Erratum to: The finiteness of the Reidemeister number of morphisms between almost-crystallographic groups

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Due to a processing error the presentation of this article was incorrect. The corrected article is given on the next pages.

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The finiteness of the Reidemeister number of morphisms between almost-crystallographic groups

Karel Dekimpe and Pieter Penninckx

Abstract. We give a practical criterion to determine whether a given pair of morphisms between almost-crystallographic groups has a finite Reidemeister coincidence number. As an application, we determine all two- and three-dimensional almost-crystallographic groups that have the R_∞ property. We also show that for a pair of continuous maps between oriented infra-nilmanifolds of equal dimension, the Nielsen coincidence number equals the Reidemeister coincidence number when the latter is finite.

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1. Introduction

In order to study the number of fixed points of a continuous self-map $f : X \rightarrow X$ on a closed, connected manifold X , the Nielsen number $N(f)$ is defined. This integer is a lower bound for the number of fixed points of f . Closely related to fixed point theory is coincidence theory. A point $x \in X$ is a coincidence of a pair of continuous maps $f, f' : X \rightarrow Y$ when $f(x) = f'(x)$. In order to study the number of coincidences of a pair of continuous maps $f, f' : X \rightarrow Y$ between oriented, closed, connected manifolds X and Y of equal dimension, one defines the Nielsen coincidence number $N(f, f')$. This integer is a lower bound for the number of coincidences of f and f' .

The computation of the Nielsen (coincidence) number is a central topic in Nielsen fixed point theory and Nielsen coincidence theory. This computation is in general very hard due to the mixture of geometry and algebra that occurs in the definition of the Nielsen (coincidence) number. For this

reason, many efforts have been made to express, at least for certain spaces, the Nielsen (coincidence) number in terms of related numbers.

One approach is to compare the Nielsen number $N(f)$ of a continuous self-map f to the absolute value of the Lefschetz number $L(f)$. An important result in this context is the theorem by Anosov [1] which states that $N(f) = |L(f)|$ for every continuous self-map f on a nilmanifold.

Another approach is to compare the Nielsen (coincidence) number to the Reidemeister (coincidence) number, which is defined on the level of the fundamental group. In this comparison, the finiteness of the Reidemeister (coincidence) number turns out to be important. Indeed, if the Reidemeister (coincidence) number is infinite, then the Nielsen (coincidence) number, which is always finite, cannot equal the Reidemeister (coincidence) number. A useful concept in the study of the finiteness of the Reidemeister number is the R_∞ property.

Definition 1.1. A closed connected manifold X has the R_∞ property if $R(f) = \infty$ for every homeomorphism $f : X \rightarrow X$.

In [9], the R_∞ property for infra-nilmanifolds is studied using geometric techniques.

The Reidemeister (coincidence) number can also be defined in an entirely algebraic setting: one can define the Reidemeister number of an endomorphism of an abstract group and the Reidemeister coincidence number of a pair of morphisms between abstract groups. The finiteness of the Reidemeister number of an automorphism of an abstract group plays an important role in a conjecture by Fel'shtyn and Hill [13]. We refer the reader to the expository paper [14] for the current state of this conjecture. One can also study the R_∞ property for abstract groups. This has been done for a large and growing number of groups; see, for instance, [32] and the references therein.

In this article, we study the finiteness of the Reidemeister coincidence number of a pair of morphisms between almost-crystallographic groups. We give a practical criterion (Theorem 3.9) to determine whether a given pair of morphisms between almost-crystallographic groups has a finite Reidemeister coincidence number, extending [9, Theorem 3.3]. As an application, we determine all two- and three-dimensional almost-crystallographic groups that have the R_∞ property. This extends the result in [19], where all two-dimensional crystallographic groups that have the R_∞ property are determined. Finally, we show that for any pair of continuous maps between oriented infra-nilmanifolds of equal dimension, the Nielsen coincidence number equals the Reidemeister coincidence number when the latter is finite. Since the fundamental groups of the infra-nilmanifolds are precisely the torsion-free almost-crystallographic groups, this result marks the importance of the finiteness of the Reidemeister coincidence number of a pair of morphisms between almost-crystallographic groups.

2. Preliminaries

In this section, we introduce basic notions concerning the Reidemeister (coincidence) number and almost-crystallographic groups. A reference on fixed point theory is [22], a reference on coincidence theory is [17] and a reference on almost-crystallographic groups is [7].

2.1. The Reidemeister coincidence number

Let X and Y be oriented, closed, connected manifolds of equal dimension. In order to study the coincidence set $\text{Coin}(f, f') = \{x \in X \mid f(x) = f'(x)\}$ of a pair of continuous maps $f, f' : X \rightarrow Y$, one splits the coincidence set into so-called coincidence classes. The notion of a coincidence class plays a crucial role in Nielsen coincidence theory as developed by Schirmer [30]. The intuition is that by homotoping f and f' , different coincidence classes cannot “merge” or “cancel each other out” or “interact” in any other way. There exist different definitions of a coincidence class. In this article, we define coincidence classes by using lifts $\tilde{f}, \tilde{f}' : \tilde{X} \rightarrow \tilde{Y}$ of f and f' , where $p_X : \tilde{X} \rightarrow X$ and $p_Y : \tilde{Y} \rightarrow Y$ are universal covers. Remark that by fixing a lift $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$ of $f : X \rightarrow Y$, the continuous map $f : X \rightarrow Y$ induces a morphism between the covering transformation groups as follows.

Definition 2.1. Let $A(\tilde{X}, p_X)$ be the covering transformation group of the cover $p_X : \tilde{X} \rightarrow X$ and $A(\tilde{Y}, p_Y)$ the covering transformation group of the cover $p_Y : \tilde{Y} \rightarrow Y$. For every $\alpha \in A(\tilde{X}, p_X)$, let $f_\times(\alpha)$ be the unique covering transformation in $A(\tilde{Y}, p_Y)$ that satisfies $\tilde{f}\alpha = f_\times(\alpha)\tilde{f}$.

Fix a lift $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$ of f and a lift $\tilde{f}' : \tilde{X} \rightarrow \tilde{Y}$ of f' . Then for any $\alpha \in A(\tilde{Y}, p_Y)$, the set $p_X(\text{Coin}(\alpha\tilde{f}, \tilde{f}'))$ is by definition a coincidence class of f, f' . Now for any pair of covering transformations $\alpha, \beta \in A(\tilde{Y}, p_Y)$, if $p_X(\text{Coin}(\alpha\tilde{f}, \tilde{f}')) \cap p_X(\text{Coin}(\beta\tilde{f}, \tilde{f}')) \neq \emptyset$, then one can show that $p_X(\text{Coin}(\alpha\tilde{f}, \tilde{f}')) = p_X(\text{Coin}(\beta\tilde{f}, \tilde{f}'))$ and that there exists $\gamma \in A(\tilde{X}, p_X)$ such that $\beta = f'_\times(\gamma)\alpha f_\times(\gamma)^{-1}$. This is the motivation for the following definitions.

Definition 2.2. Let G_1 and G_2 be groups and $\varphi, \psi : G_1 \rightarrow G_2$ morphisms of groups. Define an equivalence relation $\sim$ on G_2 by

$$g \sim g' \iff \exists h \in G_1 : g = \psi(h)g'\varphi(h)^{-1}.$$

The equivalence classes are called coincidence Reidemeister classes or (doubly) twisted conjugacy classes and $\mathcal{R}[\varphi, \psi]$ denotes the set of coincidence Reidemeister classes. For any $g \in G_2$, we use $[g]$ to denote the coincidence Reidemeister class containing g . The Reidemeister coincidence number $R(\varphi, \psi)$ of φ, ψ is defined as the cardinality of $\mathcal{R}[\varphi, \psi]$.

Definition 2.3. The Reidemeister coincidence number $R(f, f')$ of the continuous maps f and f' is defined as the Reidemeister coincidence number $R(f_\times, f'_\times)$ of $f_\times$ and $f'_\times$. For any coincidence Reidemeister class $[\alpha] \in$

$\mathcal{R}[f_\times, f'_\times]$, the set $p_X(\text{Coin}(\alpha\tilde{f}, \tilde{f}'))$ does not depend on the particular choice of the representative α of the coincidence Reidemeister class and we call $p_X(\text{Coin}(\alpha\tilde{f}, \tilde{f}'))$ the coincidence class of f, f' corresponding to $[\alpha]$.

Note that the Reidemeister coincidence number $R(f, f')$ does not depend on the particular choice of the lifts $\tilde{f}, \tilde{f}'$. Also the coincidence classes do not depend on the choice of lifts, although the corresponding coincidence Reidemeister classes do. By construction, the same nonempty coincidence class cannot be associated with different Reidemeister classes. Additionally, one can show that the number of nonempty coincidence classes is finite.

One associates a coincidence index with each coincidence class of f, f' (see [31]). If the coincidence index of a coincidence class is nonzero, then we call the coincidence class essential. Since every essential coincidence class can be shown to be nonempty, the number of essential coincidence classes is finite. The Nielsen coincidence number $N(f, f')$ is defined as the number of essential coincidence classes of f, f' . This number plays a central role in coincidence theory since $N(f, f')$ is a lower bound for the cardinality of $\text{Coin}(f, f')$.

Let $f_0, f'_0, f_1, f'_1 : X \rightarrow Y$ be continuous maps and let $F : f_0 \simeq f_1$ and $F' : f'_0 \simeq f'_1$ be homotopies. These homotopies induce a one-one correspondence (called the (F, F') -relation) between the coincidence classes of f_0, f'_0 and those of f_1, f'_1 . A coincidence class of f_0, f'_0 is essential if and only if the corresponding coincidence class of f_1, f'_1 is essential. Hence the Nielsen coincidence number is a homotopy invariant: $N(f_0, f'_0) = N(f_1, f'_1)$. It also follows that $N(f_0, f'_0)$ is a lower bound for the number of coincidences of f_1, f'_1 . Schirmer [30] proved the following important theorem.

Theorem 2.4 (see [30, Satz IX]). *Suppose that X and Y have equal dimension $n \geq 3$. Let $f_0, f'_0 : X \rightarrow Y$ be continuous maps. Then there exist homotopies $F : f_0 \simeq f_1$ and $F' : f'_0 \simeq f'_1$ such that for every coincidence class C_0 of $\text{Coin}(f_0, f'_0)$, the corresponding coincidence class of $\text{Coin}(f_1, f'_1)$ is empty if C_0 is inessential and a singleton if C_0 is essential.*

This result is often rephrased by stating that when the dimension of X and Y is at least three, then for any pair of continuous maps $f_0, f'_0 : X \rightarrow Y$, there exist maps f_1 homotopic to f_0 and f'_1 homotopic to f'_0 such that the cardinality of $\text{Coin}(f_1, f'_1)$ is precisely $N(f_0, f'_0)$. In other words: when the dimension is at least three, then the Nielsen coincidence number is the largest homotopy invariant lower bound for the number of coincidences.

The Reidemeister number in fixed point theory can be seen as a special case of the Reidemeister coincidence number.

Definition 2.5. Let G be a group. Define the Reidemeister number $R(\varphi)$ of an endomorphism $\varphi : G \rightarrow G$ by $R(\varphi) = R(\varphi, \text{Id})$. We say that G has the R_∞ property if $R(\varphi) = \infty$ for every automorphism $\varphi : G \rightarrow G$.

Definition 2.6. Let X be a closed connected manifold and $f : X \rightarrow X$ a continuous self-map. Let $p : \tilde{X} \rightarrow X$ be a universal cover and fix a lift $\tilde{f} :$

$\tilde{X} \rightarrow \tilde{X}$ of f . Then $f : X \rightarrow X$ induces a morphism $f_{\times} : A(\tilde{X}, p) \rightarrow A(\tilde{X}, p)$ of the covering transformation group. The Reidemeister number $R(f)$ of f is defined as the Reidemeister number $R(f_{\times})$ of the morphism $f_{\times}$.

Remark that the Reidemeister number of a continuous self-map on a closed connected manifold does not depend on the choice of the lift.

The definition of the Nielsen number is similar to the definition of the Nielsen coincidence number. An important difference is that the Nielsen number can be defined for self-maps on nonorientable closed connected manifolds.

2.2. Almost-crystallographic groups

Let G be a connected, simply connected, nilpotent Lie group. We use $\text{Aut}(G)$ to denote the automorphism group of G , it acts naturally on G . We define $\text{Aff}(G)$ as the semidirect product $\text{Aff}(G) = G \rtimes \text{Aut}(G)$, which is $G \times \text{Aut}(G)$ as a set, with multiplication defined by $(d_1, D_1)(d_2, D_2) = (d_1 D_1(d_2), D_1 D_2)$. The inverse of $(d, D) \in \text{Aff}(G)$ is given by $(d, D)^{-1} = (D^{-1}(d^{-1}), D^{-1})$. $\text{Aff}(G)$ acts on G by

$$(d, D)(g) = dD(g) \quad \text{for all } (d, D) \in \text{Aff}(G) \text{ and all } g \in G.$$

We will often silently consider G as a subgroup of $\text{Aff}(G)$ by identifying $g \in G$ with (g, Id_G) .

Let C be a maximal compact subgroup of $\text{Aut}(G)$, then $G \rtimes C$ is a subgroup of $\text{Aff}(G)$. A cocompact discrete subgroup Γ of $G \rtimes C$ is called an almost-crystallographic group (modelled on the Lie group G). The dimension of Γ is defined as the dimension of G . If Γ is torsion free, then we call Γ an almost-Bieberbach group. If $G = \mathbb{R}^n$, then we say that Γ is a crystallographic group. If $G = \mathbb{R}^n$ and Γ is torsion free, then we call Γ a Bieberbach group.

If $\Gamma \subset \text{Aff}(G)$ is an almost-Bieberbach group, then the quotient space $\Gamma \backslash G$ is a manifold. Such manifolds are called infra-nilmanifolds. If $\Gamma \subset G$, then we call $\Gamma \backslash G$ a nilmanifold. If $\Gamma \subset \text{Aff}(\mathbb{R}^n)$ is a Bieberbach group, then $\Gamma \backslash \mathbb{R}^n$ is a closed flat manifold.

Historically, crystallographic groups were studied first. Standard references are [6, 33]. Crystallographic groups are described by the so-called Bieberbach theorems that have been proved by Bieberbach and Fröbenius [3, 4, 15]. These theorems have been generalised to almost-crystallographic groups.

Generalised first Bieberbach theorem (see [2]). *Let $\Gamma \subset \text{Aff}(G)$ be an almost-crystallographic group. Then $\Gamma \cap G$ is a uniform lattice of G and $\Gamma \cap G$ is of finite index in Γ .*

For an almost-crystallographic group $\Gamma \subset \text{Aff}(G)$, the quotient group $\Gamma/(\Gamma \cap G)$ is isomorphic to the group

$$\{A \in \text{Aut}(G) \mid \exists a \in G : (a, A) \in \Gamma\}$$

that we call the holonomy group of Γ .

In the case $G = \mathbb{R}^n$, by conjugating, we may assume that $\Gamma \cap \mathbb{R}^n = \mathbb{Z}^n$. Because $(Az, \mathbf{1}) = (a, A)(z, \mathbf{1})(a, A)^{-1} \in \Gamma \cap \mathbb{R}^n = \mathbb{Z}^n$ for all $(a, A) \in \Gamma$ and all $z \in \mathbb{Z}^n$, this assumption implies that the holonomy group of Γ is a subgroup of $\mathrm{GL}_n(\mathbb{Z})$.

The generalised second Bieberbach theorem characterises isomorphisms between almost-crystallographic groups.

Generalised second Bieberbach theorem (see [27]). *Let $\varphi : \Gamma_1 \rightarrow \Gamma_2$ be an isomorphism between almost-crystallographic groups $\Gamma_1, \Gamma_2 \subset \mathrm{Aff}(G)$. Then there exists $\delta \in \mathrm{Aff}(G)$ such that $\varphi(\alpha) = \delta\alpha\delta^{-1}$ for all $\alpha \in \Gamma_1$.*

Since almost-crystallographic groups modelled on nonisomorphic Lie groups cannot be isomorphic, the generalised second Bieberbach theorem describes all isomorphisms between almost-crystallographic groups.

K. B. Lee [26] generalised the second Bieberbach theorem even further. To formulate this generalisation, we introduce some notation.

Notation 2.7. Let G be a group and $g \in G$. We use τ_g to denote the conjugation map $\tau_g : G \rightarrow G : h \mapsto ghg^{-1}$ and λ_g to denote the left multiplication map $\lambda_g : G \rightarrow G : h \mapsto gh$.

Theorem 2.8 (see [26, Theorem 1.1]). *Let $\Gamma_1 \subset \mathrm{Aff}(G_1)$ and $\Gamma_2 \subset \mathrm{Aff}(G_2)$ be almost-crystallographic groups and let $\varphi : \Gamma_1 \rightarrow \Gamma_2$ be a morphism of groups. Then there exist $d \in G_2$ and a morphism of Lie groups $D : G_1 \rightarrow G_2$ such that*

$$\varphi(\alpha) \circ \lambda_d \circ D = \lambda_d \circ D \circ \alpha \quad \text{for all } \alpha \in \Gamma_1.$$

Theorem 2.8 is a little more general than [26, Theorem 1.1], but the adaptation to the more general setting formulated here is straightforward. An important consequence is the following result.

Corollary 2.9 (see [26, Corollary 1.2]). *Let G_1 and G_2 be connected, simply connected, nilpotent Lie groups. Let M_1 and M_2 be infra-nilmanifolds modelled on G_1 and G_2 respectively. Let $f : M_1 \rightarrow M_2$ be a continuous map. Then there exist a morphism of Lie groups $D : G_1 \rightarrow G_2$ and $d \in G_2$ such that $\lambda_d \circ D : G_1 \rightarrow G_2$ is a homotopy lift of f (that is, $\lambda_d \circ D$ is the lift of a map homotopic to f).*

Also Corollary 2.9 is a straightforward generalisation of [26, Corollary 1.2]. There is a more precise result for nilmanifolds.

Theorem 2.10 (see [12, Proposition 3.2] and [29, Lemma 2.7]). *Let G_1 and G_2 be connected, simply connected, nilpotent Lie groups. Let M_1 and M_2 be nilmanifolds modelled on G_1 and G_2 , respectively. Let $f : M_1 \rightarrow M_2$ be a continuous map. Then there exists a morphism of Lie groups $D : G_1 \rightarrow G_2$ that is a homotopy lift of f .*

The third Bieberbach theorem states that in any given dimension, there are only finitely many crystallographic groups. The generalised third Bieberbach theorem, concerning almost-crystallographic groups, is more subtle. For more information we refer the reader to [11, 25].

3. The finiteness of the Reidemeister coincidence number of a pair of morphisms between almost-crystallographic groups

In this section, we prove a criterion that allows to easily check the finiteness of the Reidemeister coincidence number of a pair of morphisms between almost-crystallographic groups. As immediate corollaries, we obtain a criterion for the finiteness of the Reidemeister number of an endomorphism of an almost-crystallographic group, a criterion for the finiteness of the Reidemeister coincidence number of a pair of continuous maps between infra-nilmanifolds and a criterion for the finiteness of the Reidemeister number of a continuous self-map on an infra-nilmanifold.

In order to obtain these criteria, we first develop a criterion for the finiteness of the Reidemeister coincidence number of a pair of continuous maps between nilmanifolds. The algebraic counterpart is a criterion for the finiteness of the Reidemeister coincidence number of a pair of morphisms between uniform lattices in connected, simply connected, nilpotent Lie groups. For the latter criterion, we will switch from the discrete group level to the Lie group level, using the following result.

Theorem 3.1 (see [20, 28]). *Let G_1 and G_2 be connected, simply connected, nilpotent Lie groups and Λ_1 a uniform lattice of G_1 . Then every morphism of groups $\varphi : \Lambda_1 \rightarrow G_2$ is the restriction of a unique morphism of Lie groups $\Phi : G_1 \rightarrow G_2$.*

We will also switch from the Lie group level to the Lie algebra level.

Next we develop a criterion for the finiteness of the Reidemeister coincidence number of a pair of morphisms between almost-crystallographic groups by using the fact that an almost-crystallographic group has a finite index normal subgroup that is a uniform lattice of a connected, simply connected, nilpotent Lie group.

The following lemma is a first step towards a criterion for the finiteness of the Reidemeister coincidence number of a pair of morphisms between uniform lattices of connected, simply connected, nilpotent Lie groups.

Lemma 3.2. *Let G_1 and G_2 be connected, simply connected, nilpotent Lie groups with associated Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Let $\Phi, \Psi : G_1 \rightarrow G_2$ be morphisms of Lie groups inducing morphisms of Lie algebras $\Phi_*, \Psi_* : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$. Let Λ_1 be a subgroup of G_1 and Λ_2 a subgroup of G_2 and suppose that $\Phi(\Lambda_1) \subset \Lambda_2$ and $\Psi(\Lambda_1) \subset \Lambda_2$. Define $\varphi : \Lambda_1 \rightarrow \Lambda_2$ as the restriction of Φ and $\psi : \Lambda_1 \rightarrow \Lambda_2$ as the restriction of Ψ . Let*

$$\mathfrak{g}_2 = \gamma_1(\mathfrak{g}_2) \supset \gamma_2(\mathfrak{g}_2) \supset \cdots \supset \gamma_c(\mathfrak{g}_2) \supset \gamma_{c+1}(\mathfrak{g}_2) = \{0\}$$

be the lower central series of $\mathfrak{g}_2$ and suppose that each $\gamma_i(\mathfrak{g}_2)$ is spanned by $\gamma_i(\mathfrak{g}_2) \cap \log(\Lambda_2)$. Suppose that $\Psi_ - \Phi_*$ is not surjective, then $R(\varphi, \psi) = \infty$.*

Remark that the condition that each $\gamma_i(\mathfrak{g}_2)$ is spanned by $\gamma_i(\mathfrak{g}_2) \cap \log(\Lambda_2)$ is satisfied when $\Lambda_2 = G_2$ or when Λ_2 is a uniform lattice of G_2 .

In the proof of this lemma, we will switch from the Lie group level to the Lie algebra level. For this reason, let us briefly review the relation between a connected, simply connected, nilpotent Lie group G and the corresponding Lie algebra $\mathfrak{g}$. First recall that in this case, the exponential map $\exp : \mathfrak{g} \rightarrow G$ is a bijection; we denote the inverse by $\log : G \rightarrow \mathfrak{g}$. Let H also be a connected, simply connected, nilpotent Lie group and let $\mathfrak{h}$ be the associated Lie algebra, then a morphism of Lie groups $\Phi : G \rightarrow H$ induces a morphism of Lie algebras $\Phi_* : \mathfrak{g} \rightarrow \mathfrak{h}$ that satisfies $\Phi_*(X) = \log(\Phi(\exp(X)))$ for all $X \in \mathfrak{g}$.

Using the bijections $\exp$ and $\log$ between G and $\mathfrak{g}$, we can define a binary operation $*$ on $\mathfrak{g}$ by $X * Y = \log(\exp(X)\exp(Y))$ for all $X, Y \in \mathfrak{g}$. The relation between this operation and the Lie brackets is given by the Baker–Campbell–Hausdorff formula, which states that

$$X * Y = X + Y + \sum_{m=2}^{\infty} q_m(X, Y),$$

where $q_m(X, Y)$ is a rational linear combination of m -fold Lie brackets in X and Y . Since $\mathfrak{g}$ is nilpotent, this sum is actually a finite sum.

With this in mind, we can prove Lemma 3.2.

Proof of Lemma 3.2. Let i be the maximal natural number such that $\gamma_i(\mathfrak{g}_2) \not\subset \text{Im}(\Psi_* - \Phi_*)$. Now $\gamma_i(\mathfrak{g}_2) \cap \log(\Lambda_2) \subset \text{Im}(\Psi_* - \Phi_*)$ would imply that $\gamma_i(\mathfrak{g}_2) \subset \text{Im}(\Psi_* - \Phi_*)$ because $\gamma_i(\mathfrak{g}_2)$ is spanned by $\gamma_i(\mathfrak{g}_2) \cap \log(\Lambda_2)$. Hence there exists $X \in \gamma_i(\mathfrak{g}_2) \cap \log(\Lambda_2)$ such that $X \notin \text{Im}(\Psi_* - \Phi_*)$. Now for every $n \in \mathbb{Z} \setminus \{0\}$, we have that $nX \in \gamma_i(\mathfrak{g}_2) \cap \log(\Lambda_2)$ but $nX \notin \text{Im}(\Psi_* - \Phi_*)$. We claim that $\exp(n_1X)$ and $\exp(n_2X)$ represent the same Reidemeister class in $\mathcal{R}[\varphi, \psi]$ only when $n_1 = n_2$. Suppose that $\exp(n_1X)$ and $\exp(n_2X)$ represent the same Reidemeister class, then there exists $\lambda \in \Lambda_1$ such that

$$\exp(n_1X) = \psi(\lambda) \exp(n_2X) \varphi(\lambda)^{-1},$$

such that

$$\exp(n_1X) \Phi(\lambda) = \Psi(\lambda) \exp(n_2X).$$

Hence

$$n_1X * \Phi_*(\log(\lambda)) = \Psi_*(\log(\lambda)) * n_2X.$$

By the Baker–Campbell–Hausdorff formula, we can write

$$n_1X * \Phi_*(\log(\lambda)) = n_1X + \Phi_*(\log(\lambda)) + Y_1,$$

where $Y_1 = \sum_{m=2}^{\infty} q_m(n_1X, \Phi_*(\log(\lambda)))$, where $q_m(n_1X, \Phi_*(\log(\lambda)))$ is a rational linear combination of m -fold Lie brackets in n_1X and $\Phi_*(\log(\lambda))$. Because $X \in \gamma_i(\mathfrak{g}_2)$, we have that $Y_1 \in \gamma_{i+1}(\mathfrak{g}_2) \subset \text{Im}(\Psi_* - \Phi_*)$. Similarly, we can write

$$\Psi_*(\log(\lambda)) * n_2X = n_2X + \Psi_*(\log(\lambda)) + Y_2,$$

where $Y_2 \in \text{Im}(\Psi_* - \Phi_*)$. Hence

$$n_1X + \Phi_*(\log(\lambda)) + Y_1 = n_2X + \Psi_*(\log(\lambda)) + Y_2$$

and

$$(n_1 - n_2)X = (\Psi_* - \Phi_*)(\log(\lambda)) + Y_2 - Y_1 \in \text{Im}(\Psi_* - \Phi_*),$$

so that $n_1 = n_2$. We see that $\exp(n_1 X)$ and $\exp(n_2 X)$ represent different Reidemeister classes in $\mathcal{R}[\varphi, \psi]$ if $n_1 \neq n_2$, hence $R(\varphi, \psi) = \infty$. $\square$

Lemma 3.3. *Let G_1 and G_2 be connected, simply connected, nilpotent Lie groups with associated Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Let $\Phi, \Psi : G_1 \rightarrow G_2$ be morphisms of Lie groups inducing morphisms of Lie algebras $\Phi_*, \Psi_* : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$. Suppose that $\Psi_* - \Phi_* : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ is surjective, then the map $\eta : G_1 \rightarrow G_2 : x \mapsto \Psi(x)\Phi(x)^{-1}$ is surjective.*

Proof. Let us first prove that $\eta(G_1)$ contains the centre $Z(G_2)$ of G_2 . Choose $y \in Z(G_2)$, then $\log(y)$ can be written as $\log(y) = \Psi_*(X) - \Phi_*(X)$ for some $X \in \mathfrak{g}_1$. Write $x = \exp(X)$, then

$$\begin{aligned} \eta(x) &= \Psi(x)\Phi(x^{-1}) \\ &= \exp\left(\Psi_*(\log(x)) * \Phi_*(\log(x^{-1}))\right) \\ &= \exp(\Psi_*(X) * \Phi_*(-X)). \end{aligned}$$

Now because $\Psi_*(X) - \Phi_*(X) = \log(y)$ is in the centre $Z(\mathfrak{g}_2)$ of $\mathfrak{g}_2$,

$$\begin{aligned} [\Psi_*(X), \Phi_*(X)] &= [\Psi_*(X) - \Phi_*(X) + \Phi_*(X), \Phi_*(X)] \\ &= [\Psi_*(X) - \Phi_*(X), \Phi_*(X)] + [\Phi_*(X), \Phi_*(X)] \\ &= 0 + 0 = 0. \end{aligned}$$

Hence all Lie brackets in the Baker–Campbell–Hausdorff formula for $\Psi_*(X) * \Phi_*(-X)$ vanish and

$$\eta(x) = \exp(\Psi_*(X) * \Phi_*(-X)) = \exp(\Psi_*(X) - \Phi_*(X)) = y.$$

We prove the lemma by induction on the nilpotency class of G_2 . If G_2 is abelian, then $G_2 = Z(G_2) \subseteq \eta(G_1)$.

Suppose G_2 is of nilpotency class $c > 1$. Then $G_2/Z(G_2)$ is a connected, simply connected, nilpotent Lie group of nilpotency class $c - 1$. If we denote by $p : G_2 \rightarrow G_2/Z(G_2)$ the natural projection, then $\bar{\Phi} = p \circ \Phi$ and $\bar{\Psi} = p \circ \Psi$ are morphisms of Lie groups inducing morphisms of Lie algebras $\bar{\Phi}_*, \bar{\Psi}_* : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2/Z(\mathfrak{g}_2)$. It is easy to see that $\bar{\Psi}_* - \bar{\Phi}_*$ is surjective. Choose $y \in G_2$. By the induction hypothesis, $p(y)$ can be written as $p(y) = \bar{\Psi}(x)\bar{\Phi}(x)^{-1}$ for some $x \in G_1$. Hence there exists $z \in Z(G_2)$ such that $y = \Psi(x)\Phi(x)^{-1}z = \Psi(x)z\Phi(x)^{-1}$. Now because $z \in Z(G_2) \subset \eta(G_1)$, there exists $u \in G_1$ such that $z = \eta(u) = \Psi(u)\Phi(u)^{-1}$. Hence

$$\begin{aligned} y &= \Psi(x)z\Phi(x)^{-1} = \Psi(x)\Psi(u)\Phi(u)^{-1}\Phi(x)^{-1} \\ &= \Psi(xu)\Phi(xu)^{-1} = \eta(xu). \end{aligned} \quad \square$$

Lemma 3.4. *Let G_1 and G_2 be connected, simply connected, nilpotent Lie*

groups with associated Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Let $\Phi, \Psi : G_1 \rightarrow G_2$ be morphisms of Lie groups inducing morphisms of Lie algebras $\Phi_*, \Psi_* : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$. Then $R(\Phi, \Psi) = \infty$ if and only if $\Psi_* - \Phi_*$ is not surjective.

Remark that $R(\Phi, \Psi)$ is the Reidemeister coincidence number as defined in Definition 2.2 (dealing with morphisms of groups), not the one defined in Definition 2.3 (dealing with continuous maps).

Proof. If $\Psi_* - \Phi_*$ is not surjective, then $R(\Phi, \Psi) = \infty$ by Lemma 3.2. Conversely, suppose that $\Psi_* - \Phi_*$ is surjective, then by the previous lemma, the map $\eta : G_1 \rightarrow G_2 : h \mapsto \Psi(h)\Phi(h)^{-1}$ is surjective: every $g \in G_2$ can be written as $g = \Psi(h)1_{G_2}\Phi(h)^{-1}$. Hence $[g] = [1_{G_2}]$ for every $g \in G_2$ and $R(\Phi, \Psi) = 1 < \infty$. $\square$

Lemma 3.5. *Let G_1 and G_2 be connected, simply connected, nilpotent Lie groups with associated Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Let $\Phi, \Psi : G_1 \rightarrow G_2$ be morphisms of Lie groups inducing morphisms of Lie algebras $\Phi_*, \Psi_* : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$. Then for any $g \in G_2$,*

$$\Psi_* - (\tau_g)_*\Phi_* \text{ is surjective} \iff \Psi_* - \Phi_* \text{ is surjective.}$$

Proof. By the previous lemma,

$$\Psi_* - (\tau_g)_*\Phi_* \text{ is surjective} \iff R(\tau_g \circ \Phi, \Psi) < \infty$$

and

$$\Psi_* - \Phi_* \text{ is surjective} \iff R(\Phi, \Psi) < \infty.$$

Now define

$$A : \mathcal{R}[\Phi, \Psi] \rightarrow \mathcal{R}[\tau_g \Phi, \Psi] : [h] \mapsto [hg^{-1}]$$

and

$$B : \mathcal{R}[\tau_g \Phi, \Psi] \rightarrow \mathcal{R}[\Phi, \Psi] : [h] \mapsto [hg],$$

then A is a bijection with inverse B . Hence $R(\tau_g \circ \Phi, \Psi) = R(\Phi, \Psi)$ and the lemma follows. $\square$

In the case where G_1 and G_2 have equal dimension, this lemma states that for any $g \in G_2$, we have that $\det(\Psi_* - (\tau_g)_*\Phi_*) \neq 0$ if and only if $\det(\Psi_* - \Phi_*) \neq 0$. In [21], Ha, Lee and Penninckx will prove that in fact $\det(\Psi_* - (\tau_g)_*\Phi_*) = \det(\Psi_* - \Phi_*)$ for any $g \in G_2$.

Now we can formulate and prove a criterion for the finiteness of the Reidemeister coincidence number of a pair of continuous maps between nilmanifolds.

Theorem 3.6. *Let G_1 and G_2 be connected, simply connected, nilpotent Lie groups. Let N_1 and N_2 be nilmanifolds modelled on G_1 and G_2 , respectively.*

Let $f, f' : N_1 \rightarrow N_2$ be continuous maps and let $D, D' : G_1 \rightarrow G_2$ be morphisms of Lie groups that are homotopy lifts of f and f' , respectively. Then

$$R(f, f') = \infty \iff D'_* - D_* \text{ is not surjective.}$$

Proof. We may assume that D is a lift of f and D' is a lift of f' . Let Λ_1 be the uniform lattice of G_1 such that $N_1 = \Lambda_1 \backslash G_1$ and let Λ_2 be the uniform lattice of G_2 such that $N_2 = \Lambda_2 \backslash G_2$. Denote by $p_1 : G_1 \rightarrow N_1$ and $p_2 : G_2 \rightarrow N_2$ the natural covering projections, then $A(G_1, p_1) = \Lambda_1$ and $A(G_2, p_2) = \Lambda_2$. Remark that $f_\times : \Lambda_1 \rightarrow \Lambda_2$ is the restriction of $D : G_1 \rightarrow G_2$ and $f'_\times : \Lambda_1 \rightarrow \Lambda_2$ is the restriction of $D' : G_1 \rightarrow G_2$. If $R(f, f') = R(f_\times, f'_\times) < \infty$, then $D'_* - D_*$ is surjective by Lemma 3.2. Conversely, suppose that $D'_* - D_*$ is surjective, then by Lemma 3.3, the map $\eta : G_1 \rightarrow G_2 : g \mapsto D'(g)D(g)^{-1}$ is surjective. Hence for any $\lambda \in \Lambda_2 = A(G_2, p_2)$, there exists $g \in G_1$ such that $D'(g)D(g)^{-1} = \lambda$; in other words, $D'(g) = \lambda D(g)$, or $g \in \text{Coin}(\lambda D, D')$. Hence the coincidence class $p_1(\text{Coin}(\lambda D, D'))$ of f, f' is not empty. We see that for any coincidence Reidemeister class $[\lambda] \in \mathcal{R}[f_\times, f'_\times]$, the corresponding coincidence class $p_1(\text{Coin}(\lambda D, D'))$ is not empty. As the number of nonempty coincidence classes is finite, also the Reidemeister coincidence number is finite: $R(f, f') < \infty$. $\square$

Let us now formulate and prove the algebraic counterpart of the previous theorem: a criterion for the finiteness of the Reidemeister coincidence number of a pair of morphisms between uniform lattices of connected, simply connected, nilpotent Lie groups.

Corollary 3.7. *Let G_1 and G_2 be connected, simply connected, nilpotent Lie groups with associated Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$, respectively. Let $\Phi, \Psi : G_1 \rightarrow G_2$ be morphisms of Lie groups inducing morphisms of Lie algebras $\Phi_*, \Psi_* : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$. Let Λ_1 be a uniform lattice of G_1 and Λ_2 a uniform lattice of G_2 and suppose that $\Phi(\Lambda_1) \subset \Lambda_2$ and $\Psi(\Lambda_1) \subset \Lambda_2$. Define $\varphi : \Lambda_1 \rightarrow \Lambda_2$ as the restriction of Φ and $\psi : \Lambda_1 \rightarrow \Lambda_2$ as the restriction of Ψ . Then*

$$R(\varphi, \psi) = \infty \iff \Psi_* - \Phi_* \text{ is not surjective.}$$

Proof. Denote by N_1 the nilmanifold $\Lambda_1 \backslash G_1$ and by N_2 the nilmanifold $\Lambda_2 \backslash G_2$. Denote by $p_1 : G_1 \rightarrow N_1$ and $p_2 : G_2 \rightarrow N_2$ the natural projections, then p_1 and p_2 are universal covering maps and $A(G_1, p_1) = \Lambda_1$ and $A(G_2, p_2) = \Lambda_2$. Now $\Phi : G_1 \rightarrow G_2$ induces a continuous map $f : N_1 \rightarrow N_2$ and $\Psi : G_1 \rightarrow G_2$ induces a continuous map $f' : N_1 \rightarrow N_2$. If we fix Φ as a lift of f and Ψ as a lift of f' , then $f_\times = \varphi$ and $f'_\times = \psi$. Now the corollary follows from the previous theorem and the fact that $R(f, f') = R(f_\times, f'_\times) = R(\varphi, \psi)$. $\square$

Any almost-crystallographic group Γ fits in a short exact sequence

$$1 \rightarrow \Lambda \rightarrow \Gamma \rightarrow F \rightarrow 1,$$

where F is a finite group and Λ is a uniform lattice of a connected, simply connected, nilpotent Lie group. We use this fact to compare the finiteness of the Reidemeister coincidence number of a pair of morphisms between

almost-crystallographic groups to the finiteness of the Reidemeister coincidence numbers of pairs of morphisms between uniform lattices in connected, simply connected, nilpotent Lie groups. This comparison is the content of the following proposition, the proof of which incorporates the larger part of the proof of [18, Theorem 2.1].

Proposition 3.8. *Suppose we have a commutative diagram of groups*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Lambda_1 & \xrightarrow{i_1} & \Gamma_1 & \xrightarrow{p_1} & F_1 & \longrightarrow & 1 \\ & & \varphi' \downarrow \psi' & & \varphi \downarrow \psi & & \bar{\varphi} \downarrow \bar{\psi} & & \\ 1 & \longrightarrow & \Lambda_2 & \xrightarrow{i_2} & \Gamma_2 & \xrightarrow{p_2} & F_2 & \longrightarrow & 1, \end{array}$$

where F_1 and F_2 are finite groups and where the upper and lower horizontal sequences are exact. Then $R(\varphi, \psi) < \infty$ if and only if $R(\tau'_\alpha \varphi', \psi') < \infty$ for every $\alpha \in \Gamma_2$, where $\tau'_\alpha : \Lambda_2 \rightarrow \Lambda_2 : \beta \mapsto i_2^{-1}(\alpha i_2(\beta) \alpha^{-1})$.

Proof. Without loss of generality, we may assume that i_1 and i_2 are inclusion maps and $\varphi' = \varphi|_{\Lambda_1}$ and $\psi' = \psi|_{\Lambda_1}$. For any $\alpha \in \Gamma_2$, define $\bar{\tau}_\alpha : F_2 \rightarrow F_2 : \beta \mapsto p_2(\alpha) \beta p_2(\alpha^{-1})$. Then for any $\alpha \in \Gamma_2$, we have a commutative diagram as follows:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Lambda_1 & \xrightarrow{i_1} & \Gamma_1 & \xrightarrow{p_1} & F_1 & \longrightarrow & 1 \\ & & \tau'_\alpha \varphi' \downarrow \psi' & & \tau_\alpha \varphi \downarrow \psi & & \bar{\tau}_\alpha \bar{\varphi} \downarrow \bar{\psi} & & \\ 1 & \longrightarrow & \Lambda_2 & \xrightarrow{i_2} & \Gamma_2 & \xrightarrow{p_2} & F_2 & \longrightarrow & 1. \end{array}$$

For any $\alpha \in \Gamma_2$, define

$$\hat{i}^\alpha : \mathcal{R}[\tau'_\alpha \varphi', \psi'] \rightarrow \mathcal{R}[\tau_\alpha \varphi, \psi] : [\gamma] \mapsto [\gamma]$$

and

$$\hat{p}^\alpha : \mathcal{R}[\tau_\alpha \varphi, \psi] \rightarrow \mathcal{R}[\bar{\tau}_\alpha \bar{\varphi}, \bar{\psi}] : [\gamma] \mapsto [p_2(\gamma)].$$

Also write $\hat{p} = \hat{p}^{1_{\Gamma_2}}$. Now $\mathcal{R}[\varphi, \psi]$ equals the disjoint finite union

$$\coprod_{[\bar{\alpha}] \in \mathcal{R}[\bar{\varphi}, \bar{\psi}]} \hat{p}^{-1}([\bar{\alpha}]).$$

Hence it suffices to show that for any $\alpha \in \Gamma_2$, the set $\hat{p}^{-1}([p_2(\alpha)])$ is finite if and only if the cardinality $R(\tau'_\alpha \varphi', \psi')$ of $\mathcal{R}[\tau'_\alpha \varphi', \psi']$ is finite. Define

$$A : \mathcal{R}[\varphi, \psi] \rightarrow \mathcal{R}[\tau_\alpha \varphi, \psi] : [\beta] \mapsto [\beta \alpha^{-1}]$$

and

$$B : \mathcal{R}[\tau_\alpha \varphi, \psi] \rightarrow \mathcal{R}[\varphi, \psi] : [\beta] \mapsto [\beta \alpha],$$

then A is a bijection with inverse B . Moreover,

$$A(\hat{p}^{-1}([p_2(\alpha)])) = (\hat{p}^\alpha)^{-1}([1_{F_2}]) = \hat{i}^\alpha(\mathcal{R}[\tau'_\alpha \varphi', \psi'])$$

for any $\alpha \in \Gamma_2$. Hence for any $\alpha \in \Gamma_2$, the sets $\hat{p}^{-1}([p_2(\alpha)])$ and $\hat{i}^\alpha(\mathcal{R}[\tau'_\alpha \varphi', \psi'])$ have the same cardinality and it suffices to prove that $\#(\hat{i}^\alpha)^{-1}([\beta]) \leq \#F_1$ for any $[\beta] \in \mathcal{R}[\tau_\alpha \varphi, \psi]$. This follows from [23, Lemma 2.1(4)] and can also

be seen as follows. Suppose for a contradiction that there exist $\#F_1 + 1$ distinct Reidemeister classes $[\beta_1], \dots, [\beta_{\#F_1+1}] \in \mathcal{R}[\tau'_\alpha \varphi', \psi']$ that have the same image $[\beta] \in \mathcal{R}[\tau_\alpha \varphi, \psi]$ under $\hat{i}^\alpha$:

$$\hat{i}^\alpha([\beta_1]) = \dots = \hat{i}^\alpha([\beta_{\#F_1+1}]) = [\beta].$$

Then there exist $\gamma_1, \dots, \gamma_{\#F_1+1} \in \Gamma_1$ such that

$$\beta_i = \psi(\gamma_i)\beta(\tau_\alpha \circ \varphi)(\gamma_i^{-1}).$$

By the pigeonhole principle, there exist $i \neq j$ such that $p_1(\gamma_i) = p_1(\gamma_j)$. Then $\gamma_j = \delta\gamma_i$ for some $\delta \in \Lambda_1$. Now

$$\begin{aligned} \beta_j &= \psi(\gamma_j)\beta(\tau_\alpha \circ \varphi)(\gamma_j^{-1}) \\ &= \psi(\delta\gamma_i)\beta(\tau_\alpha \circ \varphi)((\delta\gamma_i)^{-1}) \\ &= \psi(\delta)\psi(\gamma_i)\beta(\tau_\alpha \circ \varphi)(\gamma_i^{-1})(\tau_\alpha \circ \varphi)(\delta^{-1}) \\ &= \psi(\delta)\beta_i(\tau_\alpha \circ \varphi)(\delta^{-1}), \end{aligned}$$

and hence $\beta_j = \psi'(\delta)\beta_i(\tau'_\alpha \circ \varphi')(\delta^{-1})$, such that $[\beta_j]$ and $[\beta_i]$ are the same Reidemeister class in $\mathcal{R}[\tau'_\alpha \varphi', \psi']$, a contradiction. $\square$

We can now formulate and prove the announced criterion for the finiteness of the Reidemeister coincidence number of a pair of morphisms between almost-crystallographic groups.

Theorem 3.9. *Let G_1 and G_2 be connected, simply connected, nilpotent Lie groups. Let $\Gamma_1 \subset \text{Aff}(G_1)$ and $\Gamma_2 \subset \text{Aff}(G_2)$ be almost-crystallographic groups and $\varphi, \psi : \Gamma_1 \rightarrow \Gamma_2$ morphisms of groups. Let $F_2 \subset \text{Aut}(G_2)$ be the holonomy group of Γ_2 . Let $D_\varphi, D_\psi : G_1 \rightarrow G_2$ be morphisms of Lie groups and $d_\varphi, d_\psi \in G_2$ such that*

$$\varphi(\beta) \circ \lambda_{d_\varphi} \circ D_\varphi = \lambda_{d_\varphi} \circ D_\varphi \circ \beta \quad \text{and} \quad \psi(\beta) \circ \lambda_{d_\psi} \circ D_\psi = \lambda_{d_\psi} \circ D_\psi \circ \beta \quad (1)$$

for every $\beta \in \Gamma_1$. Then

$$R(\varphi, \psi) = \infty \iff \exists A \in F_2 \text{ such that } D_{\psi*} - A_* D_{\varphi*} \text{ is not surjective.}$$

Remark that for any pair of morphisms $\varphi, \psi : \Gamma_1 \rightarrow \Gamma_2$, Theorem 2.8 guarantees the existence of $d_\varphi, d_\psi \in G_2$ and morphisms of Lie groups $D_\varphi, D_\psi : G_1 \rightarrow G_2$ satisfying (1).

Proof. By a straightforward generalisation of [24, Lemma 3.1], there exists a uniform lattice Λ_1 of G_1 and a uniform lattice Λ_2 of G_2 such that Λ_1 is a finite index fully characteristic subgroup of Γ_1 , Λ_2 is a finite index fully characteristic subgroup of Γ_2 and such that $\theta(\Lambda_1) \subset \Lambda_2$ for every morphism of groups $\theta : \Gamma_1 \rightarrow \Gamma_2$. Then we have a commutative diagram of the form

$$\begin{array}{ccccccc} 1 & \longrightarrow & \Lambda_1 & \longrightarrow & \Gamma_1 & \longrightarrow & \Gamma_1/\Lambda_1 \longrightarrow 1 \\ & & \varphi' \downarrow \psi' & & \varphi \downarrow \psi & & \overline{\varphi} \downarrow \overline{\psi} \\ 1 & \longrightarrow & \Lambda_2 & \longrightarrow & \Gamma_2 & \longrightarrow & \Gamma_2/\Lambda_2 \longrightarrow 1. \end{array}$$

Now choose $\lambda \in \Lambda_1$, then

$$\varphi(\lambda) \circ \lambda_{d_\varphi} \circ D_\varphi = \lambda_{d_\varphi} \circ D_\varphi \circ \lambda.$$

If we apply both sides of this equality to the identity element 1_{G_1} of G , we see that $\varphi'(\lambda)d_\varphi = d_\varphi D_\varphi(\lambda)$ and hence $\varphi'(\lambda) = (\tau_{d_\varphi} \circ D_\varphi)(\lambda)$. Define $\Phi = \tau_{d_\varphi} \circ D_\varphi : G_1 \rightarrow G_2$, then $\varphi' : \Lambda_1 \rightarrow \Lambda_2$ is the restriction of Φ . Similarly, define $\Psi = \tau_{d_\psi} \circ D_\psi$, then ψ' is the restriction of Ψ . Also, for every $(a, A) \in \Gamma_2$ and every $\lambda \in \Lambda_2$, we have that

$$\begin{aligned} \tau_{(a,A)}(\lambda) &= (a, A)(\lambda, \mathbb{1})(a, A)^{-1} \\ &= (aA(\lambda), A)(A^{-1}(a^{-1}), A^{-1}) \\ &= (aA(\lambda)a^{-1}, \mathbb{1}) \end{aligned}$$

such that $\tau_{(a,A)}(\lambda) = (\tau_a \circ A)(\lambda)$ for every $\lambda \in \Lambda_2$. Write $\tau'_{(a,A)} : \Lambda_2 \rightarrow \Lambda_2$ as the restriction of $\tau_{(a,A)} : \Gamma_2 \rightarrow \Gamma_2$, then $\tau'_{(a,A)} : \Lambda_2 \rightarrow \Lambda_2$ is also the restriction of $\tau_a \circ A : G_2 \rightarrow G_2$. Hence

$$R(\varphi, \psi) = \infty$$

$$\begin{aligned} &\iff \exists(a, A) \in \Gamma_2 : R(\tau'_{(a,A)}\varphi', \psi') = \infty \quad (\text{by Proposition 3.8}) \\ &\iff \exists(a, A) \in \Gamma_2 : \Psi_* - (\tau_a \circ A \circ \Phi)_* \text{ is not surjective} \quad (\text{by Corollary 3.7}) \\ &\iff \exists(a, A) \in \Gamma_2 : \tau_{d_\psi*} D_{\psi*} - (\tau_a \circ A \circ \tau_{d_\varphi} \circ D_\varphi)_* \text{ is not surjective} \\ &\iff \exists(a, A) \in \Gamma_2 : \tau_{d_\psi*} D_{\psi*} - \tau_{aA(d_\varphi)*} (AD_\varphi)_* \text{ is not surjective} \\ &\iff \exists A \in F_2 : D_{\psi*} - A_* D_{\varphi*} \text{ is not surjective} \quad (\text{by Lemma 3.5}). \quad \square \end{aligned}$$

As a special case, we obtain a criterion for the finiteness of the Reidemeister number of an endomorphism of an almost-crystallographic group.

Corollary 3.10. *Let G be a connected, simply connected, nilpotent Lie group and $\Gamma \subset \text{Aff}(G)$ an almost-crystallographic group with holonomy group $F \subset \text{Aut}(G)$. Let $\varphi : \Gamma \rightarrow \Gamma$ be an endomorphism. Let $D : G \rightarrow G$ be a morphism of Lie groups and $d \in G$ such that*

$$\varphi(\beta) \circ \lambda_d \circ D = \lambda_d \circ D \circ \beta$$

for every $\beta \in \Gamma$. Then

$$R(\varphi) = \infty \iff \exists A \in F \text{ such that } \det(\mathbb{1} - A_* D_*) = 0.$$

As corollaries of these algebraic criteria, we obtain a criterion for the finiteness of the Reidemeister coincidence number of a pair of continuous maps between infra-nilmanifolds and a criterion for the finiteness of the Reidemeister number of a continuous self-map on an infra-nilmanifold.

Theorem 3.11. *Let G_1 and G_2 be connected, simply connected, nilpotent Lie groups. Let M_1 and M_2 be infra-nilmanifolds modelled on G_1 and G_2 , respectively. Let $F_2 \subset \text{Aut}(G_2)$ be the holonomy group of M_2 . Let $f, f' : M_1 \rightarrow M_2$*

be continuous maps. Let $D, D' : G_1 \rightarrow G_2$ be morphisms of Lie groups and $d, d' \in G_2$ such that $\lambda_d \circ D : G_1 \rightarrow G_2$ is a homotopy lift of f and $\lambda_{d'} \circ D' : G_1 \rightarrow G_2$ is a homotopy lift of f' . Then

$$R(f, f') = \infty \iff \exists A \in F_2 \text{ such that } D'_* - A_* D_* \text{ is not surjective.}$$

Proof. Let $p_1 : G_1 \rightarrow M_1$ and $p_2 : G_2 \rightarrow M_2$ be the universal covering projections. We may assume that $\lambda_d \circ D$ is a lift of f and $\lambda_{d'} \circ D'$ is a lift of f' . Then

$$f_{\times}(\beta) \circ \lambda_d \circ D = \lambda_d \circ D \circ \beta$$

and

$$f'_{\times}(\beta) \circ \lambda_{d'} \circ D' = \lambda_{d'} \circ D' \circ \beta$$

for any $\beta \in A(G_1, p_1)$. Now the theorem follows from Theorem 3.9 and the fact that $R(f, f') = R(f_{\times}, f'_{\times})$. $\square$

Corollary 3.12. *Let G be a connected, simply connected, nilpotent Lie group. Let M be an infra-nilmanifold modelled on G . Let F be the holonomy group of M . Let $f : M \rightarrow M$ be a continuous self-map. Let $D : G \rightarrow G$ be a morphism of Lie groups and $d \in G$ such that $\lambda_d \circ D : G \rightarrow G$ is a homotopy lift of f . Then $R(f) = \infty$ if and only if there exists $A \in F$ such that $\det(1 - A_* D_*) = 0$.*

4. Determining the low-dimensional almost-crystallographic groups that have the R_{∞} property

In this section, we determine the two- and three-dimensional almost-crystallographic groups that have the R_{∞} property.

We first look at two- and three-dimensional crystallographic groups and we use the computer program CARAT [5] to list all of them. In order to make it easier to look up the groups in CARAT, for every group in the lists that follow, we give, between brackets, its CARAT name in the form “Q-class name”-“Z-class name”-“name of the affine class.”

For each crystallographic group in the lists that follow, we give a set of generators. The generators always contain all translations $(z, 1) \in \text{Aff}(\mathbb{R}^n)$, with $z \in \mathbb{Z}^n$; we denote this by adding “ $\mathbb{Z}^n$ ” to the set of generators. Often the set of generators also contains other affine transformations $(a, A) \in \text{Aff}(\mathbb{R}^n)$. In this case, we write $a \in \mathbb{R}^n$ as a column vector and we write $A \in \text{Aut}(\mathbb{R}^n)$ as a matrix in $\text{GL}_n(\mathbb{R})$, expressed with respect to the standard basis of $\mathbb{R}^n$.

For a number of two- and three-dimensional crystallographic groups, an easy computer check shows that they have the R_{∞} property. Let us briefly explain how this computer check works. Every crystallographic group Γ given by CARAT has the property that $\Gamma \cap \mathbb{R}^n = \mathbb{Z}^n$. As a consequence, the holonomy group F of Γ is a finite subgroup of $\text{GL}_n(\mathbb{Z})$. CARAT gives generators of the normaliser $N_{\text{GL}_n(\mathbb{Z})}(F)$ of F in $\text{GL}_n(\mathbb{Z})$. If this normaliser is infinite,

we proceed by hand. If the normaliser $N_{\mathrm{GL}_n(\mathbb{Z})}(F)$ of F in $\mathrm{GL}_n(\mathbb{Z})$ is finite, it is easy to check by computer whether for every D in the finite normaliser $N_{\mathrm{GL}_n(\mathbb{Z})}(F)$, there exists a matrix A in F such that $\det(\mathbf{1} - AD) = 0$. If for each $D \in N_{\mathrm{GL}_n(\mathbb{Z})}(F)$, the computer finds a matrix $A \in F$ such that $\det(\mathbf{1} - AD) = 0$, then we know that Γ has the R_∞ property. In the case where the computer finds a matrix $D \in N_{\mathrm{GL}_n(\mathbb{Z})}(F)$ such that $\det(\mathbf{1} - AD) \neq 0$ for all $A \in F$, then we cannot immediately draw a conclusion and we have to proceed by hand.

In dimension two, three of the seventeen crystallographic groups had to be checked by hand. In dimension three, 25 of the 219 crystallographic groups had to be checked by hand.

The following theorem enumerates all two-dimensional crystallographic groups that do not have the R_∞ property. Between brackets, we mention the CARAT name. This result has also been obtained in [19, Section 3].

Theorem 4.1. *Up to isomorphism, the only two-dimensional crystallographic groups that do not have the R_∞ property are*

$$\begin{aligned}\Gamma_1 &= \mathbb{Z}^2 && (\text{min.2-1.1-0}), \\ \Gamma_2 &= \langle \mathbb{Z}^2, (0, -1_2) \rangle && (\text{group.1-1.1-0}), \\ \Gamma_3 &= \langle \mathbb{Z}^2, (0, \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}) \rangle && (\text{min.5-1.1-0}).\end{aligned}$$

Proof. Let Γ be one of the three groups listed above, then there exists an automorphism $\varphi : \Gamma \rightarrow \Gamma$ with finite Reidemeister number. Indeed, define $\delta \in \mathrm{Aff}(\mathbb{R}^2)$ by

- $\delta = (0, \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix})$ if $\Gamma = \Gamma_1$ (min.2-1.1-0) or Γ_2 (group.1-1.1-0),
- $\delta = (0, \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix})$ if $\Gamma = \Gamma_3$ (min.5-1.1-0).

One verifies that $\varphi : \Gamma \rightarrow \Gamma : \alpha \mapsto \delta\alpha\delta^{-1}$ is an automorphism and by applying Corollary 3.10, one shows that $R(\varphi) < \infty$.

Let Γ be one of the remaining two-dimensional crystallographic groups. Without loss of generality, we may assume that $\Gamma \cap \mathbb{R}^2 = \mathbb{Z}^2$ and that the holonomy group F of Γ is a subgroup of $\mathrm{GL}_2(\mathbb{Z})$. Let $\varphi : \Gamma \rightarrow \Gamma$ be an automorphism. By the second Bieberbach theorem, there exists $\delta \in \mathrm{Aff}(\mathbb{R}^2)$ such that $\varphi(\alpha) = \delta\alpha\delta^{-1}$ for all $\alpha \in \Gamma$. Now $\delta = (d, D)$ for some $D \in \mathrm{GL}_2(\mathbb{Z})$. It is easy to see that D belongs to the normaliser $N_{\mathrm{GL}_2(\mathbb{Z})}(F)$ of F in $\mathrm{GL}_2(\mathbb{Z})$. One verifies with the aid of a computer that for any $\hat{D} \in N_{\mathrm{GL}_2(\mathbb{Z})}(F)$, there exists $A \in F$ such that $\det(\mathbf{1}_2 - A\hat{D}) = 0$. Because this applies to D , by Corollary 3.10, $R(\varphi) = \infty$. Because this holds for an arbitrary automorphism $\varphi : \Gamma \rightarrow \Gamma$, we conclude that Γ has the R_∞ property. $\square$

Theorem 4.2. *Up to isomorphism, the only three-dimensional crystallographic*

groups that do not have the R_∞ property are

$$\begin{aligned}
 \Gamma_1 &= \mathbb{Z}^3 & (\text{min.6-1.1-0}), \\
 \Gamma_2 &= \left\langle \mathbb{Z}^3, \left(0, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \right\rangle & (\text{min.7-1.1-0}), \\
 \Gamma_3 &= \left\langle \mathbb{Z}^3, \left(\begin{pmatrix} 0 \\ 0 \\ 1/2 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \right\rangle & (\text{min.7-1.1-1}), \\
 \Gamma_4 &= \left\langle \mathbb{Z}^3, \left(0, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \right\rangle & (\text{min.7-1.2-0}), \\
 \Gamma_5 &= \langle \mathbb{Z}^3, (0, -\mathbb{1}_3) \rangle & (\text{group.5-1.1-0}), \\
 \Gamma_6 &= \left\langle \mathbb{Z}^3, \left(0, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\right), \left(0, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\right) \right\rangle & (\text{min.10-1.1-0}), \\
 \Gamma_7 &= \left\langle \mathbb{Z}^3, \left(\begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\right), \left(\begin{pmatrix} 1/2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\right) \right\rangle & (\text{min.10-1.1-3}), \\
 \Gamma_8 &= \left\langle \mathbb{Z}^3, \left(0, \begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}\right), \left(0, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\right) \right\rangle & (\text{min.10-1.3-0}), \\
 \Gamma_9 &= \left\langle \mathbb{Z}^3, \left(0, \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & -1 \end{pmatrix}\right), \left(0, \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}\right) \right\rangle & (\text{min.10-1.4-0}), \\
 \Gamma_{10} &= \left\langle \mathbb{Z}^3, \left(\begin{pmatrix} -1/2 \\ 1/2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & -1 \end{pmatrix}\right), \left(0, \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}\right) \right\rangle & (\text{min.10-1.4-1}), \\
 \Gamma_{11} &= \left\langle \mathbb{Z}^3, \left(0, \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \right\rangle & (\text{min.13-1.1-0}), \\
 \Gamma_{12} &= \left\langle \mathbb{Z}^3, \left(0, \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}\right) \right\rangle & (\text{min.13-1.2-0}).
 \end{aligned}$$

Proof. Let Γ be one of the twelve groups listed above, then there exists an automorphism $\varphi : \Gamma \rightarrow \Gamma$ with finite Reidemeister number. Indeed, define $\delta \in \text{Aff}(\mathbb{R}^3)$ by

- $\delta = \left(0, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}\right)$ if $\Gamma = \Gamma_1$ (min.6-1.1-0), Γ_2 (min.7-1.1-0) or Γ_3 (min.7-1.1-1),
- $\delta = \left(0, \begin{pmatrix} -1 & -1 & -1 \\ -1 & -2 & -1 \\ -1 & -1 & -2 \end{pmatrix}\right)$ if $\Gamma = \Gamma_4$ (min.7-1.2-0),
- $\delta = \left(0, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}\right)$ if $\Gamma = \Gamma_5$ (group.5-1.1-0),
- $\delta = \left(0, \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}\right)$ if $\Gamma = \Gamma_6$ (min.10-1.1-0),
- $\delta = \left(\begin{pmatrix} 0 \\ 1/4 \\ 1/4 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}\right)$ if $\Gamma = \Gamma_7$ (min.10-1.1-3),
- $\delta = \left(0, \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}\right)$ if $\Gamma = \Gamma_8$ (min.10-1.3-0),

- $\delta = \left(0, \begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix}\right)$ if $\Gamma = \Gamma_9$ (min.10-1.4-0),
- $\delta = \left(\begin{pmatrix} 1/2 \\ 1/4 \end{pmatrix}, \begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix}\right)$ if $\Gamma = \Gamma_{10}$ (min.10-1.4-1),
- $\delta = (0, -\mathbb{1}_3)$ if $\Gamma = \Gamma_{11}$ (min.13-1.1-0) or Γ_{12} (min.13-1.2-0).

One verifies that $\varphi : \Gamma \rightarrow \Gamma : \alpha \mapsto \delta\alpha\delta^{-1}$ is an automorphism and by applying Corollary 3.10, one shows that $R(\varphi) < \infty$.

There are thirteen three-dimensional crystallographic groups for which we will prove by hand that they have the R_∞ property, namely

$$\begin{aligned}
\Gamma_{13} &= \left\langle \mathbb{Z}^3, \left(0, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right), \left(0, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\right) \right\rangle && \text{(group.6-1.1-0),} \\
\Gamma_{14} &= \left\langle \mathbb{Z}^3, \left(\begin{pmatrix} 0 \\ 1/2 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right), \left(0, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\right) \right\rangle && \text{(group.6-1.1-1),} \\
\Gamma_{15} &= \left\langle \mathbb{Z}^3, \left(0, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right), \left(\begin{pmatrix} 1/2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\right) \right\rangle && \text{(group.6-1.1-2),} \\
\Gamma_{16} &= \left\langle \mathbb{Z}^3, \left(\begin{pmatrix} 0 \\ 1/2 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right), \left(\begin{pmatrix} 1/2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\right) \right\rangle && \text{(group.6-1.1-3),} \\
\Gamma_{17} &= \left\langle \mathbb{Z}^3, \left(0, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\right) \right\rangle && \text{(min.8-1.1-0)} \\
\Gamma_{18} &= \left\langle \mathbb{Z}^3, \left(\begin{pmatrix} 1/2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\right) \right\rangle && \text{(min.8-1.1-1),} \\
\Gamma_{19} &= \left\langle \mathbb{Z}^3, \left(0, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\right) \right\rangle && \text{(min.8-1.2-0),} \\
\Gamma_{20} &= \left\langle \mathbb{Z}^3, \left(\begin{pmatrix} 1/2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\right) \right\rangle && \text{(min.8-1.2-1),} \\
\Gamma_{21} &= \left\langle \mathbb{Z}^3, \left(0, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\right), \left(0, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\right) \right\rangle && \text{(group.6-1.2-0),} \\
\Gamma_{22} &= \left\langle \mathbb{Z}^3, \left(0, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\right), \left(\begin{pmatrix} 1/2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\right) \right\rangle && \text{(group.6-1.2-1),} \\
\Gamma_{23} &= \left\langle \mathbb{Z}^3, \left(\begin{pmatrix} 0 \\ 1/2 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\right), \left(0, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\right) \right\rangle && \text{(min.10-1.1-1),} \\
\Gamma_{24} &= \left\langle \mathbb{Z}^3, \left(\begin{pmatrix} 0 \\ 1/2 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\right), \left(0, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\right) \right\rangle && \text{(min.10-1.1-2),} \\
\Gamma_{25} &= \left\langle \mathbb{Z}^3, \left(\begin{pmatrix} 0 \\ 1/3 \end{pmatrix}, \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \right\rangle && \text{(min.13-1.1-1).}
\end{aligned}$$

Let Γ be a three-dimensional crystallographic group not isomorphic to $\Gamma_1, \dots, \Gamma_{11}$ or Γ_{12} . Without loss of generality, we may assume that $\Gamma \cap \mathbb{R}^3 = \mathbb{Z}^3$ and that the holonomy group F of Γ is a subgroup of $\text{GL}_3(\mathbb{Z})$. Let $\varphi : \Gamma \rightarrow \Gamma$ be an automorphism. Then there exists $\delta \in \text{Aff}(\mathbb{R}^3)$ such that $\varphi(\alpha) = \delta\alpha\delta^{-1}$

for all $\alpha \in \Gamma$. Again, $\delta = (d, D)$ for some $D \in N_{\mathrm{GL}_3(\mathbb{Z})}(F)$. We distinguish the following five cases.

- Γ is isomorphic to Γ_{13} (group.6-1.1-0), Γ_{14} (group.6-1.1-1), Γ_{15} (group.6-1.1-2), Γ_{16} (group.6-1.1-3), Γ_{17} (min.8-1.1-0) or Γ_{18} (min.8-1.1-1). Without loss of generality, we may assume that Γ equals Γ_{13} , Γ_{14} , Γ_{15} , Γ_{16} , Γ_{17} or Γ_{18} . Then

$$D \in N_{\mathrm{GL}_3(\mathbb{Z})}(F) = \left\{ \begin{pmatrix} x & y & 0 \\ z & t & 0 \\ 0 & 0 & \epsilon \end{pmatrix} \mid \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}), \epsilon = \pm 1 \right\},$$

and it is easy to see that we can always find $A \in F$ such that $\det(\mathbb{1}_3 - AD) = 0$.

- Γ is isomorphic to Γ_{19} (min.8-1.2-0), Γ_{20} (min.8-1.2-1), Γ_{21} (group.6-1.2-0) or Γ_{22} (group.6-1.2-1). Again, we assume that Γ equals Γ_{19} , Γ_{20} , Γ_{21} or Γ_{22} . In this case, write

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \in F.$$

Then $DAD^{-1} = A$. Indeed, DAD^{-1} has determinant -1 and clearly DAD^{-1} cannot equal $-\mathbb{1}_3$. The equality $DAD^{-1} = A$ implies that D has the form

$$D = \begin{pmatrix} d_{1,1} & d_{1,2} & d_{1,2} \\ d_{2,1} & d_{2,2} & d_{2,3} \\ d_{2,1} & d_{2,3} & d_{2,2} \end{pmatrix}$$

with $d_{1,1}, d_{1,2}, d_{2,1}, d_{2,2}, d_{2,3} \in \mathbb{Z}$. Put

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix},$$

then PFP^{-1} contains

$$PAP^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Also, PDP^{-1} is of the form

$$PDP^{-1} = \begin{pmatrix} d_{1,1} & d_{1,2} & 0 \\ 2d_{2,1} & d_{2,3} + d_{2,2} & 0 \\ 0 & 0 & d_{2,2} - d_{2,3} \end{pmatrix}.$$

Because $\det(D) = \pm 1$, we must have that $d_{2,2} - d_{2,3} = \pm 1$. Now it is easy to see that either $\det(\mathbb{1}_3 - D) = 0$ or $\det(\mathbb{1}_3 - AD) = 0$.

- Γ is isomorphic to Γ_{23} (min.10-1.1-1) or Γ_{24} (min.10-1.1-2). Again, we assume that $\Gamma = \Gamma_{23}$ or $\Gamma = \Gamma_{24}$. By the proof of [9, Proposition 5.6] or [8, Proposition 3.3], there exists a permutation σ of the set $\{1, 2, 3\}$ and $\epsilon_1, \epsilon_2, \epsilon_3 \in \{-1, 1\}$ such that the only nonzero entries of $D = (D_{i,j})_{i,j}$ are given by $D_{\sigma(1),1} = \epsilon_1$, $D_{\sigma(2),2} = \epsilon_2$ and $D_{\sigma(3),3} = \epsilon_3$. If there

exists $j \in \{1, 2, 3\}$ such that $\sigma(j) = j$, then it is clear that we can find $A \in F$ such that $\det(\mathbb{1}_3 - AD) = 0$. Suppose for a contradiction that $\sigma = (1\ 2\ 3)$ or $\sigma = (1\ 2\ 3)^{-1}$. Without loss of generality, we may assume that $\sigma = (1\ 2\ 3)$ (take the inverse of φ otherwise). Then δ has the form

$$\delta = \left(\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \epsilon_3 \\ \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \end{pmatrix} \right),$$

where $d_1, d_2, d_3 \in \mathbb{R}$. Then

$$\delta \left(0, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right) \delta^{-1} = \left(\begin{pmatrix} 2d_1 \\ 0 \\ 2d_3 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right) \notin \Gamma,$$

a contradiction.

- Γ is isomorphic to Γ_{25} (min.13-1.1-1). Again, we assume that $\Gamma = \Gamma_{25}$. Then $D \in N_{\text{GL}_3(\mathbb{Z})}(F)$ must have the form of a block matrix

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & \epsilon \end{pmatrix},$$

where $D_1 \in N_{\text{GL}_2(\mathbb{Z})}(\langle A_1 \rangle)$, $A_1 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$ and $\epsilon = \pm 1$. If $\epsilon = 1$, then $\det(\mathbb{1}_3 - D) = 0$. Suppose that $\epsilon = -1$. Put

$$(a, A) = \left(\begin{pmatrix} 0 \\ 0 \\ 1/3 \end{pmatrix}, \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \in \Gamma.$$

Write $(a', A') = (d, D)(a, A)(d, D)^{-1}$, then $a' + A'(d) = d + D(a)$. Write

$$a' = \begin{pmatrix} a'_1 \\ a'_2 \\ a'_3 \end{pmatrix} \quad \text{and} \quad d = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}.$$

Using the assumption that $\epsilon = -1$, we get that

$$a'_3 + d_3 = d_3 - 1/3.$$

Hence $A' = A^{-1}$. Hence $D_1 A_1 = A_1^{-1} D_1$ and it follows that

$$D_1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \\ \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In all these cases, $\det(\mathbb{1}_2 - D_1) = 0$. Hence $\det(\mathbb{1}_3 - D) = 0$.

- Γ is not isomorphic to $\Gamma_1, \dots, \Gamma_{24}$ or Γ_{25} . In this case, one verifies with the aid of a computer that for any $\hat{D} \in N_{\text{GL}_3(\mathbb{Z})}(F)$, there exists $A \in F$

such that $\det(\mathbb{1}_3 - A\hat{D}) = 0$. Because this applies to D , by Corollary 3.10, $R(\varphi) = \infty$. Because this holds for an arbitrary automorphism $\varphi : \Gamma \rightarrow \Gamma$, we see that Γ has the R_∞ property. $\square$

We now consider the almost-crystallographic groups up to dimension three that are not crystallographic. The lowest dimension in which one finds almost-crystallographic groups that are not crystallographic is dimension three. These three-dimensional almost-crystallographic groups are all modelled on the three-dimensional Heisenberg group H , which is the only nilpotent and non-abelian simply connected Lie group up to dimension three. Recall from [7] or [10] the following result.

Lemma 4.3. *Let Γ be an almost-crystallographic group modelled on the three-dimensional Heisenberg group H . Then $\Gamma \cap Z(H)$ is a fully characteristic subgroup of Γ that is isomorphic to $\mathbb{Z}$ and the quotient $\Gamma/(\Gamma \cap Z(H))$ is a two-dimensional crystallographic group. Additionally, if $\varphi : \Gamma \rightarrow \Gamma'$ is an isomorphism between almost-crystallographic groups modelled on H , then $\varphi(\Gamma \cap Z(H)) = \Gamma' \cap Z(H)$.*

The following theorem determines the almost-crystallographic groups that are modelled on H and that have the R_∞ property.

Theorem 4.4. *Let Γ be an almost-crystallographic group modelled on the three-dimensional Heisenberg group H . Then Γ does not have the R_∞ property if and only if the quotient $\Gamma/(\Gamma \cap Z(H))$ is isomorphic to the group Γ_1 from Theorem 4.1 or to the group Γ_2 from Theorem 4.1.*

Since all three-dimensional almost-crystallographic groups are either crystallographic or modelled on H , this theorem, together with Theorem 4.2, determines all three-dimensional almost-crystallographic groups that have the R_∞ property.

The almost-crystallographic groups modelled on H are divided into 17 families, according to the isomorphism type of the quotient $\Gamma/(\Gamma \cap Z(H))$. Each such family is nonempty (and in fact contains an infinite number of nonisomorphic groups) and groups belonging to different families are not isomorphic.

In [7, pp. 160–166] and [10], these families are numbered family 1 up to family 17 and for each of them, these references give a description of the presentations of the almost-crystallographic groups contained therein. As our proof of Theorem 4.4 will extensively make use of these presentations, let us, as an example, recall here partly the description of the presentations of the almost-crystallographic groups in family 2. For each group Γ in family 2, there exist integers k_1, k_2, k_3, k_4 such that $k_1 \neq 0$ and such that Γ is isomorphic to the group

$$\begin{aligned} \langle a, b, c, \alpha \mid [b, a] = c^{k_1}, [c, a] = 1, [c, b] = 1, \\ \alpha a = a^{-1} \alpha c^{k_2}, \alpha b = b^{-1} \alpha c^{k_3}, \alpha c = c \alpha, \alpha^2 = c^{k_4} \rangle. \end{aligned}$$

This group can be embedded in $\mathrm{GL}_4(\mathbb{R})$ via the morphism $\lambda : \Gamma \rightarrow \mathrm{GL}_4(\mathbb{R})$

defined by

$$\begin{aligned}\lambda(a) &= \begin{pmatrix} 1 & 0 & -k_1/2 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & \lambda(b) &= \begin{pmatrix} 1 & k_1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ \lambda(c) &= \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & \lambda(\alpha) &= \begin{pmatrix} 1 & k_2 & k_3 & k_4/2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.\end{aligned}$$

Conversely, for every $k_1 \in \mathbb{Z} \setminus \{0\}$ and $k_2, k_3, k_4 \in \mathbb{Z}$, the group described above is an almost-crystallographic group in family 2. Remark that different choices of the parameters k_1, k_2, k_3, k_4 may give rise to isomorphic groups.

The groups Γ in the other families are described in a similar fashion, where the notation is always chosen such that $\Gamma \cap H$ is the group generated by a, b and c and such that $\Gamma \cap Z(H)$ is the group generated by c .

Proof of Theorem 4.4. As $\Gamma \cap Z(H)$ is a fully characteristic subgroup of Γ , each automorphism of Γ induces an automorphism of the quotient (crystallographic) group $\Gamma/(\Gamma \cap Z(H))$. It follows that when this quotient group has the R_∞ property, then also Γ has the R_∞ property. By Theorem 4.1, we know that 14 out of the 17 two-dimensional crystallographic groups have the R_∞ property, so also all three-dimensional almost-crystallographic groups belonging to the corresponding 14 families have the R_∞ property.

The three remaining families correspond to the groups Γ_1, Γ_2 and Γ_3 in Theorem 4.1. These families are numbered family 1, family 2 and family 13, respectively in [7, 10].

Let us first consider any group Γ of family 2 and fix a choice of parameters k_1, k_2, k_3, k_4 . Define $\varphi : \Gamma \rightarrow \Gamma$ by

$$\begin{aligned}\varphi(a) &= ab^2c^{k_1-k_2-k_3}, & \varphi(b) &= a^2b^3c^{3k_1-k_2-2k_3}, \\ \varphi(c) &= c^{-1}, & \varphi(\alpha) &= c^{-k_4}\alpha,\end{aligned}$$

then φ is an automorphism. This is easy to check since $\lambda(\varphi(\gamma)) = \delta\lambda(\gamma)\delta^{-1}$ for every $\gamma \in \Gamma$, where

$$\delta = \begin{pmatrix} -1 & -k_2 - k_3 & -k_2 - 2k_3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

One can apply Corollary 3.12 to show that $R(\varphi) < \infty$ and hence Γ does not have the R_∞ property.

An analogous argument (e.g., using the restriction of the above φ to the subgroup generated by a, b and c) shows that also the groups in family 1 do not have the R_∞ property.

The situation for family 13 is more complicated. Let Γ be an almost-crystallographic group in family 13 and $\varphi : \Gamma \rightarrow \Gamma$ an automorphism. Fix a choice of parameters k_1, k_2, k_3 and k_4 . Let a, b, c and α be as in the presentation of Γ in [7, p. 164] or [10]. By using the presentation of Γ and the fact that the subgroup generated by a, b and c is characteristic, one can see that

$$\begin{aligned}\varphi(a) &= a^{m_1} b^{m_2} c^{l_1}, & \varphi(b) &= a^{m_3} b^{m_4} c^{l_2}, \\ \varphi(c) &= c^{m_1 m_4 - m_2 m_3}, & \varphi(\alpha) &= a^{l_3} b^{l_4} c^{l_5} \alpha^\epsilon,\end{aligned}$$

where

$$\begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}), \quad l_1, l_2, l_3, l_4, l_5 \in \mathbb{Z} \text{ and } \epsilon \in \{1, 2\}.$$

It follows that $\varphi(c) = c^{\pm 1}$. For any automorphism φ with $\varphi(c) = c$, the corresponding matrix D_* from Corollary 3.12 has eigenvalue 1 and hence $R(\varphi) = \infty$.

On the other hand, using the presentation of Γ , one can compute that the case $\varphi(c) = c^{-1}$, only occurs when $m_3 = -m_1 + m_2$ and $m_4 = -m_1$ (and $\epsilon = 2$). In this case, it follows that

$$\begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}$$

always has eigenvalue 1 and so the corresponding matrix D_* also has eigenvalue 1. Hence also in this case $R(\varphi) = \infty$.

We conclude that all groups in family 13 have the R_∞ property. $\square$

5. The relation between the Nielsen number and the Reidemeister number

In this section, we show that for any pair of continuous maps between oriented infra-nilmanifolds of equal dimension, the Nielsen coincidence number equals the Reidemeister coincidence number when the latter is finite. This theorem has been suggested by Gonçalves, who independently proved it using similar techniques. The theorem can also be deduced from [18, Corollary 4.2].

Theorem 5.1. *Let M_1 and M_2 be oriented infra-nilmanifolds of equal dimension n . Let $f, f' : M_1 \rightarrow M_2$ be continuous maps. Suppose that $R(f, f') < \infty$. Then $N(f, f') = R(f, f')$.*

Proof. If $n = 1$ or $n = 2$, then either M_1 and M_2 are the circle or M_1 and M_2 are the 2-torus. In either case, the result is well known. Hence we may assume, without loss of generality, that the dimension n of M_1 and M_2 is at

least 3. Let G_1 be the connected, simply connected, nilpotent Lie group on which M_1 is modelled and G_2 the connected, simply connected, nilpotent Lie group on which M_2 is modelled. Then we have natural covering projections $p_1 : G_1 \rightarrow M_1$ and $p_2 : G_2 \rightarrow M_2$. Let $\Gamma_1 \subset \text{Aff}(G_1)$ be the almost-Bieberbach group corresponding to M_1 and $\Gamma_2 \subset \text{Aff}(G_2)$ the almost-Bieberbach group corresponding to M_2 , then $A(G_1, p_1) = \Gamma_1$ and $A(G_2, p_2) = \Gamma_2$. Choose a lift $\tilde{f} : G_1 \rightarrow G_2$ of f and a lift $\tilde{f}' : G_1 \rightarrow G_2$ of f' . Then $f : M_1 \rightarrow M_2$ induces a morphism of groups $f_\times : \Gamma_1 \rightarrow \Gamma_2$ and $f' : M_1 \rightarrow M_2$ induces a morphism of groups $f'_\times : \Gamma_1 \rightarrow \Gamma_2$. By a straightforward generalisation of [24, Lemma 3.1], there exist a uniform lattice Λ_1 of G_1 and a uniform lattice Λ_2 of G_2 such that Λ_1 is a finite index fully characteristic subgroup of Γ_1 , Λ_2 is a finite index fully characteristic subgroup of Γ_2 and such that $\theta(\Lambda_1) \subset \Lambda_2$ for every morphism of groups $\theta : \Gamma_1 \rightarrow \Gamma_2$. Then $N_1 = \Lambda_1 \backslash G_1$ and $N_2 = \Lambda_2 \backslash G_2$ are nilmanifolds and f lifts to a map $\bar{f} : N_1 \rightarrow N_2$ and f' lifts to a map $\bar{f}' : N_1 \rightarrow N_2$ such that the diagram

$$\begin{array}{ccccc}
 G_1 & \xrightarrow{\tilde{f}} & & \xrightarrow{\tilde{f}'} & G_2 \\
 & \searrow p'_1 & & \swarrow p'_2 & \\
 & & N_1 & \xrightarrow[\bar{f}']{\bar{f}} & N_2 \\
 & \swarrow \bar{p}_1 & & \searrow \bar{p}_2 & \\
 M_1 & \xrightarrow[f']{f} & & \xrightarrow{f} & M_2
 \end{array}$$

(Note: The diagram above is a simplified representation of the commutative diagram in the image. The actual diagram has more arrows and labels.)

commutes. The morphism $\bar{f}_\times : \Lambda_1 \rightarrow \Lambda_2$ induced by $\bar{f} : N_1 \rightarrow N_2$ is the restriction of $f_\times : \Gamma_1 \rightarrow \Gamma_2$ and the morphism $\bar{f}'_\times : \Lambda_1 \rightarrow \Lambda_2$ induced by $\bar{f}' : N_1 \rightarrow N_2$ is the restriction of $f'_\times : \Gamma_1 \rightarrow \Gamma_2$. Hence we have a commutative diagram of the form

$$\begin{array}{ccccccc}
 1 & \longrightarrow & \Lambda_1 & \longrightarrow & \Gamma_1 & \longrightarrow & \Gamma_1/\Lambda_1 \longrightarrow 1 \\
 & & \bar{f}_\times \downarrow & & f_\times \downarrow & & \downarrow \\
 1 & \longrightarrow & \Lambda_2 & \longrightarrow & \Gamma_2 & \longrightarrow & \Gamma_2/\Lambda_2 \longrightarrow 1,
 \end{array}$$

where Γ_1/Λ_1 and Γ_2/Λ_2 are finite groups and where the upper and lower horizontal sequences are exact. Now suppose that $N(f, f') \neq R(f, f')$. Then f, f' have an inessential coincidence class $C = p_1(\text{Coin}(\alpha f, f'))$ for some $\alpha \in A(G_2, p_2) = \Gamma_2$. Without loss of generality, by Theorem 2.4, we may replace f and f' by homotopic maps such that we may assume that $C = \emptyset$. Then $\bar{C} = p'_1(\text{Coin}(\alpha \tilde{f}, \tilde{f}'))$ is an empty (and hence inessential) coincidence class of $\bar{\alpha} \bar{f}, \bar{f}'$ for some $\bar{\alpha} \in A(N_2, \bar{p}_2)$ such that α is a lift of $\bar{\alpha}$. Hence $N(\bar{\alpha} \bar{f}, \bar{f}') < R(\bar{\alpha} \bar{f}, \bar{f}')$. By [16, Theorem 2.6], this implies that $R(\bar{\alpha} \bar{f}, \bar{f}') = \infty$. Now $\alpha \tilde{f}$ is a lift of $\bar{\alpha} \bar{f}$ and one calculates that $(\bar{\alpha} \bar{f})_\times = \tau'_\alpha \bar{f}_\times$, where $\tau'_\alpha : \Lambda_2 \rightarrow$

$\Lambda_2 : \gamma \mapsto \alpha\gamma\alpha^{-1}$. Hence $R(\tau'_\alpha \bar{f}_\times, \bar{f}'_\times) = R((\bar{\alpha} \bar{f})_\times, \bar{f}'_\times) = R(\bar{\alpha} \bar{f}, \bar{f}') = \infty$. By Proposition 3.8, $R(f_\times, f'_\times) = \infty$. Hence $R(f, f') = \infty$. $\square$

Similarly, one can prove the following result.

Theorem 5.2. *Let $f : M \rightarrow M$ be a continuous self-map on an infra-nilmanifold M . If $R(f) < \infty$, then $N(f) = R(f)$.*

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