; \mathbb{Z})$ is a generator if and only if $i_{\mathbb{Z}}^*(u)$ is a generator which, in turn, is a generator if and only if $i'_{\mathbb{Z}}^*(u)$ is. Let $\mathbb{Z} \rightarrow R$ be the canonical homomorphism. It follows that the natural map ρ_* between the cohomologies with coefficients in $\mathbb{Z}$ and in R induced by ρ transforms the above isomorphisms into isomorphisms and the generators into the generators. Moreover, ρ_* transforms some orientation o' of $\mathbb{R}^k$ over $\mathbb{Z}$ into the orientation o . It follows by (XI) that

$$\Sigma_{(o, f, U, V)}(\rho_*(u)) = \rho(\Sigma_{(o', f, U, V)}(u)),$$

hence we can assume $R = \mathbb{Z}$.

Let τ denote the translation of $\mathbb{R}^n \times \mathbb{R}^k$ which sends 0 to x . Choose vector orientations ω of W and ξ of $\mathbb{R}^n$ such that

$$\omega \wedge \Omega(\tau^* i'^*(u)) = \xi \times \bar{\Omega}(o_0).$$

Proposition 11.1. *Let Σ satisfy properties (I)–(XI). If $(w_1, \dots, w_n)$ is a basis of W which represents ω and u is a generator of $H^k(U, V)$, then*

$$\begin{aligned} \Sigma_{(o, f, U, V)}(u) &= \sigma_{(o, f, x)}(i^*(u)) \\ &= \begin{cases} 1 & \text{if } ((j - d_x f)(w_1), \dots, (j - d_x f)(w_n)) \text{ represents } \xi, \\ -1 & \text{otherwise.} \end{cases} \end{aligned}$$

Proof. By the remarks above, we assume $R = \mathbb{Z}$. The translation τ is the composition of two translations; the first one is equal to the identity with respect to the first variable, and the second one with respect to the second variable. Therefore, by (VII) and (X) we can assume $x = 0$. By (III), f can be replaced by its differential $d_x f$, hence the result follows directly from Lemma 11.1. $\square$

12. Proof of Uniqueness

Now we are able to provide a proof of Theorem 2.2. Let Λ be a C^∞ -manifold, let X be an ANR, and let $u \in H^k(U, V)$. By (X), we can assume that X is a retract of an open subset of a normed space, by (IX) we can assume that X is an open subset of a normed space, and again by (IX) we can assume that X is a normed space. Let $K := \text{Fix}(f) \cap U \setminus V$. By (II), the local compactness of Λ , and Lemma 5.4 we can assume that f is defined on $\overline{U}$, $K \subset U$, $\overline{p(U)}$ is compact, and $\text{Fix}(f) \cap \overline{U} \setminus \overline{V} \setminus U$ is empty. It follows by Lemmas 5.1(c) and 5.3(b), Schauder Approximation Theorem, and by (III) that we can assume f is a finite-dimensional map. Again by (IX), we can assume that X is a finite-dimensional vector space, hence as a consequence of (X) we can assume $X = \mathbb{R}^n$. By (III) and Sard Theorem, f can be replaced by a C^∞ -approximation g such that 0 is a regular value of $j - g$. Thus we can assume $f = g$ and

$$F := \text{Fix}(f)$$

is a k -dimensional submanifold of $\mathbb{R}^n \times \Lambda$. Let $i: (F, F \setminus K) \hookrightarrow (U, V)$ be the inclusion. Since F is a manifold, we do not distinguish between singular and Čech cohomologies;

$$\sigma_{(f,K)}(i^*(u)) = \Sigma_{(f,U,V)}(u).$$

Let $B = \bigcup_{i=1,\dots,r} B_i$ be the union of compact contractible balls B_i in F such that $K \subset B$. By (II $^\sigma$), we have

$$\sigma(f, K)(i^*(u)) = \sigma_{(f,B)}(v)$$

for the restriction v of $i^*(u)$. Since F is k -dimensional, $H^{k+1}(F, U) = 0$ for every open set $U \subset F$. (Indeed, it suffices to consider F connected; $H^{k+1}(F) = 0$ and $H^k(U) \neq 0$ only in the case $U = F$ is compact, hence the result follows by the exact sequence of (F, U) .) Thus

$$H^{k+1} \left(F, F \setminus \left(B_s \cap \bigcup_{i=1}^{s-1} B_i \right) \right) = 0$$

for $s = 2, \dots, r$, hence, by consecutive application of the cohomology Mayer-Vietoris sequence for the triads $(F, F \setminus B_s, F \setminus \bigcup_{i=1}^{s-1} B_i)$, we conclude that v is the sum of restrictions of some classes $v_i \in H^k(F, F \setminus B_i)$, hence it suffices to determine $\sigma_{(f,B_i)}(v_i)$. Since B_i is contractible, by (II $^\sigma$) it suffices to replace B_i by its center b_i . By (I $^\sigma$) and (VII $^\sigma$) we can replace Λ by $\mathbb{R}^k$. Now both Theorems 2.2 and 2.4 follow by Proposition 11.1.

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Note added in proof

Other invariants for fixed points of parameterized maps are considered, for example, in [C], [CJ], and [Di].

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Roman Srzednicki
Institute of Mathematics
Jagiellonian University
6 Łojasiewicza St.
30-348 Kraków
Poland
e-mail: srzednicki@im.uj.edu.pl

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