Robust H_∞ Filter Design for Time-delay Systems with Saturation

Xiao-Fang Hong^{1,2} Yu-Zhen Wang¹ Hai-Tao Li¹

¹School of Control Science and Engineering, Shandong University, Jinan 250061, China

²Department of Electrical and Automation, Shandong Labor Vocational and Technical College, Jinan 250100, China

Abstract: This paper investigates the H_∞ filtering problem for a class of linear continuous-time systems with both time delay and saturation. Such systems have time delay in their state equations and saturation in their output equations, and their process and measurement noises have unknown statistical characteristics and bounded energies. Based on the Lyapunov-Krasovskii stability theorem and the linear matrix inequalities (LMIs) technique, a generalized dynamic filter architecture is proposed, and a filter design method is developed. The linear H_∞ filter designed by the method can guarantee the H_∞ performance. The parameters of the designed filter can be obtained by solving a kind of LMI. An illustrative example shows that the design method proposed in this paper is very effective.

Keywords: H_∞ filter, saturation, time delay, nonlinear system, linear matrix inequality.

1 Introduction

The H_∞ filtering problem is of both theoretical and practical importance for control design and signal processing^[1-4]. It is concerned with the design of estimators guaranteeing that the L_2 gain from disturbances to estimation errors is within a prescribed level^[5-7]. In contrast with the well-known Kalman filter, the H_∞ filtering makes no assumptions on the statistics of the process or the measurement noise except for the only bounded energy one. Moreover, H_∞ filters have been demonstrated more useful in industrial applications. Up to now, various approaches to the H_∞ filter design have been proposed, such as the algebraic Riccati-equation method, the interpolation method, the linear matrix inequality (LMI) method, etc.

Practical filtering systems are inherently subject to many nonlinear effects, such as adder overflows and amplitude saturations^[8, 9]. These nonlinearities may bring limit cycles or other unexpected influences on the systems, and thus undermine the stability of the systems. In [10] and [11], some asymptotical stability results were given for the systems with generalized overflow characteristics and partial state saturation, respectively. Besides these, another matter to cause instability of the filtering systems under study is time delay. It is well known that time delay is frequently encountered in various practical systems, such as chemical systems, mechanical systems, communication network systems, etc.^[12-16]. The presence of time delay may result in instability, chaotic mode, and poor performance of the systems. Thus, the control of dynamic systems with time delay has been a research subject of great practical and theoretical significance, which has received considerable attention in the past decades^[17-22]. Over the past few years,

increasing attention has been drawn to the study of stability analysis and controller design for the systems with time delay. At the same time, the H_∞ filter design problem for the systems with delay-dependent and delay-independent designs has received much attention.

However, to our best knowledge, in contrast to the existing results on the analysis and synthesis of H_∞ filters for systems either with time delay or with saturation, there are fewer results for the systems with both time delay and saturation. This is because such systems have more complicated dynamic behaviors than the ones with only time delay or only saturation. Accordingly, the analysis and synthesis of H_∞ filters for the systems with both time delay and saturation are more difficult than those with only time delay or only saturation.

In this paper, we study the robust H_∞ filter problem for a class of time-delay continuous-time systems with saturation. Using the Lyapunov-Krasovskii stability theory and the dissipative theory (mainly L_2 gain analysis), we will derive some sufficient conditions for the existence of a feasible solution to the problem, which guarantees that the filtering error system is asymptotically stable and satisfies the H_∞ robustness performance constraint for all admissible disturbances. In addition, we also establish an LMI approach to the robust H_∞ filter design.

The main contributions of this paper are as follows: 1) The H_∞ filtering problem for a class of linear continuous-time systems with both time delay and saturation is considered, and a new design method is established for this problem. 2) A new stability condition is proposed by constructing a proper Lyapunov-Krasovskii function, which has less conservatism. 3) By introducing two non-singular matrices, a set of formulae is obtained to handle the parameters of the filter, which is predigested by the characteristic of the transfer function.

The remainder of the paper is organized as follows. Section 2 is the problem formulation. In Section 3, an internal stability result is proposed, and the H_∞ filter design problem is discussed. In Section 4, we give an illustrative

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example to support our results, which is followed by the conclusions in Section 5.

Notation. The superscript T stands for matrix transposition. $\mathbf{R}^n$ is the n -dimensional Euclidean space, and $\mathbf{R}^{n \times m}$ is the set of all $n \times m$ real matrices. $L_2[0, \infty)$ is the space of square integrable vectors. $\|x\|$ denotes the Euclidean norm of vector x , I_n is the $n \times n$ identity matrix, $*$ denotes the symmetric entries of a matrix, and $\text{diag}\{\dots\}$ stands for a diagonal matrix. Matrix inequalities $X > Y$ and $X \geq Y$ mean that $X - Y$ is positive definite and positive semi-definite, respectively.

2 Problem formulation

In this section, we give the problem formulation, including the model of the system, the designed filter and the objective of this paper.

Consider the following continuous-time system with time delay and saturation

$$\begin{cases} \dot{x}(t) &= Ax(t) + A_d x(t - \tau) + B_1 w(t) \\ y(t) &= \text{sat}(Cx(t)) + B_2 w(t) \\ z(t) &= Lx(t) \\ x(\theta) &= \tilde{\varphi}(\theta) \end{cases} \quad (1)$$

where $\tilde{\varphi}(\theta)$ is a given continuous function on $[-\tau, 0]$, the time delay $\tau > 0$ is a constant number, $x(t) \in \mathbf{R}^n$, $y(t) \in \mathbf{R}^p$ and $z(t) \in \mathbf{R}^r$ are the state of the system, the measurement output and the unmeasurable signal to be estimated, respectively, $w(t) \in \mathbf{R}^m$ is the noise signal which contains process and measurement noises and is assumed to belong to $L_2[0, \infty)$, and $A \in \mathbf{R}^{n \times n}$, $A_d \in \mathbf{R}^{n \times n}$, $B_1 \in \mathbf{R}^{n \times m}$, $B_2 \in \mathbf{R}^{p \times m}$, $C \in \mathbf{R}^{p \times n}$ and $L \in \mathbf{R}^{r \times n}$ are known constant matrices. The saturation function $\text{sat}: \mathbf{R}^p \rightarrow [-1, 1]^p$ is defined as ^[10]

$$\text{sat}(u) = [\text{sat}(u_1), \text{sat}(u_2), \dots, \text{sat}(u_p)]^T \quad (2)$$

where $\text{sat}(u_i) = \text{sgn}(u_i) \min(1, |u_i|)$, $i = 1, 2, \dots, p$.

We have also slightly abused the notation by using “sat” to denote both the scalar valued and the vector valued saturation functions.

In this paper, we design a filter in the following form for system (1):

$$\begin{cases} \dot{\hat{x}}(t) &= A_f \hat{x}(t) + B_f y(t), \quad \hat{x}(0) = 0 \\ \hat{z}(t) &= C_f \hat{x}(t) \end{cases} \quad (3)$$

where A_f , B_f and C_f are constant filter matrices with appropriate dimensions, which are to be designed.

To handle the saturation term in system (1), we introduce a new matrix $D(\alpha) = \text{diag}\{\alpha_i(x)\}$, where the diagonal elements $\alpha_i(x)$, $i = 1, 2, \dots, p$, satisfy

$$\alpha_i(x) = \begin{cases} -\frac{1}{C_i x}, & C_i x < -1 \\ 1, & -1 \leq C_i x \leq 1 \\ \frac{1}{C_i x}, & C_i x > 1 \end{cases} \quad (4)$$

which implies that $0 < \alpha_i(x) \leq 1$ ($i = 1, 2, \dots, p$). Here, C_i is the i -th row of C .

With the above notation, systems (1) and (3) can be rewritten as

$$\begin{cases} \dot{x}(t) &= Ax(t) + A_d x(t - \tau) + B_1 w(t) \\ y(t) &= D(\alpha)Cx(t) + B_2 w(t) \\ z(t) &= Lx(t) \end{cases} \quad (5)$$

and

$$\begin{cases} \dot{\hat{x}}(t) &= A_f \hat{x}(t) + B_f(D(\alpha)Cx(t) + B_2 w(t)) \\ \hat{z}(t) &= C_f \hat{x}(t) \end{cases} \quad (6)$$

respectively.

Combining (5) and (6), we obtain a filtering augment system as

$$\begin{cases} \dot{e}(t) &= \tilde{A}e(t) + \tilde{A}_d H e(t - \tau) + \tilde{B}w(t) \\ \tilde{z}(t) &= \tilde{C}e(t) \end{cases} \quad (7)$$

where $e(t) = \begin{bmatrix} x^T(t) & \hat{x}^T(t) \end{bmatrix}^T$, $\tilde{z}(t) = z(t) - \hat{z}(t)$ is the estimation error, and

$$\tilde{A} = \begin{bmatrix} A & 0 \\ B_f D(\alpha)C & A_f \end{bmatrix}, \quad \tilde{A}_d = \begin{bmatrix} A_d \\ 0 \end{bmatrix},$$

$$H = \begin{bmatrix} I_n & 0 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B_1 \\ B_f B_2 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} L & -C_f \end{bmatrix}. \quad (8)$$

The objective of this paper is to develop an H_∞ filter of the form (3) such that 1) system (7) with $w(t) = 0$ is asymptotically stable, and 2) the L_2 gain of the estimation error vector $\tilde{z}(t)$ is bounded by γ , i.e., under the zero initial condition $e(t) = 0$ ($t \in [-\tau, 0]$),

$$\|\tilde{z}\|_2 \leq \gamma \|w\|_2 \quad (9)$$

is satisfied for any nonzero $w(t) \in L_2[0, \infty)$.

Moreover, if there exists a minimal positive scalar γ satisfying 1) and 2), system (3) is an optimal H_∞ filter of $z(t)$.

3 H_∞ filter design

In this section, we design the H_∞ filter (3) such that both 1) and 2) hold.

First, we present a result on the stability of the augment system (7).

Consider the internal stability of system (7) in the absence of noise $w(t)$ as

$$\begin{cases} \dot{e}(t) &= \tilde{A}e(t) + \tilde{A}_d H e(t - \tau) \\ \tilde{z}(t) &= \tilde{C}e(t). \end{cases} \quad (10)$$

We have the following result.

Theorem 1. If there exist symmetric positive definite matrices $P \in \mathbf{R}^{2n \times 2n}$ and $Q \in \mathbf{R}^{n \times n}$ such that

$$\tilde{S}_0 = \begin{bmatrix} \tilde{A}^T P + P \tilde{A} + H^T Q H & P \tilde{A}_d \\ \tilde{A}_d^T P & -Q \end{bmatrix} < 0 \quad (11)$$

then system (10) is asymptotically stable.

Proof. For system (10), construct the following Lyapunov-Krasovskii functional:

$$V(e_t) = e^T(t)Pe(t) + \int_{t-\tau}^t e^T(s)H^TQH e(s)ds \quad (12)$$

where $e_t(\theta) = e(t+\theta)$, $\forall \theta \in [-\tau, 0]$. Computing the derivative of $V(e_t)$ along the trajectory of system (10), we have

$$\begin{aligned} \dot{V}(e_t) &= 2e^T(t)P\dot{e}(t) + e^T(t)H^TQH e(t) - \\ &\quad e^T(t-\tau)H^TQH e(t-\tau) = \\ &\quad 2e^T(t)P[\tilde{A}e(t) + \tilde{A}_dHe(t-\tau)] + \\ &\quad e^T(t)H^TQH e(t) - e^T(t-\tau)H^TQH e(t-\tau) = \\ &\quad \begin{bmatrix} e(t) \\ He(t-\tau) \end{bmatrix}^T \tilde{S}_0 \begin{bmatrix} e(t) \\ He(t-\tau) \end{bmatrix} \end{aligned}$$

where

$$\tilde{S}_0 = \begin{bmatrix} \tilde{A}^TP + P\tilde{A} + H^TQH & P\tilde{A}_d \\ \tilde{A}_d^TP & -Q \end{bmatrix}.$$

Therefore, if $\tilde{S}_0 < 0$, then we can get $\dot{V}(e_t) < 0$.

According to the Lyapunov-Krasovskii stability theorem, system (10) is asymptotically stable. $\square$

In the following, we consider the internal stability of system (7) with noise $w(t)$, and propose a method to determine the parameters of filter (3) in terms of LMI.

In order to design the optimal H_∞ filter, we present the following result on the H_∞ performance.

Theorem 2. If there exist symmetric positive definite matrices $P \in \mathbf{R}^{2n \times 2n}$ and $Q \in \mathbf{R}^{n \times n}$, and a positive scalar γ such that

$$\tilde{S}_1 = \begin{bmatrix} \tilde{A}^TP + P\tilde{A} + H^TQH & P\tilde{B} & \tilde{C}^T & P\tilde{A}_d \\ * & -\gamma^2 I & 0 & 0 \\ * & 0 & -I & 0 \\ * & 0 & 0 & -Q \end{bmatrix} < 0 \quad (13)$$

then there exists an optimal H_∞ filter such that system (7) is asymptotically stable when $w(t) = 0$, and the upper bound of the L_2 gain for the estimation error vector $\tilde{z}(t)$ is finite. Moreover, the upper bound γ^* is obtained by

$$\begin{aligned} \min \gamma^2 \\ \text{s.t. } \tilde{S}_1 < 0, \quad P > 0, \quad Q > 0. \end{aligned} \quad (14)$$

Proof. Considering system (7), and using the Lyapunov-Krasovskii functional in (12), it is easy to see that (9) is equivalent to

$$J(w(t)) = \int_0^\infty [\tilde{z}^T \tilde{z} - \gamma^2 w^T(t)w(t)]dt < 0$$

and

$$\begin{aligned} J(w(t)) &= \int_0^\infty [\tilde{z}^T \tilde{z} - \gamma^2 w^T(t)w(t) + \dot{V}(e_t)]dt - \\ &\quad V(\infty) + V(0) \leq \\ &\quad \int_0^\infty [\tilde{z}^T \tilde{z} - \gamma^2 w^T(t)w(t) + \dot{V}(e_t)]dt = \\ &\quad \int_0^\infty [e^T(t)\tilde{C}^T\tilde{C}e(t) - \gamma^2 w^T(t)w(t) + \\ &\quad 2e^T(t)P\dot{e}(t) + e^T(t)H^TQH e(t) - \\ &\quad e^T(t-\tau)H^TQH e(t-\tau)]dt. \end{aligned} \quad (15)$$

On the other hand, we have

$$\begin{aligned} \dot{V}(e_t) &= \begin{bmatrix} e(t) \\ w(t) \\ He(t-\tau) \end{bmatrix}^T \times \\ &\quad \begin{bmatrix} \tilde{A}^TP + P\tilde{A} + H^TQH & P\tilde{B} & P\tilde{A}_d \\ * & 0 & 0 \\ * & 0 & -Q \end{bmatrix} \begin{bmatrix} e(t) \\ w(t) \\ He(t-\tau) \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} e^T(t)\tilde{C}^T\tilde{C}e(t) - \gamma^2 w^T(t)w(t) &= \begin{bmatrix} e(t) \\ w(t) \\ He(t-\tau) \end{bmatrix}^T \times \\ &\quad \begin{bmatrix} \tilde{C}^T\tilde{C} & 0 & 0 \\ 0 & -\gamma^2 I & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e(t) \\ w(t) \\ He(t-\tau) \end{bmatrix} \end{aligned}$$

with which it is easy to obtain that

$$\begin{aligned} J(w(t)) &\leq \int_0^\infty [\tilde{z}^T \tilde{z} - \gamma^2 w^T(t)w(t) + \dot{V}(e_t)]dt = \\ &\quad \int_0^\infty \begin{bmatrix} e(t) \\ w(t) \\ He(t-\tau) \end{bmatrix}^T S_1 \begin{bmatrix} e(t) \\ w(t) \\ He(t-\tau) \end{bmatrix} dt \end{aligned}$$

where

$$S_1 = \begin{bmatrix} \tilde{A}^TP + P\tilde{A} + H^TQH + \tilde{C}^T\tilde{C} & P\tilde{B} & P\tilde{A}_d \\ * & -\gamma^2 I & 0 \\ * & 0 & -Q \end{bmatrix}.$$

Therefore, if $S_1 < 0$, then we can get $J(w(t)) < 0$. According to the matrix Schur complement,

$$\begin{aligned} S_1 < 0 &\Leftrightarrow \\ \tilde{S}_1 &= \begin{bmatrix} \tilde{A}^TP + P\tilde{A} + H^TQH & P\tilde{B} & \tilde{C}^T & P\tilde{A}_d \\ * & -\gamma^2 I & 0 & 0 \\ * & 0 & -I & 0 \\ * & 0 & 0 & -Q \end{bmatrix} < 0. \end{aligned}$$

Then, if $\tilde{S}_1 < 0$, $J(w(t)) < 0$. Hence, the L_2 gain of the estimation error vector $\tilde{z}(t)$ does not exceed γ , i.e., the gain is finite. Moreover, it is easy to see that such γ can be minimized by the optimization problem (14). On the other

hand, $\tilde{S}_1 < 0$ implies that $\tilde{S}_0 < 0$. Thus, system (7) is asymptotically stable when $w(t) = 0$. $\square$

Based on Theorem 2, we can obtain the H_∞ performance of system (7). However, since (13) is a nonlinear matrix inequality, one can hardly obtain the parameters of the filter. Next, in order to design the filter by using the LMI technique, we need to change (13) into a linear one. To do this, we give an important lemma first.

Let $S_0 \subset \mathbf{R}^n$ be any compact set and note that (4) holds. For any $x(t) \in S_0$, one can define a lower bound of $\alpha_i(x)$ as $\alpha_{\min(i)} = \min\{\alpha_i(x), x \in S_0\}$, which implies that scalars $\alpha_i(x)$, $i = 1, \dots, p$, satisfy $\alpha_{\min(i)} \leq \alpha_i(x) \leq 1$ and $D(\alpha_{\min}) \leq D(\alpha(x)) \leq I, \forall x(t) \in S_0$.

Lemma 1.^[9] Let $D(\gamma_j) = \text{diag}\{\gamma_{j(i)}\}$, where $\gamma_{j(i)} = 1$ or $\alpha_{\min(i)}$, $i = 1, 2, \dots, p$. Then $D(\alpha)$ can be written as $\sum_{j=1}^{2^p} \lambda_{j,t} D(\gamma_j)$ with $\sum_{j=1}^{2^p} \lambda_{j,t} = 1, \lambda_{j,t} \geq 0$.

Now, we give a result, which provides a design method for the parameters of the desired H_∞ filter law (3).

Theorem 3. Assume that $D(\alpha_{\min})$ is given. Suppose that there exist a positive number γ , three matrices $M \in \mathbf{R}^{n \times n}$, $N \in \mathbf{R}^{r \times n}$ and $Z \in \mathbf{R}^{n \times p}$, and three symmetric positive definite matrices $Q \in \mathbf{R}^{n \times n}$, $R \in \mathbf{R}^{n \times n}$ and $X \in \mathbf{R}^{n \times n}$, such that

$$X - R > 0 \quad (16)$$

$$\begin{bmatrix} -I & \Psi_j^T \\ * & \Psi_j + \Psi_j^T \end{bmatrix} \leq 0, \quad j = 1, \dots, 2^p \quad (17)$$

and the following optimization problem has a solution γ^* :

$$\begin{aligned} \min \quad & \gamma^2 \\ \text{s.t.} \quad & \end{aligned}$$

$$\begin{bmatrix} RA + A^T R + Q + I & \Xi & RB_1 & L^T - N^T & RA_d \\ * & \Xi_1 & XB_1 + ZB_2 & L^T & XA_d \\ * & * & -\gamma^2 I & 0 & 0 \\ * & * & 0 & -I & 0 \\ * & * & 0 & 0 & -Q \end{bmatrix} < 0 \quad (18)$$

where $\Psi_j = ZD(\gamma_j)C - ZD(\alpha_{\min})C$, $\Xi = RA + A^T X + (ZD(\alpha_{\min})C)^T + M^T + Q$ and $\Xi_1 = XA + A^T X + ZD(\alpha_{\min})C + (ZD(\alpha_{\min})C)^T + Q$. Then, there exists an optimal H_∞ filter in the form (3) such that system (7) is asymptotically stable when $w(t) = 0$, and the upper bound of the L_2 gain given in (9) is γ^* . Furthermore, the parameters of the desired H_∞ filter (3) can be given as

$$\begin{cases} A_f = (R - X)^{-1}M \\ B_f = (R - X)^{-1}Z \\ C_f = N. \end{cases} \quad (19)$$

Proof. Based on Lemma 1, (17) is equivalent to

$$\sum_{j=1}^{2^p} \lambda_{j,t} \begin{bmatrix} -I & \Psi_j^T \\ * & \Psi_j + \Psi_j^T \end{bmatrix} \leq 0, \quad j = 1, \dots, 2^p$$

i.e.,

$$\begin{bmatrix} -I & \Phi^T \\ * & \Phi + \Phi^T \end{bmatrix} \leq 0$$

from which it follows that

$$\begin{bmatrix} -I & \Phi^T & 0 & 0 & 0 \\ * & \Phi + \Phi^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \leq 0 \quad (20)$$

where $\Phi = ZD(\alpha)C - ZD(\alpha_{\min})C$.

According to (18) and (20), we have

$$\begin{bmatrix} RA + A^T R + Q & \tilde{\Xi} & RB_1 & L^T - N^T & RA_d \\ * & \tilde{\Xi}_1 & XB_1 + ZB_2 & L^T & XA_d \\ * & * & -\gamma^2 I & 0 & 0 \\ * & * & 0 & -I & 0 \\ * & * & 0 & 0 & -Q \end{bmatrix} < 0 \quad (21)$$

where $\tilde{\Xi} = RA + A^T X + (ZD(\alpha)C)^T + M^T + Q$ and $\tilde{\Xi}_1 = XA + A^T X + ZD(\alpha)C + (ZD(\alpha)C)^T + Q$.

From (16), it is easy to obtain that XR^{-1} is nonsingular, which implies that there exist two nonsingular matrices $S \in \mathbf{R}^{n \times n}$ and $W \in \mathbf{R}^{n \times n}$ satisfying $XR^{-1} + SW^T = I$.

Letting

$$J = \begin{bmatrix} R^{-1} & I \\ W^T & 0 \end{bmatrix}, \quad \tilde{J} = \begin{bmatrix} I & X \\ 0 & S^T \end{bmatrix},$$

$$\tilde{J}J^{-1} = \begin{bmatrix} X & S \\ S^T & W^{-1}R^{-1}(X - R)R^{-1}W^{-T} \end{bmatrix} = P$$

we have

$$X - R > 0 \Leftrightarrow \begin{bmatrix} R^{-1} & I \\ I & X \end{bmatrix} = J^T P J > 0 \Leftrightarrow P > 0$$

i.e., P is a symmetric positive definite matrix.

Pre-multiplying and post-multiplying (21) by $\text{diag}\{R^{-1}, I, I, I, I\}$, we readily obtain

$$\begin{bmatrix} \Theta_1 & \Theta_2 & B_1 & R^{-1}L^T - \tilde{Z}^T & A_d \\ * & \tilde{\Xi}_1 & XB_1 + ZB_2 & L^T & XA_d \\ * & * & -\gamma^2 I & 0 & 0 \\ * & * & 0 & -I & 0 \\ * & * & 0 & 0 & -Q \end{bmatrix} < 0 \quad (22)$$

where $\Theta_1 = AR^{-1} + R^{-1}A^T + R^{-1}QR^{-1}$, $\Theta_2 = A + R^{-1}A^T X + R^{-1}C^T(ZD(\alpha))^T + \hat{Z}^T + R^{-1}Q$, and

$$\begin{cases} \hat{Z} = MR^{-1} \\ \tilde{Z} = NR^{-1}. \end{cases}$$

Let

$$\begin{cases} SB_f = Z \\ SA_f W^T = \hat{Z} \\ C_f W^T = \tilde{Z}. \end{cases} \quad (23)$$

Pre-multiplying and post-multiplying (22) by $\text{diag}\{J^{-T}, I, I, I, I\}$ and $\text{diag}\{J^{-1}, I, I, I, I\}$, respectively,

and using (8) and (23), we have

$$\begin{bmatrix} \tilde{A}^T P + P \tilde{A} + H^T Q H & P \tilde{B} & \tilde{C}^T & P \tilde{A}_d \\ * & -\gamma^2 I & 0 & 0 \\ * & 0 & -I & 0 \\ * & 0 & 0 & -Q \end{bmatrix} < 0$$

from which it follows that the conditions of Theorem 2 hold. From Theorem 2, there exists an optimal H_∞ filter in the form of (3) such that system (7) is asymptotically stable when $w(t) = 0$, and the upper bound of the L_2 gain given in (9) is the minimal γ^* under the conditions of Theorem 3.

Finally, we give the parameters of the desired filter. From (23), the filter parameters A_f , B_f and C_f can be designed as

$$\begin{cases} A_f = S^{-1} \tilde{Z} W^{-T} \\ B_f = S^{-1} Z \\ C_f = \tilde{Z} W^{-T}. \end{cases} \quad (24)$$

On the other hand, notice that the transfer function matrix of the filter is $G(s) = C_f(sI - A_f)^{-1} B_f$. Thus, by substituting (22) and (24) into $G(s)$, we obtain

$$\begin{aligned} G(s) &= \tilde{Z}(W^T)^{-1} [sI - S^{-1} \tilde{Z}(W^T)^{-1}]^{-1} S^{-1} Z = \\ &= \tilde{Z} [s S W^T - \tilde{Z}]^{-1} Z = \\ &= N R^{-1} [s(I - X R^{-1}) - M R^{-1}]^{-1} Z = \\ &= N [sI - (R - X)^{-1} M]^{-1} (R - X)^{-1} Z. \end{aligned} \quad (25)$$

Therefore, it is easy to see from (25) that the filter of system (1) in the form of (3) can be designed as (19), which is our desired H_∞ filter. $\square$

4 Illustrative example

In the section, we give an illustrative example to show how to apply the method proposed in this paper to design H_∞ filters for some time-delay systems with saturation.

Consider the following time-delay system with saturation, which was also studied in [8]:

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} -2 & -1.5 \\ 0.3 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} x(t-1) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w(t) \\ y(t) = \text{sat}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(t)\right) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w(t) \\ z(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(t) \end{cases} \quad (26)$$

where the saturation functions $\text{sat}(\cdot)$ is defined as (2) and (4), and $w(t)$ is the external disturbance.

Now, we apply Theorem 3 to design an optimal H_∞ filter for system (26).

For this example, by choosing $\alpha_{\min} = 0.1$ and using Matlab LMI toolbox to solve (17), it is easy to obtain

$$Z = \begin{bmatrix} -0.5455 & 0 \\ 0 & -0.5455 \end{bmatrix}.$$

Then using Matlab LMI toolbox to solve (16) and (18) in Theorem 3, we can obtain $\gamma^* = 0.6708$ and

$$X = \begin{bmatrix} 1.2691 & -0.6891 \\ -0.6891 & 2.3316 \end{bmatrix}$$

$$R = \begin{bmatrix} 0.7367 & -0.1727 \\ -0.1727 & 0.6297 \end{bmatrix}$$

$$M = \begin{bmatrix} 2.0503 & 0.5006 \\ -1.4770 & 2.8397 \end{bmatrix}$$

$$N = \begin{bmatrix} 1.0000 & 1.0000 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0.0513 & -0.0339 \\ -0.0339 & 0.4525 \end{bmatrix}.$$

Obviously, $X > 0$, $R > 0$ and $Q > 0$. Thus, all the conditions of Theorem 3 are satisfied. From Theorem 3, we can design an optimal H_∞ filter for system (26) as

$$\begin{cases} \dot{\hat{x}}(t) = \begin{bmatrix} -4.2635 & -3.6250 \\ -0.4257 & -2.7685 \end{bmatrix} \hat{x}(t) + \begin{bmatrix} 1.4516 & 0.4404 \\ 0.4404 & 0.4542 \end{bmatrix} y(t) \\ \hat{z}(t) = \begin{bmatrix} 1.0000 & 1.0000 \end{bmatrix} \hat{x}(t). \end{cases}$$

To show the correctness of the above conclusion, we carry out some simulation results with the following choices:

1) Let initial conditions be $x(0) = [0.4, -0.3]^T$ and $\hat{x}(0) = [0, 0]^T$. To test the robustness of the filter, we add a disturbance of amplitude 5 to the original system in the time duration 7s–9s. The simulation results of $z(t)$ and its estimator $\hat{z}(t)$ are shown in Fig. 1.

2) Let initial conditions be $x(0) = [0, 0]^T$ and $\hat{x}(0) = [0, 0]^T$. We add a disturbance of $w(t) = 5 \exp(-0.05t) \sin(0.02t)$. The simulation results of $z(t)$ and its estimator $\hat{z}(t)$ are shown in Fig. 2.

Fig. 1 Estimation signals

It can be observed from Fig. 1 that the estimation signal $\hat{z}(t)$ is close to $z(t)$ quickly when taking out the external disturbance, and the estimation error $\tilde{z}(t) = z(t) - \hat{z}(t)$ is

asymptotically convergent to 0 as $w(t) \rightarrow 0$. From Fig. 2, one can see that the performance of H_∞ disturbance is attenuated. The simulation results show that the filter of (26) is very effective and has strong robustness against external disturbances.

To show the less conservatism of the paper, we also plotted the simulation results in [8], with the same simulation conditions 1) and 2) in Figs. 1 and 2. Compared with [8], our results have the following advantages: 1) The upper bound of the L_2 gain for our filter is $\gamma^* = 0.6708$, which is better than the optimal H_∞ filter in [8] with $\gamma^* = 1.1868$, 2) From Figs. 1 and 2, one can see that the output of our filter is much closer to $z(t)$ than that in [8].

Fig. 2 Estimation signals

5 Conclusions

In this paper, we have investigated the H_∞ filtering problem for a class of linear continuous-time systems with both time delay and saturation. Based on the Lyapunov-Krasovskii stability theorem and the linear matrix inequalities technique, we have proposed a generalized dynamic filter architecture and developed a new filter design method. By this method, the designed optimal H_∞ filter can ensure that the gain γ from external disturbances to penalty signals is minimized, and the parameters of the designed filter can be obtained by solving a kind of LMIs. The study of an illustrative example with simulations has shown that the designed H_∞ filter is very effective and has robustness against external disturbances. It is noted that the results obtained in this paper are delay-independent. In future works, to further reduce the design conservatism, one can develop new methods to obtain delay-dependent results.

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Xiao-Fang Hong graduated from Shandong University, China in 2004. She received the M.Sc. degree from Shandong University in 2007. She is currently a Ph. D. candidate at School of Control Science and Engineering, Shandong University, China.

Her research interests include the robust control and filtering for the systems with both time delay and saturation.

E-mail: qingfeng3582@sina.com (Corre-

sponding author)

Yu-Zhen Wang graduated from Taian Teachers College in 1986, received his M.Sc. degree from Shandong University of Science & Technology in 1995, and his Ph.D. degree from the Institute of Systems Science, Chinese Academy of Sciences in 2001. From 2001 to 2003, he worked as a postdoctoral fellow in Tsinghua University, China. Since 2003, he has been a professor with the School of Control Science and

Engineering, Shandong University, China, and the director of the Institute of Automatic Control, Shandong University. From March 2004 to June 2004, February 2006 to May 2006, and from November 2008 to January 2009, he visited City University of Hong Kong as a research fellow. From September 2004 to May 2005, he worked as a visiting research fellow at the National University of Singapore. He received the Prize of Guan Zhaozhi in 2002, the Prize of Huawei from the Chinese Academy of Sciences in 2001, the Prize of Natural Science from the Chinese Education Ministry in 2005, and the National Prize of Natural Science of China in 2008. Currently, he is a Taishan Scholar of Shandong Province, China, an associate editor of the *Asian Journal of Control*, and the *IMA Journal of Mathematical Control and Information*.

His research interests include nonlinear control systems, hamiltonian systems, boolean networks, and multi-agent systems.

E-mail: yzwang@sdu.edu.cn

Hai-Tao Li graduated from Shandong Normal University, China in 2007. He received the M.Sc. degree from Shandong Normal University, China in 2010. He is currently a Ph.D. candidate at School of Control Science and Engineering, Shandong University, China. He received the Prize of Guan Zhaozhi in 2012, and the Best Student Paper Award in the 10th World Congress on Intelligent Control and Au-

tomation.

His research interests include logical dynamic systems and switched systems.

E-mail: haitaoli09@gmail.com