A Continuous State Feedback Controller Design for High-order Nonlinear Systems with Polynomial Growth Nonlinearities

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Abstract: In this paper, we investigate the problem of global stabilization for a general class of high-order and non-smoothly stabilizable nonlinear systems with both lower-order and higher-order growth conditions. The designed continuous state feedback controller is recursively constructed to guarantee the global strong stabilization of the closed-loop system.

Keywords: Nonlinear systems, lower-order and high-order nonlinearities, state feedback, adding a power integrator, global strong stabilization.

1 Introduction

In this paper, we consider a class of high-order nonlinear systems

$$\begin{aligned}\dot{x}_i(t) &= x_{i+1}^{p_i}(t) + f_i(x(t), t), \quad i = 1, \dots, n-1 \\ \dot{x}_n(t) &= u^{p_n}(t) + f_n(x(t), t)\end{aligned}\quad (1)$$

where $x(t) = (x_1(t), \dots, x_n(t))^T \in \mathbf{R}^n$ and $u(t) \in \mathbf{R}$ are the system state and control input, respectively. $p_i \in \mathbf{R}_{\text{odd}}^{\geq 1} = \{\tilde{q} \in \mathbf{R}_+ : \tilde{q} \geq 1 \text{ is a ratio of odd integers}\}$ is said to be the high-order of system (1). When $p_i \geq 1$, system (1) is called high-order nonlinear system if at least one p_i in system (1) satisfies $p_i > 1$. The functions $f_i(x, t) (i = 1, \dots, n) : \mathbf{R}^n \times \mathbf{R}_+ \rightarrow \mathbf{R}$ are continuous in (x, t) and $f_i(0, t) = 0$.

In recent years, with the aid of the adding a power integrator scheme and backstepping method, the stabilization of the so-called high-order nonlinear system (1) has been an active area of research, and tremendous progress has been made toward the development of systematic design methodologies^[1-3]. Following the state-feedback stabilization results^[4,5], for a family of systems (1) with uncontrollable/unobservable linearization, Yang et al.^[6,7] studied the robust output feedback stabilization with f_i satisfying the higher-order growth condition and $p_1 = \dots = p_{n-1} = p$ being odd integers. Recently, by adopting a subtle homogeneous observer/controller construction and homogeneous domination design, the higher-order growth condition was replaced by higher-order homogeneous growth condition^[8-10]. For some special nonlinear systems with $p_i = 1$ ^[11], both linear and higher-order growth terms on f_i were taken into account simultaneously.

It is well-known that the classical Lyapunov stability theory and the concepts, such as stability and asymptotic sta-

bility in the sense of Lyapunov, can only be applied to a nonlinear differential equation which has a unique solution. In [12], Kurzweil introduced the concepts related to the notion of global strong stability and Lyapunov stability theory in a continuous framework, without requiring uniqueness of the trajectories of a nonlinear differential equation. With the aid of Lyapunov stability theory for continuous system, Qian and Lin^[13] presented a new feedback design tool to solve global strong stabilization problem for system (1) with more general higher-order growth condition which is impossible to stabilize using any smooth feedback. Subsequently, they considered the stabilization problem for system (1) under the lower-order and linear growth condition, and only global strong stabilization via nonsmooth feedback was shown to be possible^[14]. For more general nonlinear systems with both lower-order and higher-order growth conditions, Poindot et al.^[15-17] further investigated the global strong stabilization. Naturally, one may ask the following important and interesting questions:

Whether the growth condition can be further relaxed? Under the weaker condition, can a continuous state feedback controller be designed?

In this paper, we will address the above problems and provide a satisfactory answer. For system (1), by relaxing the polynomial growth condition, combining the adding a power integrator scheme with backstepping method, skillfully constructing C^1 Lyapunov functions, and employing a more flexible controller, we design a continuous state feedback controller to guarantee the global strong stabilization of the closed-loop system.

The paper is organized as follows. Section 2 provides the mathematical preliminaries. The design and analysis of the state feedback controller are given in Section 3, followed by a simulation example in Section 4. Finally, Section 5 concludes this paper.

2 Mathematical preliminaries

The following notations will be used throughout the paper. $\mathbf{R}_+$ denotes the set of all nonnegative real numbers,

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$\mathbf{R}^n$ denotes the real n -dimensional space, $\mathbf{R}_{\text{odd}}^{\geq 1} = \{\bar{q} \in \mathbf{R}_+ : \bar{q} \geq 1 \text{ is a ratio of odd integers}\}$, and $\mathbf{R}_{\text{odd}} = \{\bar{q} \in \mathbf{R}_+ : \bar{q} \text{ is a ratio of odd integers}\}$. For a given vector or matrix X , X^T denotes its transpose, and $|X|$ is the Euclidean norm of vector X . $\mathcal{C}^i$ denotes the set of all functions with continuous i -th partial derivatives, $\mathcal{C}^{1,1}$ denotes the family of all functions which are $\mathcal{C}^1$ in the first argument for each fixed second argument, and $\mathcal{C}^1$ in the second argument for each fixed first argument. And $\mathcal{C}^0$ denotes the set of all continuous functions. To simplify the procedure, we sometimes denote $x(t)$ by x for any variable $x(t)$.

Then, we give some definition and lemmas which will be frequently used in this paper for the design and analysis of controller.

Consider the following nonlinear systems

$$\dot{x} = f(x, t), \quad x \in \mathbf{R}^n, \quad t \in \mathbf{R} \quad (2)$$

with $f : \mathbf{R}^n \times \mathbf{R} \rightarrow \mathbf{R}^n$ being continuous in (x, t) and $f(0, t) = 0, \forall t \in \mathbf{R}$. And we do not need uniqueness of the trajectories of system (2) from any initial condition. Namely, system (2) may have infinite many solutions, which is why the notion of global strong stability is introduced in [12].

Definition 1^[12]. The trivial solution $x = 0$ of system (2) is said to be globally strongly stable if there are two functions $B : (0, +\infty) \rightarrow (0, +\infty)$ and $T : (0, +\infty) \times (0, +\infty) \rightarrow (0, +\infty)$ with B being increasing and $\lim_{s \rightarrow 0} B(s) = 0$, such that $\forall \alpha > 0$ and $\forall \varepsilon > 0$, for every solution $x(t)$ of system (2) defined on $[0, t_1]$, $0 < t_1 \leq +\infty$ with $|x(0)| \leq \alpha$, there is a solution $z(t)$ of system (2) defined on $[0, +\infty)$ satisfying: 1) $z(t) = x(t), t \in [0, t_1]$, 2) $|z(t)| \leq B(\alpha), \forall t \geq 0$, and 3) $|z(t)| \leq \varepsilon, \forall t \geq T(\alpha, \varepsilon)$.

Lemma 1^[12]. We suppose there exist a $\mathcal{C}^{1,1}$ function $V(x, t) : \mathbf{R}^n \times \mathbf{R} \rightarrow \mathbf{R}$, a $\mathcal{C}^0$ function $U_1 : \mathbf{R}^n \rightarrow \mathbf{R}$, which is positive definite and proper, and $\mathcal{C}^0$ functions $U_i : \mathbf{R}^n \rightarrow \mathbf{R}, i = 2, 3$, which are positive definite, such that $U_1(x) \leq V(x, t) \leq U_2(x), \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x, t) \leq -U_3(x)$. Then, the trivial solution $x = 0$ of system (2) is globally strongly stable.

Lemma 2^[16]. Let $0 \leq \delta_1 \leq \dots \leq \delta_n$ be the real numbers and $c_j > 0, j = 1, \dots, n$. Then, $\forall x \in \mathbf{R}, c_1|x|^{\delta_1} + c_n|x|^{\delta_n} \leq \sum_{j=1}^n c_j|x|^{\delta_j} \leq (\sum_{j=1}^n c_j)(|x|^{\delta_1} + |x|^{\delta_n})$.

Lemma 3^[16]. For any $x, y \in \mathbf{R}$, if $p \geq 1$ is a constant, then $|x+y|^p \leq 2^{p-1}|x^p+y^p|$ and $(|x|+|y|)^{\frac{1}{p}} \leq |x|^{\frac{1}{p}} + |y|^{\frac{1}{p}} \leq 2^{\frac{p-1}{p}}(|x|+|y|)^{\frac{1}{p}}$. Moreover, if $p \in \mathbf{R}_{\text{odd}}^{\geq 1}$, then $|x-y|^p \leq 2^{p-1}|x^p-y^p|$ and $|x|^{\frac{1}{p}} - |y|^{\frac{1}{p}} \leq 2^{1-\frac{1}{p}}|x-y|^{\frac{1}{p}}$.

Lemma 4^[16]. If c and d are positive constants, then for any positive number $\bar{\gamma}$, $|x|^c|y|^d \leq \frac{c}{c+d}\bar{\gamma}|x|^{c+d} + \frac{d}{c+d}\bar{\gamma}^{-\frac{c}{d}}|y|^{c+d}$.

Lemma 5 (Young's inequality). For any positive constant ϵ , $|x||y| \leq \frac{\epsilon^p}{p}|x|^p + \frac{1}{q\epsilon^q}|y|^q$, where $p > 1, q > 1$, and $(p-1)(q-1) = 1$.

3 Controller design and analysis

3.1 Problem formulation

The research purpose of this paper is to design a continuous state feedback controller for system (1) such that the closed-loop system is globally strongly stable. To achieve

the control purpose, we need the following assumption.

Assumption 1. For $f_i(x, t) (i = 1, \dots, n)$ in system (1), there exist positive constants c, r_i and τ_i such that

$$|f_i(x, t)| \leq c \left(|x_1|^{\frac{1}{p_1 \cdots p_{i-1}}} + \dots + |x_{i-1}|^{\frac{1}{p_{i-1}}} + |x_i| + |x_1|^{\frac{r_i+\tau_i}{r_1}} + \dots + |x_i|^{\frac{r_i+\tau_i}{r_i}} \right) \quad (3)$$

where r_i and τ_i are defined as

$$\begin{aligned} r_1 &= 1, \tau_1 \geq 1, \quad r_{i+1}p_i = r_i + \tau_i, \\ \tau_k &\geq \max \left\{ \left(\sum_{j=1}^{k-1} \frac{\tau_j}{r_j} \right) p_1 \cdots p_{k-1} r_k, \tau_{k-1} \right\}, \\ k &= 2, \dots, n. \end{aligned} \quad (4)$$

Without loss of generality, we assume $\tau_i = \frac{n_i}{m_i} (i = 1, \dots, n)$ with n_i being an even integer and m_i being an odd integer. Under this assumption, we know that $r_1, \dots, r_{n+1} \in \mathbf{R}_{\text{odd}}$.

Remark 1. Let us discuss Assumption 1. Compared with [17], Assumption 1 is a more general and weaker condition. The assumption on f_i in [17] is

$$|f_i(x, t)| \leq c_1 \left(|x_1|^{\frac{1}{p_1 \cdots p_{i-1}}} + \dots + |x_{i-1}|^{\frac{1}{p_{i-1}}} + |x_i| + |x_1|^{\frac{r_i+\tau}{r_1}} + \dots + |x_i|^{\frac{r_i+\tau}{r_i}} \right), \quad i = 1, \dots, n \quad (5)$$

where τ is a positive constant. By using Lemma 2 and choosing appropriate positive constants c and τ_i , it can be obtained that

$$\begin{aligned} c_1 \left(|x_1|^{\frac{1}{p_1 \cdots p_{i-1}}} + \dots + |x_{i-1}|^{\frac{1}{p_{i-1}}} + |x_i| + |x_1|^{\frac{r_i+\tau}{r_1}} + \dots + |x_i|^{\frac{r_i+\tau}{r_i}} \right) &\leq \\ c \left(|x_1|^{\frac{1}{p_1 \cdots p_{i-1}}} + \dots + |x_{i-1}|^{\frac{1}{p_{i-1}}} + |x_i| + |x_1|^{\frac{r_i+\tau_i}{r_1}} + \dots + |x_i|^{\frac{r_i+\tau_i}{r_i}} \right). \end{aligned} \quad (6)$$

Combining (6) with (5) leads to (3) in this paper. In other word, Assumption 1 includes the growth condition in [17].

On the other hand, when $\{\tau_i\}$ of Assumption 1 is any non-negative real number, we can still design the desired state feedback controller for system (1) without any modification, whose simple explanation is shown below. Since there exists $\bar{\tau}_i > 0$ satisfying $\bar{\tau}_i = \frac{n_i}{m_i} \geq \tau_i$ with n_i being an even integer and m_i being an odd integer for $i = 1, \dots, n$, it is not difficult to deduce from Lemma 2 that

$$\begin{aligned} |f_i(x, t)| &\leq c_2 \left(|x_1|^{\frac{1}{p_1 \cdots p_{i-1}}} + \dots + |x_{i-1}|^{\frac{1}{p_{i-1}}} + |x_i| + |x_1|^{\frac{r_i+\bar{\tau}_i}{r_1}} + \dots + |x_i|^{\frac{r_i+\bar{\tau}_i}{r_i}} \right) \leq \\ c_3 \left(|x_1|^{\frac{1}{p_1 \cdots p_{i-1}}} + \dots + |x_{i-1}|^{\frac{1}{p_{i-1}}} + |x_i| + |x_1|^{\frac{r_i+\tau_i}{r_1}} + \dots + |x_i|^{\frac{r_i+\tau_i}{r_i}} \right) \end{aligned} \quad (7)$$

where c_2 and c_3 are positive constants.

Thus, under the general condition on f_i , one can give the flexible controller design and analysis for system (1).

3.2 Design of state feedback controller

Next, we construct a continuous state feedback controller for system (1) by combining the adding a power integrator with the backstepping technique.

Step 1. Let $\xi_1 = x_1$ and $\mu = p_1 \cdots p_n r_{n+1}$. Since the nonlinearities of system (1) consist of lower-order and higher-order parts, it is natural to construct a Lyapunov function with two parts: One is to deal with the lower-order nonlinearities and the other is to deal with the higher-order nonlinearities. Now, construct the positive definite and proper Lyapunov function

$$V_1 = V_{L1} + V_{H1} = \frac{1}{2}\xi_1^2 + \frac{r_1}{2\mu - \tau_1}\xi_1^{\frac{2\mu - \tau_1}{r_1}} \quad (8)$$

whose time derivative along the trajectories of system (1) is

$$\dot{V}_1 = (\xi_1 + \xi_1^{2\mu - \tau_1 - 1})(x_2^{p_1} + f_1(\cdot)). \quad (9)$$

By (3), Lemma 2, and the definitions of μ , τ_1 and r_1 in (4), we have

$$\begin{aligned} (\xi_1 + \xi_1^{2\mu - \tau_1 - 1})f_1(\cdot) &\leq \\ c(|\xi_1| + |\xi_1|^{2\mu - \tau_1 - 1}) \times (|\xi_1| + |\xi_1|^{1 + \tau_1}) &\leq \\ a_{1,1}(\xi_1^2 + \xi_1^{2\mu}) \end{aligned} \quad (10)$$

where $a_{1,1}$ is a positive constant. Using (10), (9) turns into

$$\dot{V}_1 \leq (\xi_1 + \xi_1^{2\mu - \tau_1 - 1})x_2^{p_1} + a_{1,1}(\xi_1^2 + \xi_1^{2\mu}). \quad (11)$$

Thus, it is not difficult to see that the first virtual control

$$\begin{aligned} x_2^* &= -\alpha_1 \left(\xi_1 + \xi_1^{\frac{r_2 p_1}{r_1}} \right)^{\frac{1}{p_1}} \\ \alpha_1 &= -(c_{1,1} + a_{1,1})^{\frac{1}{p_1}} \end{aligned} \quad (12)$$

renders $\dot{V}_1 \leq -c_{1,1}(\xi_1^2 + \xi_1^{\frac{2\mu}{r_1}}) + (\xi_1 + \xi_1^{2\mu - \tau_1 - 1})(x_2^{p_1} - x_2^{*p_1})$, where $c_{1,1}$ is a positive real number.

Step i ($i = 2, \dots, n$). At this step, we can obtain a property similar to Step 1, which is presented by the following lemma.

Lemma 6. Suppose at Step $i - 1$, there are a $\mathcal{C}^1$ Lyapunov function $V_{i-1}: \mathbf{R}^{i-1} \rightarrow \mathbf{R}$, which is positive definite and proper, and a set of virtual controllers $x_1^*, x_2^*, \dots, x_i^*$ defined by

$$\begin{aligned} x_1^* &= 0 \\ \xi_1 &= x_1 - x_1^* \\ x_2^* &= -\alpha_1 \left(\xi_1 + \xi_1^{\frac{r_2 p_1}{r_1}} \right)^{\frac{1}{p_1}} \\ \xi_2 &= x_2^{p_1} - x_2^{*p_1} \\ &\vdots \\ x_i^* &= -\alpha_{i-1} \left(\xi_{i-1} + \xi_{i-1}^{\frac{r_i p_{i-1}}{r_{i-1}}} \right)^{\frac{1}{p_{i-1}}} \\ \xi_i &= x_i^{p_1 \cdots p_{i-1}} - x_i^{*p_1 \cdots p_{i-1}} \end{aligned} \quad (13)$$

with appropriate constants $\alpha_j > 0$ ($j = 1, \dots, i - 1$), such

that

$$\begin{aligned} \dot{V}_{i-1} &\leq -\sum_{j=1}^{i-1} c_{i-1,j} \left(\xi_j^2 + \xi_j^{\frac{2\mu}{p_1 \cdots p_{j-1} r_j}} \right) + \\ &\left(\xi_{i-1}^{2 - \frac{1}{p_1 \cdots p_{i-2}}} + \xi_{i-1}^{\frac{2\mu - r_i p_{i-1}}{p_1 \cdots p_{i-2} r_{i-1}}} \right) \times (x_i^{p_i} - x_i^{*p_i}) \end{aligned} \quad (14)$$

where $c_{i-1,1}, \dots, c_{i-1,i-1}$ are positive constants. Then, there is a virtual control law $x_{i+1}^* = -\alpha_i(\xi_i + \xi_i^{\frac{r_{i+1} p_i}{r_i}})^{\frac{1}{p_1 \cdots p_i}}$, such that

$$\begin{aligned} \dot{V}_i &\leq -\sum_{j=1}^i c_{i,j} \left(\xi_j^2 + \xi_j^{\frac{2\mu}{p_1 \cdots p_{j-1} r_j}} \right) + \\ &\left(\xi_i^{2 - \frac{1}{p_1 \cdots p_{i-1}}} + \xi_i^{\frac{2\mu - r_{i+1} p_i}{p_1 \cdots p_{i-1} r_i}} \right) \times (x_{i+1}^{p_{i+1}} - x_{i+1}^{*p_{i+1}}) \end{aligned} \quad (15)$$

where the i -th Lyapunov function V_i defined by

$$\begin{aligned} V_i &= V_{i-1} + V_{Li} + V_{Hi} \\ V_{Li} &= \int_{x_i^*}^{x_i} (s^{p_1 \cdots p_{i-1}} - x_i^{*p_1 \cdots p_{i-1}})^{2 - \frac{1}{p_1 \cdots p_{i-1}}} ds \\ V_{Hi} &= \int_{x_i^*}^{x_i} (s^{p_1 \cdots p_{i-1}} - x_i^{*p_1 \cdots p_{i-1}})^{\frac{2\mu - r_{i+1} p_i}{p_1 \cdots p_{i-1} r_i}} ds \end{aligned} \quad (16)$$

is $\mathcal{C}^1$, positive definite and proper, and $c_{i,1}, \dots, c_{i,i}$ are positive constants.

The proof of Lemma 6 can be seen in Appendix.

Hence, at Step n , choosing the state feedback controller of the form

$$u^{p_n} = x_{n+1}^{*p_n} = -\alpha_n \left(\xi_n^{2 - \frac{1}{p_1 \cdots p_{n-1}}} + \xi_n^{\frac{r_{n+1} p_n}{p_1 \cdots p_{n-1} r_n}} \right) \quad (17)$$

yields

$$\dot{V}_n \leq -\sum_{j=1}^n c_{n,j} \left(\xi_j^2 + \xi_j^{\frac{2\mu}{p_1 \cdots p_{j-1} r_j}} \right) \quad (18)$$

where $c_{n,1}, \dots, c_{n,n}$ are positive constants.

3.3 Stability analysis

We are now in the position to state the main theorem in this paper.

Theorem 1. If Assumption 1 holds for the high-order nonlinear system (1), under the state feedback controller (13) and (17), then the closed-loop systems (1), (13) and (17) are globally strongly stable.

Proof. Defining $U_1 = \sum_{i=1}^n m_i((x_i - x_i^*)^{2p_1 \cdots p_{i-1}} + (x_i - x_i^*)^{\frac{2\mu - r_i + 1 p_i}{r_i} + 1})$, $U_2 = \sum_{i=1}^n \bar{m}_i(\xi_i^2 + \xi_i^{\frac{2\mu - \tau_i}{p_1 \cdots p_{i-1} r_i}})$, and $U_3 = \sum_{j=1}^i c_{i,j}(\xi_j^2 + \xi_j^{\frac{2\mu}{p_1 \cdots p_{j-1} r_j}})$, together with Lemma 1, (18) and (24), imply that Theorem 1 holds. $\square$

Remark 2. We emphasize three points.

1) By introducing a more flexible definition of $\{\tau_i\}$ in Assumption 1 instead of τ in [17], we greatly enlarge the class of systems that we can deal with. Thus, the existing results in [17] can be viewed as a special case of this result. Furthermore, we can present a flexible controller design on

state feedback for a more general class of systems compared with [17].

2) What should be stressed is that how to skillfully design the rigorous state feedback controller to dominate the nonlinearities and inequations is not a trivial work.

3) It is difficult to choose an appropriate Lyapunov function in the design and analysis of controller by combining the adding a power integrator scheme and backstepping approach. The high-order of system (1), and lower-order and higher-order nonlinearities in the system, and the repeated use of some inequalities will inevitably lead to the difficulty in choosing the Lyapunov function and controller.

4 Simulation example

Consider the nonlinear system

$$\begin{aligned}\dot{x}_1 &= x_2 + x_1 \\ \dot{x}_2 &= u^3 + x_1 + |x_1|^{\frac{1}{2}} |x_2|^{\frac{3}{2}}.\end{aligned}\quad (19)$$

One can choose $\tau_1 = \frac{2}{3}$, $r_2 = r_1 + \tau_1 = \frac{5}{3}$, $\tau_2 = \frac{10}{3} \geq \max\{(\frac{\tau_1}{r_1})p_1r_2, \tau_1\} = \frac{16}{9}$, and $r_3 = 5$. Then, one gets $|f_1| \leq |x_1|$ and $|f_2| \leq \frac{3}{2}(|x_1| + |x_2|^3)$. It is obvious that Assumption 1 holds.

Choose $\mu = p_1p_2r_3 = 5$. According to the design procedure in Section 3, one constructs the controller as

$$\begin{aligned}\xi_1 &= x_1 \\ x_2^* &= -3 \left(\xi_1 + \xi_1^{\frac{5}{3}} \right) \\ \xi_2 &= x_2 - x_2^* \\ u &= -92^{\frac{1}{3}} \left(\xi_2 + \xi_2^3 \right)^{\frac{1}{3}}.\end{aligned}\quad (20)$$

In the simulation, we choose the initial values $x_1(0) = 1$ and $x_2(0) = -1$. Fig. 1 demonstrates the effectiveness of the controller.

Fig. 1 The responses of the closed-loop systems (19) and (20)

5 Conclusions

In this paper, by combining the adding a power integrator method and backstepping approach, the continuous state feedback control problem is solved successfully for a class of high-order nonlinear systems under the polynomial growth condition.

Recently, for stochastic nonlinear systems and time-delay nonlinear systems, references [18–48] discussed the different control problems. However, all these references only

considered the systems whose bounding condition is the allowed linear growth condition or higher-order growth condition. Therefore, an important problem is how to establish stochastic strong stability theory and give the design and analysis of the controller for stochastic nonlinear systems with both lower-order and higher-order nonlinearities.

Appendix

Proof of Lemma 6. By the definitions of μ , p_i , r_i in (4) and (16), it is easy to prove that V_{Li} and V_{Hi} are $\mathcal{C}^1$. We can deduce from (13) and $|x^{\frac{1}{p}} - y^{\frac{1}{p}}| \leq 2^{1-\frac{1}{p}} |x - y|^{\frac{1}{p}}$ in Lemma 3 that

$$|x_i - x_i^*| \leq 2^{1-\frac{1}{p_1 \cdots p_{i-1}}} |\xi_i|^{\frac{1}{p_1 \cdots p_{i-1}}}. \quad (21)$$

Then, we divide the proof into two cases to prove that V_{Li} and V_{Hi} are positive definite and proper.

Case 1. When $x_i^* \leq x_i$, using (16), (21), $p_i \in \mathbf{R}_{\text{odd}}^{\geq 1}$ and $|x - y|^p \leq 2^{p-1} |x^p - y^p|$ in Lemma 3, we get

$$\begin{aligned}V_{Li} + V_{Hi} &= \int_{x_i^*}^{x_i} (s^{p_1 \cdots p_{i-1}} - x_i^{*p_1 \cdots p_{i-1}})^{2-\frac{1}{p_1 \cdots p_{i-1}}} ds + \\ &\int_{x_i^*}^{x_i} (s^{p_1 \cdots p_{i-1}} - x_i^{*p_1 \cdots p_{i-1}})^{\frac{2\mu-r_i+1}{p_1 \cdots p_{i-1} r_i}} ds \geq \\ &2^{1-p_1 \cdots p_{i-1}} \left(\int_{x_i^*}^{x_i} (s - x_i^*)^{2p_1 \cdots p_{i-1}-1} ds + \right. \\ &\left. \int_{x_i^*}^{x_i} (s - x_i^*)^{\frac{2\mu-r_i+1}{r_i}} ds \right) \geq \\ &m_i \left((x_i - x_i^*)^{2p_1 \cdots p_{i-1}} + (x_i - x_i^*)^{\frac{2\mu-r_i+1}{r_i}+1} \right) \quad (22)\end{aligned}$$

where m_i is a positive constant. By (16) and $|x^{\frac{1}{p}} - y^{\frac{1}{p}}| \leq 2^{1-\frac{1}{p}} |x - y|^{\frac{1}{p}}$ in Lemma 3, we have

$$\begin{aligned}V_{Li} + V_{Hi} &\leq (x_i^{p_1 \cdots p_{i-1}} - x_i^{*p_1 \cdots p_{i-1}})^{2-\frac{1}{p_1 \cdots p_{i-1}}} \times \\ &\left(x_i^{\frac{p_1 \cdots p_{i-1}}{p_1 \cdots p_{i-1}}} - x_i^{*\frac{p_1 \cdots p_{i-1}}{p_1 \cdots p_{i-1}}} \right) + \\ &(x_i^{p_1 \cdots p_{i-1}} - x_i^{*p_1 \cdots p_{i-1}})^{\frac{2\mu-r_i+1}{p_1 \cdots p_{i-1} r_i}} \times \\ &\left(x_i^{\frac{p_1 \cdots p_{i-1}}{p_1 \cdots p_{i-1}}} - x_i^{*\frac{p_1 \cdots p_{i-1}}{p_1 \cdots p_{i-1}}} \right) \leq \\ &\bar{m}_i \left(\xi_i^2 + \xi_i^{\frac{2\mu-r_i}{p_1 \cdots p_{i-1} r_i}} \right) \quad (23)\end{aligned}$$

where $\bar{m}_i$ is a positive constant.

Case 2. When $x_i^* \geq x_i$, (22) and (23) can be proved in a fashion similar to Case 1.

From Cases 1 and 2, it follows that

$$\begin{aligned}\sum_{j=1}^i m_j \left((x_j - x_j^*)^{2p_1 \cdots p_{j-1}} + (x_j - x_j^*)^{\frac{2\mu-r_j+1}{r_j}+1} \right) &\leq \\ V_i &\leq \sum_{j=1}^i \bar{m}_j \left(\xi_j^2 + \xi_j^{\frac{2\mu-r_j}{p_1 \cdots p_{j-1} r_j}} \right).\end{aligned}\quad (24)$$

(24) combined with (23) and the definitions of μ , r_i and p_i in (4), implies that V_i is positive definite and proper.

Now, we prove that (15) holds. By (1), (14) and (16), we have

$$\begin{aligned} \dot{V}_i &= \dot{V}_{i-1} + \sum_{j=1}^i \frac{\partial(V_{Li} + V_{Hi})}{\partial x_j} (x_{j+1}^{p_j} + f_j(\cdot)) \leq \\ &- \sum_{j=1}^{i-1} c_{i-1,j} \left(\xi_j^2 + \xi_j^{\frac{2\mu}{p_1 \cdots p_{j-1} r_j}} \right) + \frac{\partial(V_{Li} + V_{Hi})}{\partial x_i} x_{i+1}^{p_i} + \\ &\left(\xi_{i-1}^{2 - \frac{1}{p_1 \cdots p_{i-2}}} + \xi_{i-1}^{\frac{2\mu - r_i p_{i-1}}{p_1 \cdots p_{i-2} r_{i-1}}} \right) (x_i^{p_{i-1}} - x_i^{*p_{i-1}}) + \\ &\frac{\partial(V_{Li} + V_{Hi})}{\partial x_i} f_i(\cdot) + \sum_{j=1}^{i-1} \frac{\partial(V_{Li} + V_{Hi})}{\partial x_j} (x_{j+1}^{p_j} + f_j(\cdot)). \end{aligned} \quad (25)$$

Then, we concentrate on the last three terms on the right-hand side of (25).

Before estimating the other terms, we present three useful inequations as follows. By (13), $|x + y|^p \leq 2^{p-1}|x^p + y^p|$ and $(|x| + |y|)^{\frac{1}{p}} \leq |x|^{\frac{1}{p}} + |y|^{\frac{1}{p}}$ in Lemma 3, we have

$$\begin{aligned} |x_j| &\leq \tilde{b}_{j,1} \left(|\xi_j|^{\frac{1}{p_1 \cdots p_{j-1}}} + |\xi_{j-1}|^{\frac{1}{p_1 \cdots p_{j-1}}} + \right. \\ &\quad \left. |\xi_{j-1}|^{\frac{r_j}{p_1 \cdots p_{j-2} r_{j-1}}} \right) \\ |x_1| &\leq |\xi_1| \end{aligned} \quad (26)$$

where $\tilde{b}_{j,1}$ is a positive constant, for $i \geq j \geq 2$. Using (3), (26), Lemma 2 and $|x + y|^p \leq 2^{p-1}|x^p + y^p|$ in Lemma 3, we obtain

$$|f_i(\cdot)| \leq \hat{b}_i \sum_{j=1}^i \left(|\xi_j|^{\frac{1}{p_1 \cdots p_{i-1}}} + |\xi_j|^{\frac{r_{i+1} p_i}{p_1 \cdots p_{j-1} r_j}} \right) \quad (27)$$

where $\hat{b}_i$ is a positive constant. Putting (13) and $|x^{\frac{1}{p}} - y^{\frac{1}{p}}| \leq 2^{1-\frac{1}{p}}|x - y|^{\frac{1}{p}}$ in Lemma 3 together, we arrive at

$$(x_i^{p_{i-1}} - x_i^{*p_{i-1}}) \leq 2^{1 - \frac{1}{p_1 \cdots p_{i-2}}} |\xi_i|^{\frac{1}{p_1 \cdots p_{i-2}}}. \quad (28)$$

Using (28), Lemmas 4 and 5 ($p = \frac{2\mu}{2\mu - r_i p_{i-1}}$ and $q = \frac{2\mu}{r_i p_{i-1}}$ in Lemma 5), it is not difficult to verify that

$$\begin{aligned} &\left(\xi_{i-1}^{2 - \frac{1}{p_1 \cdots p_{i-2}}} + \xi_{i-1}^{\frac{2\mu - r_i p_{i-1}}{p_1 \cdots p_{i-2} r_{i-1}}} \right) (x_i^{p_{i-1}} - x_i^{*p_{i-1}}) \leq \\ &a_{i,0} \left(\xi_{i-1}^2 + \xi_{i-1}^{\frac{2\mu}{p_1 \cdots p_{i-2} r_{i-1}}} \right) + \bar{a}_{i,0} \left(\xi_i^2 + \xi_i^{\frac{2\mu}{p_1 \cdots p_{i-1} r_i}} \right) \end{aligned} \quad (29)$$

where $a_{i,0}$ and $\bar{a}_{i,0}$ are positive constants.

By (13), (16), (27), Lemmas 2, 4 and 5 with $p =$

$\frac{2\mu}{2\mu - r_{i+1} p_i}$ and $q = \frac{2\mu}{r_{i+1} p_i}$, we get

$$\begin{aligned} &\frac{\partial(V_{Hi} + V_{Li})}{\partial x_i} f_i(\cdot) = \\ &\left(\xi_i^{2 - \frac{1}{p_1 \cdots p_{i-1}}} + \xi_i^{\frac{2\mu - r_{i+1} p_i}{p_1 \cdots p_{i-1} r_i}} \right) f_i(\cdot) \leq \\ &\hat{b}_i \left(\xi_i^{2 - \frac{1}{p_1 \cdots p_{i-1}}} + \xi_i^{\frac{2\mu - r_{i+1} p_i}{p_1 \cdots p_{i-1} r_i}} \right) \times \\ &\sum_{j=1}^i \left(|\xi_j|^{\frac{1}{p_1 \cdots p_{i-1}}} + |\xi_j|^{\frac{r_{i+1} p_i}{p_1 \cdots p_{j-1} r_j}} \right) \leq \\ &\sum_{j=1}^i a_{i,j} \left(\xi_j^2 + \xi_j^{\frac{2\mu}{p_1 \cdots p_{j-1} r_j}} \right) \end{aligned} \quad (30)$$

where $a_{i,1}, \dots, a_{i,i}$ are positive constants.

At last, we concentrate on the term $\sum_{j=1}^{i-1} \frac{\partial(V_{Li} + V_{Hi})}{\partial x_j} (x_{j+1}^{p_j} + f_j(\cdot))$. For $j = 1, \dots, i-1$, by (13), (16) and the mean value theorem of integrals, we have

$$\begin{aligned} &\left(\frac{\partial V_{Li}}{\partial x_j} + \frac{\partial V_{Hi}}{\partial x_j} \right) (x_{j+1}^{p_j} + f_j(\cdot)) = \\ &- \frac{\partial x_i^{*p_1 \cdots p_{i-1}}}{\partial x_j} \left(\int_{x_i^*}^{x_i} \left(2 - \frac{1}{p_1 \cdots p_{i-1}} \right) \times \right. \\ &\quad \left(s^{p_1 \cdots p_{i-1}} - x_i^{*p_1 \cdots p_{i-1}} \right)^{1 - \frac{1}{p_1 \cdots p_{i-1}}} ds + \\ &\quad \int_{x_i^*}^{x_i} \frac{2\mu - r_{i+1} p_i}{p_1 \cdots p_{i-1} r_i} (s^{p_1 \cdots p_{i-1}} - \\ &\quad \left. x_i^{*p_1 \cdots p_{i-1}})^{\frac{2\mu - r_{i+1} p_i}{p_1 \cdots p_{i-1} r_i} - 1} ds \right) (x_{j+1}^{p_j} + f_j(\cdot)) \leq \\ &\tilde{b}_{j,2} \left| \frac{\partial x_i^{*p_1 \cdots p_{i-1}}}{\partial x_j} \right| |x_{j+1}^{p_j} + f_j(\cdot)| \times \\ &\left(|\xi_i|^{1 - \frac{1}{p_1 \cdots p_{i-1}}} + |\xi_i|^{\frac{2\mu - r_{i+1} p_i}{p_1 \cdots p_{i-1} r_i} - 1} \right) |x_i - x_i^*| \end{aligned} \quad (31)$$

where $\tilde{b}_{j,2}$ is a positive constant.

Next, we can prove that there exists a positive constant $\bar{l}_{i,j,6}$ such that

$$\begin{aligned} &\left| \frac{\partial x_i^{*p_1 \cdots p_{i-1}}}{\partial x_j} \right| |x_{j+1}^{p_j} + f_j(\cdot)| \leq \\ &\bar{l}_{i,j,6} \sum_{j=1}^i \left(|\xi_j| + |\xi_j|^{\frac{r_i p_{i-1} \cdots p_1 + r_i}{p_1 \cdots p_{j-1} r_j}} \right). \end{aligned} \quad (32)$$

Indeed, by (4), (13), (26), (27), Lemmas 2–4 and 5 with

$p = \frac{p_1 \cdots p_{j-1} r_j + \tau_j}{p_1 \cdots p_{j-1} r_j - r_j}$ and $q = \frac{p_1 \cdots p_{j-1} r_j + \tau_j}{r_j + \tau_j}$, we get

$$\begin{aligned} & \left| \frac{\partial x_i^{*p_1 \cdots p_{i-1}}}{\partial x_j} \right| |x_{j+1}^{p_j} + f_j(\cdot)| \leq \\ & l_{i,j,1} \left(1 + |\xi_{i-1}|^{\frac{\tau_{i-1}}{r_{i-1}}} \right) \cdots \left(1 + |\xi_j|^{\frac{\tau_j}{r_j}} \right) \times \\ & |x_j|^{p_1 \cdots p_{j-1}-1} (|x_{j+1}|^{p_j} + |f_j(\cdot)|) \leq \\ & l_{i,j,2} \left(1 + |\xi_{i-1}|^{\frac{\tau_{i-1}}{r_{i-1}}} \right) \cdots \left(1 + |\xi_j|^{\frac{\tau_j}{r_j}} \right) (|\xi_j| + |\xi_{j-1}| + \\ & |\xi_{j-1}|^{\frac{r_j p_{j-1}}{r_{j-1}}})^{1 - \frac{1}{p_1 \cdots p_{j-1}}} \left(|\xi_{j+1}|^{\frac{1}{p_1 \cdots p_{j-1}}} + |\xi_j|^{\frac{1}{p_1 \cdots p_{j-1}}} + \right. \\ & \left. |\xi_j|^{\frac{r_j+1 p_j}{p_1 \cdots p_{j-1} r_j}} + \sum_{k=1}^j \left(|\xi_k|^{\frac{1}{p_1 \cdots p_{j-1}}} + |\xi_k|^{\frac{r_j+1 p_j}{p_1 \cdots p_{k-1} r_k}} \right) \right) \leq \\ & l_{i,j,3} \left(1 + |\xi_{i-1}|^{\frac{\tau_{i-1}}{r_{i-1}}} \right) \cdots \left(1 + |\xi_j|^{\frac{\tau_j}{r_j}} \right) \times \\ & \left(|\xi_j| + |\xi_{j-1}| + |\xi_{j-1}|^{\frac{r_j p_{j-1}}{r_{j-1}}} \right)^{1 - \frac{1}{p_1 \cdots p_{j-1}}} \times \\ & \left(|\xi_{j+1}|^{\frac{1}{p_1 \cdots p_{j-1}}} + \sum_{k=1}^j \left(|\xi_k|^{\frac{1}{p_1 \cdots p_{j-1}}} + |\xi_k|^{\frac{r_j+1 p_j}{p_1 \cdots p_{k-1} r_k}} \right) \right) \leq \\ & l_{i,j,4} \left(1 + |\xi_{i-1}|^{\frac{\tau_{i-1}}{r_{i-1}}} \right) \cdots \left(1 + |\xi_j|^{\frac{\tau_j}{r_j}} \right) \times \\ & \left(|\xi_{j+1}| + \sum_{k=1}^j \left(|\xi_k| + |\xi_k|^{\frac{p_1 \cdots p_{j-1} r_j + \tau_j}{p_1 \cdots p_{k-1} r_k}} \right) \right) = \\ & \underbrace{l_{i,j,4} \left(1 + |\xi_{i-1}|^{\frac{\tau_{i-1}}{r_{i-1}}} \right) \cdots \left(1 + |\xi_j|^{\frac{\tau_j}{r_j}} \right) |\xi_{j+1}| +}_{\Xi.1} \\ & l_{i,j,4} \sum_{k=1}^j \left(|\xi_k| + |\xi_k|^{\frac{p_1 \cdots p_{j-1} r_j + \tau_j}{p_1 \cdots p_{k-1} r_k}} \right) \left(1 + |\xi_j|^{\frac{\tau_j}{r_j}} \right) \times \\ & \left(1 + |\xi_{j+1}|^{\frac{\tau_{j+1}}{r_{j+1}}} \right) \cdots \left(1 + |\xi_{i-1}|^{\frac{\tau_{i-1}}{r_{i-1}}} \right) \end{aligned} \quad (33)$$

where $l_{i,j,1}$, $l_{i,j,2}$, $l_{i,j,3}$ and $l_{i,j,4}$ are positive constants. For $\Xi.1$ of (33), by (4), Lemmas 2 and 4, we obtain

$$\begin{aligned} & l_{i,j,4} \left(1 + |\xi_{i-1}|^{\frac{\tau_{i-1}}{r_{i-1}}} \right) \cdots \left(1 + |\xi_j|^{\frac{\tau_j}{r_j}} \right) |\xi_{j+1}| \leq \\ & \bar{l}_{i,j,1} \left(1 + \sum_{k=j}^{i-1} |\xi_k|^{\sum_{m=j}^{i-1} \frac{\tau_m}{r_m}} \right) |\xi_{j+1}| \leq \\ & \bar{l}_{i,j,2} \left(|\xi_{j+1}| + |\xi_{j+1}|^{1 + \sum_{m=j}^{i-1} \frac{\tau_m}{r_m}} + \sum_{k=j}^{i-1} |\xi_k|^{1 + \sum_{m=j}^{i-1} \frac{\tau_m}{r_m}} \right) \leq \\ & \bar{l}_{i,j,3} \left(\sum_{j=1}^i \left(|\xi_j| + |\xi_j|^{\frac{r_i p_{i-1} \cdots p_1 + \tau_i}{p_1 \cdots p_{j-1} r_j}} \right) \right) \end{aligned} \quad (34)$$

where $\bar{l}_{i,j,1}$, $\bar{l}_{i,j,2}$ and $\bar{l}_{i,j,3}$ are positive constants. To estimate the last term of (33), by using Lemmas 2 and 5 repeatedly, we observe from the definitions of μ, τ_i in (4)

that

$$\begin{aligned} & l_{i,j,4} \sum_{k=1}^j \underbrace{\left(|\xi_k| + |\xi_k|^{\frac{p_1 \cdots p_{j-1} r_j + \tau_j}{p_1 \cdots p_{k-1} r_k}} \right)}_{\Xi.2} \left(1 + |\xi_j|^{\frac{\tau_j}{r_j}} \right) \times \\ & \left(1 + |\xi_{j+1}|^{\frac{\tau_{j+1}}{r_{j+1}}} \right) \cdots \left(1 + |\xi_{i-1}|^{\frac{\tau_{i-1}}{r_{i-1}}} \right) \leq \\ & \bar{l}_{i,j,4} \sum_{k=1}^j \left(|\xi_k| + |\xi_k|^{\frac{p_1 \cdots p_j r_{j+1} + \tau_{j+1}}{p_1 \cdots p_{k-1} r_k}} \right) \times \\ & \left(1 + |\xi_{j+1}|^{\frac{p_1 \cdots p_j r_{j+1}}{p_1 \cdots p_j r_{j+1}}} \right) \cdots \left(1 + |\xi_{i-1}|^{\frac{p_1 \cdots p_{i-2} \tau_{i-1}}{p_1 \cdots p_{i-2} r_{i-1}}} \right) \leq \\ & \bar{l}_{i,j,5} \sum_{k=1}^{i-1} \left(|\xi_k| + |\xi_k|^{\frac{r_i p_{i-1} \cdots p_1 + \tau_i}{p_1 \cdots p_{k-1} r_k}} \right) \end{aligned} \quad (35)$$

where the first inequality is obtained by dealing with $\Xi.2$ ($p = \frac{p_1 \cdots p_j r_{j+1} + \tau_{j+1}}{p_1 \cdots p_{j-1} r_j}$ and $q = \frac{p_1 \cdots p_j r_{j+1} + \tau_{j+1}}{p_1 \cdots p_{j-1} r_j + \tau_j}$ in Lemma 5). After that, $\sum_{k=1}^j (|\xi_k| + |\xi_k|^{\frac{p_1 \cdots p_j r_{j+1} + \tau_{j+1}}{p_1 \cdots p_{k-1} r_k}}) (1 + |\xi_{j+1}|^{\frac{p_1 \cdots p_j r_{j+1}}{p_1 \cdots p_j r_{j+1}}})$ ($p = \frac{p_1 \cdots p_{j+1} r_{j+2} + \tau_{j+2}}{p_1 \cdots p_j r_{j+1}}$ and $q = \frac{p_1 \cdots p_{j+1} r_{j+2} + \tau_{j+2}}{p_1 \cdots p_j r_{j+1} + \tau_{j+2}}$ in Lemma 5), $\bar{l}_{i,j,4}$ and $\bar{l}_{i,j,5}$ are positive constants. Substituting (34) and (35) into (33) yields (32). Hence, by (21), (32), Lemmas 2, 4 and 5 with $p = \frac{2\mu}{2\mu - \tau_i - r_i p_{i-1} \cdots p_1}$ and $q = \frac{2\mu}{p_1 \cdots p_{i-1} r_i + \tau_i}$, there are positive constants $\bar{b}_{i,1}, \dots, \bar{b}_{i,i}, b_{i,1}, \dots, b_{i,i}$ and $\bar{l}_{i,j,7}$ such that

$$\begin{aligned} & \left(\frac{\partial V_{Li}}{\partial x_j} + \frac{\partial V_{Hi}}{\partial x_j} \right) (x_{j+1}^{p_j} + f_j(\cdot)) \leq \\ & \bar{b}_{j,2} \bar{l}_{i,j,6} \left(|\xi_i|^{1 - \frac{1}{p_1 \cdots p_{i-1}}} + |\xi_i|^{\frac{2\mu - r_{i+1} p_i}{p_1 \cdots p_{i-1} r_i} - 1} \right) \times \\ & |x_i - x_i^*| \sum_{k=1}^i \left(|\xi_k| + |\xi_k|^{\frac{r_i p_{i-1} \cdots p_1 + \tau_i}{p_1 \cdots p_{k-1} r_k}} \right) \leq \\ & \bar{l}_{i,j,7} \left(|\xi_i| + |\xi_i|^{\frac{2\mu - \tau_i - p_1 \cdots p_{i-1} r_i}{p_1 \cdots p_{i-1} r_i}} \right) \times \\ & \sum_{k=1}^i \left(|\xi_k| + |\xi_k|^{\frac{r_i p_{i-1} \cdots p_1 + \tau_i}{p_1 \cdots p_{k-1} r_k}} \right) \leq \\ & \sum_{j=1}^i \bar{b}_{i,j} \left(\xi_j^2 + \xi_j^{\frac{2\mu}{p_1 \cdots p_{j-1} r_j}} \right). \end{aligned}$$

Moreover,

$$\begin{aligned} & \sum_{j=1}^{i-1} \frac{\partial (V_{Li} + V_{Hi})}{\partial x_j} (x_{j+1}^{p_j} + f_j(\cdot)) \leq \\ & \sum_{j=1}^i b_{i,j} \left(\xi_j^2 + \xi_j^{\frac{2\mu}{p_1 \cdots p_{j-1} r_j}} \right). \end{aligned} \quad (36)$$

With (25), (29), (30) and (36) in mind, it can be con-

cluded that

$$\begin{aligned}
\dot{V} \leq & -\sum_{j=1}^{i-1} c_{i-1,j} \left(\xi_j^2 + \xi_j^{\frac{2\mu}{p_1 \cdots p_{j-1} r_j}} \right) + \\
& a_{i,0} \left(\xi_{i-1}^2 + \xi_{i-1}^{\frac{2\mu}{p_1 \cdots p_{i-2} r_{i-1}}} \right) + \bar{a}_{i,0} \left(\xi_i^2 + \xi_i^{\frac{2\mu}{p_1 \cdots p_{i-1} r_i}} \right) + \\
& \sum_{j=1}^i a_{i,j} \left(\xi_j^2 + \xi_j^{\frac{2\mu}{p_1 \cdots p_{j-1} r_j}} \right) + \\
& \sum_{j=1}^i b_{i,j} \left(\xi_j^2 + \xi_j^{\frac{2\mu}{p_1 \cdots p_{j-1} r_j}} \right) + \\
& \left(\xi_i^{2-\frac{1}{p_1 \cdots p_{i-1}}} + \xi_i^{\frac{2\mu-r_{i+1}p_i}{p_1 \cdots p_{i-1} r_i}} \right) x_{i+1}^{p_i} = \\
& -\sum_{j=1}^{i-1} c_{i,j} \left(\xi_j^2 + \xi_j^{\frac{2\mu}{p_1 \cdots p_{j-1} r_j}} \right) + \\
& (a_{i,i} + b_{i,i} + \bar{a}_{i,0}) \left(\xi_i^2 + \xi_i^{\frac{2\mu}{p_1 \cdots p_{i-1} r_i}} \right) + \\
& \left(\xi_i^{2-\frac{1}{p_1 \cdots p_{i-1}}} + \xi_i^{\frac{2\mu-r_{i+1}p_i}{p_1 \cdots p_{i-1} r_i}} \right) x_{i+1}^{*p_i} + \\
& \left(\xi_i^{2-\frac{1}{p_1 \cdots p_{i-1}}} + \xi_i^{\frac{2\mu-r_{i+1}p_i}{p_1 \cdots p_{i-1} r_i}} \right) (x_{i+1}^{p_i} - x_{i+1}^{*p_i}) \quad (37)
\end{aligned}$$

where $c_{i,j} = c_{i-1,j} - a_{i,j} - b_{i,j}$, $j = 1, \dots, i-2$, $c_{i,i-1} = c_{i-1,i-1} - a_{i,0} - a_{i,i-1} - b_{i,i-1}$. Finally, a virtual controller of the form $x_{i+1}^{*p_i} = -\alpha_i^{\frac{1}{p_i}} (\xi_i + \xi_i^{\frac{r_{i+1}p_i}{r_i}})$, $\alpha_i = (a_{i,i} + b_{i,i} + \bar{a}_{i,0} + c_{i,i})$, results in (15), which can be divided into the following two cases to prove.

1) When $\xi_i < 0$, by $p_i \in \mathbf{R}_{\text{odd}}^{\geq 1}$, $(|x| + |y|)^{\frac{1}{p}} \leq |x|^{\frac{1}{p}} + |y|^{\frac{1}{p}}$ in Lemma 3 and the definitions of μ and r_i , we have

$$\begin{aligned}
& \left(\xi_i^{2-\frac{1}{p_1 \cdots p_{i-1}}} + \xi_i^{\frac{2\mu-r_{i+1}p_i}{p_1 \cdots p_{i-1} r_i}} \right) x_{i+1}^{*p_i} = \\
& - \left((-\xi_i)^{2-\frac{1}{p_1 \cdots p_{i-1}}} + (-\xi_i)^{\frac{2\mu-r_{i+1}p_i}{p_1 \cdots p_{i-1} r_i}} \right) \times \\
& (a_{i,i} + b_{i,i} + \bar{a}_{i,0} + c_{i,i}) \left((-\xi_i) + (-\xi_i)^{\frac{r_{i+1}p_i}{r_i}} \right)^{\frac{1}{p_1 \cdots p_{i-1}}} \leq \\
& -(a_{i,i} + b_{i,i} + \bar{a}_{i,0} + c_{i,i}) \left(\xi_i^2 + \xi_i^{\frac{2\mu}{p_1 \cdots p_{i-1} r_i}} \right). \quad (38)
\end{aligned}$$

2) When $\xi_i \geq 0$, similar to 1), (38) also holds. $\square$

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