Projective Lag Synchronization and Parameter Identification of a New Hyperchaotic System

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Abstract: In this paper, the projective lag synchronization of a new hyperchaotic system with certain/uncertain parameters is addressed. Based on Lyapunov stability theory, a generic and simple controller is designed for the projective lag synchronization. Furthermore, with LaSalle's invariance principle, an adaptive method is proposed to identify the unknown parameters of the new hyperchaotic system based on the projective lag synchronization. Finally, numerical simulations are given to support the analytical approach.

Keywords: Projective lag synchronization, hyperchaotic system, parameter identification, Lyapunov stability theory, LaSalle's invariance principle.

1 Introduction

Many hyperchaotic systems, such as Lorenz hyperchaotic system^[1], Chen hyperchaotic system^[2], Lü hyperchaotic system^[3], the generalized Hénon-Heiles system^[4], have been extensively studied since the pioneering work of [5]. Hyperchaotic systems have at least two positive Lyapunov exponents. They have more complex behavior and abundant dynamics than a chaotic system. So they are more suitable for some engineering applications such as cryptography and secure communication. Hence, how to realize synchronization of hyperchaotic systems is an interesting and challenging task.

Since the notation of chaos synchronization was introduced by Pecora and Carroll^[6], it has been an active research topic and received extensive studies. It has many applications, e.g., chemical oscillators, population dynamics, physiological interactions, coupled neurons to mechanical oscillators, and lasers. Up to now, many effective control methods have been proposed to synchronize chaotic systems, e.g., generalized synchronization^[7], phase synchronization^[8], lag synchronization^[9], Q-S synchronization^[10], projective synchronization^[11], H_∞ synchronization^[12], anticipating synchronization^[13], exponential synchronization^[14], etc. Among these regimes of synchronization, projective synchronization means that the corresponding state variables of drive-response hyperchaotic (chaotic) systems evolve in a proportion scale, and it includes the complete synchronization and the anti-synchronization. On the other hand, the lag synchronization is more reasonable in our engineering applications, because time delay exists inevitably for finite signal transmission speeds or memory effects in communication.

Therefore, based on these two points, we firstly proposed the notation of projective lag synchronization which includes the projective synchronization and the lag syn-

chronization at the same time. It is more meaningful because it has a more general synchronization form and can suit for more real cases. Moreover, in practical situations, there exist many cases with uncertain information, e.g., uncertain system parameters may destroy the synchronization. Therefore, how to effectively synchronize two hyperchaotic systems with unknown parameters is an important problem for theoretical research and practical applications. Parameter estimations of single or coupled nonlinear system(s) based on the complete synchronization method have been investigated^[15–19]. In this paper, we will propose a novel method to identify the unknown parameters based on the projective lag synchronization of a new hyperchaotic system^[20], and introduce a key condition that uncertain parameters can be identified correctly when projective lag synchronization is achieved.

This paper is organized as follows. The hyperchaotic system is introduced in Section 2. The projective lag synchronization and parameter identification of the hyperchaotic system are discussed in Sections 3 and 4, respectively. Numerical simulations are given in Section 5. Finally, the paper is concluded in Section 6.

2 System description

Recently, a new four-dimensional continuous autonomous hyperchaotic system^[20] was constructed as

$$\begin{cases} \dot{x}_1 = a(x_2 - x_1) + x_4 \\ \dot{x}_2 = x_1x_3 - x_2 \\ \dot{x}_3 = b - x_1x_2 - cx_3 \\ \dot{x}_4 = dx_4 - x_2x_3 \end{cases} \quad (1)$$

where x_1, x_2, x_3 and x_4 are state variables, and a, b, c and d are real constant parameters. After 5×10^4 iterations, we have four Lyapunov exponents which are 1.400240, 0.313208, -0.968585 and -3.327758 when $a = 5$, $b = 16$, $c = 1$ and $d = -0.5$. It implies that there exists a hyperchaotic attractor^[20]. The three-dimensional phase portrait of the hyperchaotic attractor is illustrated in Fig. 1.

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Fig. 1 Hyperchaotic attractor

2.1 Projective lag synchronization of the new hyperchaotic system

For the drive hyperchaotic system (1), the response system can be described as

$$\begin{cases} \dot{y}_1 = a(y_2 - y_1) + y_4 \\ \dot{y}_2 = y_1 y_3 - y_2 + u_1 \\ \dot{y}_3 = b - y_1 y_2 - c y_3 + u_2 \\ \dot{y}_4 = d y_4 - y_2 y_3 + u_3 \end{cases} \quad (2)$$

where u_1, u_2 and u_3 are nonlinear controllers to be designed so that systems (1) and (2) can be synchronized.

Let the error variables be

$$\begin{cases} e_1 = y_1 - \lambda x_1(t - \tau) \\ e_2 = y_2 - \lambda x_2(t - \tau) \\ e_3 = y_3 - \lambda x_3(t - \tau) \\ e_4 = y_4 - \lambda x_4(t - \tau) \end{cases} \quad (3)$$

where $\lambda \neq 0$ is the proportion scale, and τ is constant, representing the time delay or lag.

Substituting (1) and (2) into (3), we have the following error dynamical system:

$$\begin{cases} \dot{e}_1 = -a e_1 + a e_2 + e_4 \\ \dot{e}_2 = -e_2 + y_1 y_3 - \lambda x_1(t - \tau) x_3(t - \tau) + u_1 \\ \dot{e}_3 = -c e_3 - y_1 y_2 + \lambda x_1(t - \tau) x_2(t - \tau) + (1 - \lambda) b + u_2 \\ \dot{e}_4 = d e_4 - y_2 y_3 + \lambda x_2(t - \tau) x_3(t - \tau) + u_3. \end{cases} \quad (4)$$

Definition 1. The drive system (1) and the response system (2) are said to achieve projective lag synchronization, if the solution of the error dynamical system (4) satisfies $\lim_{t \rightarrow \infty} e_i = 0$ ($i = 1, 2, 3, 4$) for arbitrary initial conditions $x(0) = (x_1(0), x_2(0), x_3(0), x_4(0))^T$ and $y(0) = (y_1(0), y_2(0), y_3(0), y_4(0))^T$.

Specially, for $\tau = 0$, it is called the projective synchronization. For $\lambda = 1$, it is called the lag synchronization.

Theorem 1. Systems (1) and (2) will approach exponential projective lag synchronization for any initial condition with the following control law:

$$\begin{cases} u_1 = -y_1 y_3 + \lambda x_1(t - \tau) x_3(t - \tau) - a e_1 \\ u_2 = y_1 y_2 - \lambda x_1(t - \tau) x_2(t - \tau) + (\lambda - 1) b \\ u_3 = y_2 y_3 - \lambda x_2(t - \tau) x_3(t - \tau) - e_1. \end{cases} \quad (5)$$

Proof. Construct a Lyapunov function

$$V(t) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_4^2).$$

With (5), the time derivative of the Lyapunov function along the trajectories of system (4) is

$$\begin{aligned} \dot{V}(t) &= e_1 \dot{e}_1 + e_2 \dot{e}_2 + e_3 \dot{e}_3 + e_4 \dot{e}_4 = \\ &= e_1(-a e_1 + a e_2 + e_4) + \\ &= e_2(-e_2 + y_1 y_3 - \lambda x_1(t - \tau) x_3(t - \tau) + u_1) + \\ &= e_3(-c e_3 - y_1 y_2 + \lambda x_1(t - \tau) x_2(t - \tau) + \\ &= (1 - \lambda) b + u_2) + \\ &= e_4(y_2 y_3 - \lambda x_2(t - \tau) x_3(t - \tau) - e_1) = \\ &= -a e_1^2 - e_2^2 - c e_3^2 + d e_4^2. \end{aligned}$$

Let $\eta = \min(a, 1, c, -d)$, then we have

$$\dot{V}(t) \leq -\eta V(t)$$

which yields

$$V(t) \leq V(0) e^{-2\eta t}.$$

This implies that $\lim_{t \rightarrow \infty} e_i = 0$ ($i = 1, 2, 3, 4$), which guarantees the global and exponential asymptotical stability of the origin of system (4). In other word, systems (1) and (2) achieve global and exponential projective lag synchronization. $\square$

Remark 1. If $\tau = 0$, then systems (1) and (2) asymptotically achieve projective synchronization. If $\lambda = 1$, then systems (1) and (2) asymptotically achieve lag synchronization.

3 Parameter identification based on projective lag synchronization

In this section, suppose parameters a, b, c and d are unknown in the hyperchaotic system (1). We will focus on identifying the unknown parameters via using the projective lag synchronization theory and adaptive control techniques.

The response system is described by

$$\begin{cases} \dot{y}_1 = \hat{a}(y_2 - y_1) + y_4 \\ \dot{y}_2 = y_1 y_3 - y_2 + u_1 \\ \dot{y}_3 = \hat{b} - y_1 y_2 - \hat{c} y_3 + u_2 \\ \dot{y}_4 = \hat{d} y_4 - y_2 y_3 + u_3 \end{cases} \quad (6)$$

where $\hat{a}, \hat{b}, \hat{c}$ and $\hat{d}$ are the estimations of the uncertain parameters a, b, c and d , and u_i ($i = 1, 2, 3$) are the controllers such that the hyperchaotic systems get projective lag synchronization.

Denote $\tilde{a} = \hat{a} - a$, $\tilde{b} = \hat{b} - b$, $\tilde{c} = \hat{c} - c$ and $\tilde{d} = \hat{d} - d$. Substituting (1) and (5) into (3), we have the error dynamical system as

$$\begin{cases} \dot{e}_1 = -a e_1 + a e_2 + \tilde{a}(y_2 - y_1) + e_4 \\ \dot{e}_2 = -e_2 + y_1 y_3 - \lambda x_1(t - \tau) x_3(t - \tau) + u_1 \\ \dot{e}_3 = -c e_3 + \tilde{b} + (1 - \lambda) b - y_1 y_2 + \lambda x_1(t - \tau) x_2(t - \tau) - \tilde{c} y_3 + u_2 \\ \dot{e}_4 = d e_4 + \tilde{d} y_4 - y_2 y_3 + \lambda x_2(t - \tau) x_3(t - \tau) + u_3. \end{cases} \quad (7)$$

Theorem 2. We select the controllers in (6) as

$$\begin{cases} u_1 = -y_1 y_3 + \lambda x_1(t-\tau)x_3(t-\tau) - ae_1 \\ u_2 = y_1 y_2 - \lambda x_1(t-\tau)x_2(t-\tau) + (\lambda-1)b \\ u_3 = y_2 y_3 - \lambda x_2(t-\tau)x_3(t-\tau) - e_1. \end{cases} \quad (8)$$

And we use the following updating laws as

$$\begin{cases} \dot{\hat{a}} = -e_1(y_2 - y_1) \\ \dot{\hat{b}} = -e_3 \\ \dot{\hat{c}} = e_3 y_3 \\ \dot{\hat{d}} = -e_4 y_4. \end{cases} \quad (9)$$

Then, systems (1) and (6) achieve projective lag synchronization. And the uncertain parameters are well estimated from the system parameters a, b, c and d in the sense that $\lim_{t \rightarrow \infty} \hat{a} = a, \lim_{t \rightarrow \infty} \hat{b} = b, \lim_{t \rightarrow \infty} \hat{c} = c$ and $\lim_{t \rightarrow \infty} \hat{d} = d$.

Proof. Choose the following Lyapunov functional candidate:

$$V(t) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_4^2 + \tilde{a}^2 + \tilde{b}^2 + \tilde{c}^2 + \tilde{d}^2).$$

Differentiating $V(t)$ with respect to time along the solution of (7) yields

$$\begin{aligned} \dot{V}(t) &= e_1^2 + e_2^2 + e_3^2 + e_4^2 + \tilde{a}^2 + \tilde{b}^2 + \tilde{c}^2 + \tilde{d}^2 = \\ &= e_1 \dot{e}_1 + e_2 \dot{e}_2 + e_3 \dot{e}_3 + e_4 \dot{e}_4 + \tilde{a} \dot{\tilde{a}} + \tilde{b} \dot{\tilde{b}} + \tilde{c} \dot{\tilde{c}} + \tilde{d} \dot{\tilde{d}} = \\ &= e_1(-ae_1 + ae_2 + \tilde{a}(y_2 - y_1) + e_4) + \\ &= e_2(-e_2 + y_1 y_3 - \lambda x_1(t-\tau)x_3(t-\tau) + u_1) + \\ &= e_3(-ce_3 + \tilde{b} + (1-\lambda)b - y_1 y_2 + \\ &= \lambda x_1(t-\tau)x_2(t-\tau) - \tilde{c}y_3 + u_2) + \\ &= e_4(de_4 + \tilde{d}y_4 - y_2 y_3 + \lambda x_2(t-\tau)x_3(t-\tau) + u_3) - \\ &= \tilde{a}e_1(y_2 - y_1) - \tilde{b}e_3 + \tilde{c}e_3 y_3 - \tilde{d}e_4 y_4 = \\ &= -ae_1^2 - e_2^2 - ce_3^2 + de_4^2. \end{aligned}$$

Obviously, $\dot{V} = 0$ if and only if $e_i = 0, i = 1, 2, 3, 4$, thus $E = \{\dot{V} = 0\} = \{e(t) = 0\}$. So set $M = \{e(t) = 0, \hat{a} = A, \hat{b} = b, \hat{c} = c, \hat{d} = d\}$ is the largest invariant set of set E for the error system (7). According to LaSalle's invariance principle^[1], starting with arbitrary initial values, the trajectory of error system (7) asymptotically converges to set M , i.e., $e_i \rightarrow 0, \hat{a} \rightarrow a, \hat{b} \rightarrow b, \hat{c} \rightarrow c$ and $\hat{d} \rightarrow d$ as $t \rightarrow \infty$. This implies that these unknown parameters can be successfully estimated using the adaptive feedback method proposed in Theorem 2. $\square$

4 Numerical simulation

In this section, we will show the numerical results for projective lag synchronization and parameter identification of the new hyperchaotic system to verify and illustrate the effectiveness of the theoretical analysis in Sections 3 and 4.

4.1 Projective lag synchronization of the new hyperchaotic system

In this subsection, we set $a = 5, b = 16, c = 1$ and $d = -0.5$ in system (1). By the control method (5), one has the projective lag synchronization of systems (1) and (2).

Fig.2 displays the trajectories of systems (1) and (2) when $\lambda = 2$ and $\tau = 1$. Fig.3 shows the trajectories of

systems (1) and (2) when $\lambda = -1$ and $\tau = 1$. From these two figures, we can see that systems (1) and (2) get projective lag synchronization.

Fig. 2 The trajectories of systems (1) and (2) when $\lambda = 2$ and $\tau = 1$

Fig. 3 The trajectories of systems (1) and (2) when $\lambda = -1$ and $\tau = 1$

4.2 Parameter identification based on projective lag synchronization

In this subsection, we also set $a = 5, b = 16, c = 1$ and $d = -0.5$ in system (1). By the updating laws (8) and (9), the numerical results are shown in Figs.4 and 5. It can be seen that the drive system (1) and the response system (6) get projective lag synchronization, and at the same time, unknown parameters a, b, c and d are identified correctly.

5 Conclusions

In this paper, the projective lag synchronization of a new hyperchaotic system with certain/uncertain parameters was addressed. Based on the Lyapunov stability theory, a generic and simple controller was designed for the projective lag synchronization. Moreover, with the LaSalle's invariance principle, an adaptive method was proposed to identify the unknown parameters of the new hyperchaotic system based on projective lag synchronization. Finally, numerical simulations fully supported the analytical approach.

Fig. 4 The trajectories of systems (1) and (6) when $\lambda = 3$ and $\tau = 0.5$

Fig. 5 Parameter identification of system (6)

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