A brief discussion on the relationship between apparent stress and slip-weakening law based on the energy partition criteria*

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Abstract According to the representation theorem of seismic energy radiation, we know that, at any point on the fault, the instantaneous seismic radiation energy, E_s , the seismic moment M_0 , and the apparent stress $\sigma_a = \mu E_s / M_0$ (μ is the shear modulus on the fault plane) should be positive values at any time during an earthquake faulting. However, we have noticed that, in recent source parameter inversion scheme for deriving the critical slip-weakening distance, the apparent stress used as a constraint condition on the fault plane could be less than zero or negative value, and the negative part was considered as dissipation energy and incorporated to the fracture energy. Although the mathematical formula in such case has no influence to the final resolution, however, the earthquake dynamic source process violates obviously the basic physical law, which could results in the overestimating of radiated seismic energy. In this study, we have proposed an alternative way to take account of the apparent stress expression based on the earthquake energy partition principle, and at the same time, we have also suggested that an additional constraint regarding to the radiation energy conservation on the fault could be added into the source parameter inversion in order to estimate the critical slip-weakening distance D_c .

Key words: apparent stress; seismic radiation energy; fracture energy; critical slip-weakening distance

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1 Introduction

From our current understanding of earthquake source mechanism, we know that, as fault ruptures with associated slip along the fault plane, the radiated seismic energy, E_s , from the fault should not be less than zero, so does the seismic moment, M_0 . Correspondingly, apparent stress, σ_a , ($\sigma_a = \mu E_s / M_0$, μ is a material shear modulus on the fault) should also be positive. Apparent stress σ_a , [$\sigma_a = E_s / (\Sigma_0 D^f)$, where Σ_0 and D^f are the fault ruptured area and averaged slip displacement, respectively] describes the amount of radiation energy per unit area and per unit slip. Therefore, the apparent stress σ_a could not be less than zero at any time during fault rupture process.

In recent study given by Pulido and Irikura (2000), apparent stress, σ_a , has been used as a constraint to resolve the critical slip-weakening distance, D_c , for the 1992 $M_w 7.3$ Landers earthquake, California. In their

inversion algorithm, they mentioned that the apparent stress, σ_a , could be a negative at some times (Madariaga, 1977), and the negative part does not contribute to the radiated seismic energy, but belongs to an energy dissipation which could be combined to the fracture energy used to advance the rupture propagation. A generalized mathematical description is as follows (Madariaga, 1977; Pulido and Irikura, 2000)

$$\sigma_a(D) = \frac{1}{D \Sigma_0} \int_{\Sigma_0} \int_0^D du [\bar{\sigma}_{ij} - \sigma_{ij}(u)] n_j - \frac{\gamma_{\text{eff}}}{D}, \quad (1)$$

where $\bar{\sigma}_{ij} = (\sigma_{ij}^0 + \sigma_{ij}^1) / 2$ is an average shear stress on fault, and σ_{ij}^0 and σ_{ij}^1 are the shear stresses before and after earthquake, respectively; D is an accumulation slip displacement on fault surface of Σ_0 , and $\sigma_{ij}(u)$ is a time-dependent/slip-dependent frictional stress on fault. n_j is a unit vector which is perpendicular to fault surface of Σ_0 , γ_{eff} is the effective surface energy on Σ_0 . In general, because the effective surface energy ' γ_{eff} ' is very difficult to estimate, it can usually be treated as part of fracture energy (Rivera and Kanamori, 2005). Combined with

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simplified slip-weakening frictional law indicated in Figure 1, the apparent stress, σ_a , derived from equation (1) increases with slip (see Figure 1) gradually. In Figure 1, when $D \leq D'$, then $\sigma_a \leq 0$, the area given by triangle AOD' corresponds to a dissipation energy; while $D \geq D'$, then $\sigma_a \geq 0$, it corresponds to radiated seismic energy. Once D is equal to D_c (D_c is called a critical slip-weakening distance), σ_a reaches the maximum value. The resultant total radiated seismic energy is given by the area of trapeziform $BD'D'C$. According to earthquake energy partition principle (Rivera, Kanamori, 2005), the actual radiated seismic energy should be equal to the area given by a rectangle $BD_cD'C$ ($E_S = (\sigma_0 - \sigma_1)(D' - D_c)\Sigma_0/2$). In this case, the radiated seismic energy calculated by equation (1) does not match with the energy partition principle for a source parameter inversion scheme in an implementation level.

Figure 1 The changes of apparent stress and average stress in the slipping. D_c is the critical slip weakening distance, σ_0 and σ_1 are the initial stress and final stress on the fault, $\bar{\sigma}$ is the average stress of σ_0 and σ_1 . The upper and lower lines are frictional stress and apparent stress, respectively. Horizontal axis represents fault slip.

Obviously, the negative apparent stress appeared in Figure 1 is conflicted with earthquake energy partition principle. In this study, by using earthquake energy partition principle, we try to point out the shortage of equation (1) used in source parameter inversion without any modification. We also give an alternative and correct description of apparent stress, σ_a , in order to derive source parameters correctly without violating earthquake energy partition principle.

2 Energy partition principle

Theoretically, the fault is covered in a closed sur-

face S_0 (outer boundary), with a characteristic scale Σ (internal boundary) ($\Sigma = \Sigma^- + \Sigma^+$). When a relative motion of both fault sides occurs, some part of energy radiates to S_0 in the form of seismic waves. If the distance r from any point ξ on the fault to S_0 is much larger than fault size, the radiation energy from far field can be written as follows (Rudnicki and Freund, 1981):

$$E_S = \int_{t_0}^{t_1} \int_{S_0} \rho [\alpha (\dot{u}_j n_j)^2 + \beta (\dot{u}_i - \dot{u}_j n_j n_i)^2] dS dt, \quad (2)$$

where ρ is density of homogeneous isotropic medium, α and β are P-wave and S-wave speed, respectively, u_j is the displacement on the boundary of S_0 , n_j is its unit normal vector, t_0 is time before earthquake, and t_1 is stop time of particle motion on S_0 after earthquakes.

On the other side, if the distance r from any point ξ on the fault to S_0 is much larger than fault size, the seismic radiation energy (only considering S-wave) led from the change of elastic potential energy before and after earthquakes can be calculated (Kostrov, 1974; Dahlen, 1977):

$$E_S = \frac{1}{2} \int_{x_0} (\sigma_{ij}^0 + \sigma_{ij}^1) \Delta u_i v_j dS - \int_{\Sigma} 2\gamma_{\text{eff}} dS - \int_{\Sigma} dS \int_{t_r(\xi)}^{t_1} \sigma_{ij} \Delta \dot{u}_i v_j dt, \quad (3)$$

where σ_{ij}^0 and σ_{ij}^1 are the shear stresses before and after earthquake, respectively. σ_{ij} is the stress on the fault at any time during fault rupture. $\Delta u_i = u_i^+ - u_i^-$ is a relative slip displacement of the two sides of the fault. Δu_i^1 is the final slip displacement. v_j is a unit vector perpendicular to the fault Σ_0 . $t_r(\xi)$ is the time on which the slip begins from any point ξ on the fault. γ_{eff} is effective surface energy on Σ_0 (Kostrov, 1974). Combining with equations (2) and (3), the observation seismic signals gathered on the ground surface can be used to estimate the source dynamic rupture parameters on the fault (energy of P-wave radiated from double couple source is much less than the energy of S-wave). More detail discussions can be founded in Rivera and Kanamori (2005). Furthermore, equation (3) can also be written to (Rivera and Kanamori, 2005):

$$E_S = \frac{1}{2} \int_{x_0} (\sigma_{ij}^0 + \sigma_{ij}^1) \Delta u_i^1 v_j dS - \int_{x_0} 2\gamma_{\text{eff}} dS - \int_{x_0} dS \int_0^{\Delta u_i^1} \sigma_{ij} (\Delta u_i) v_j d(\Delta u_i). \quad (4)$$

In fact, we know that, from equation (4), the radiated seismic energy not only depends on slip behavior at

a single point on fault, but also depends on the slip processes on the whole fault surface, which was expounded by Rivera and Kanamori (2005). Therefore, the theoretical study given by Rivera and Kanamori (2005) indicated that a simple summation of seismic energy flow from an individual point on the fault resulted in an in-corrected estimating of total radiated seismic energy. In this way, we can get this by the first integral of equation (4), which describes the volume integral of any pipe covered with dS , rather than the integral of dS (see Figure 2, Rivera and Kanamori, 2005).

If the stress change on the fault surface and the particle displacement obey simplified slip-weakening principle (Figure 2), the equation (4) can be rewritten as

$$E_S = \frac{1}{2} \int_{x_0} (\sigma_0 + \sigma_1) \Delta u^1 dS - \int_{x_0} dS \int_0^{\Delta u^1} \sigma(u) du - \int_{x_0} 2\gamma_{\text{eff}} dS. \quad (5)$$

In equation (5), if we change the surface integral into the direction of fault surface, the σ_{ij}^0 and σ_{ij}^1 can be replaced by σ_0 and σ_1 , in which σ_0 and σ_1 are the initial and final stress on the fault, respectively, Δu^1 is the final slip gap (single point source approach), $\sigma(u)$ is a time-dependent/slip-dependent stress (dynamic frictional stress) on the fault during fault rupture propagation. According to the definition of apparent stress (Wyss and Brune, 1968), we have:

$$\begin{aligned} \sigma_a &= \mu \frac{E_S}{M_0} = \frac{E_S}{\Delta u^1 \Sigma_0} \\ &= \frac{1}{\Delta u^1 \Sigma_0} \int_{x_0} dS \int_0^{\Delta u^1} du [\bar{\sigma} - \sigma(u)] - \frac{2}{\Delta u^1} \gamma_{\text{eff}}, \quad (6) \end{aligned}$$

where $\bar{\sigma} = (\sigma_0 + \sigma_1)/2$. In general, the surface energy γ_{eff} in equations (5) and (6) can be involved into fracture energy for the specific calculation, which means that the last term in the right side of equations (5) and (6) can be neglected. In contrasting, although the form of equation (6) and equation (1) seems similar with each other, the physical meaning related to the rupture process is the kind of difference. In equation (6), the integral upper limit of slip is final slip Δu^1 , but the integral upper limit of slip in equation (1) is an instantaneous accumulation slip D (the slip of each point on the fault). Therefore, by neglecting the surface energy or distributing it to the rupture energy E_G (see Figure 2) (Rivera and Kanamori, 2005), the apparent stress, σ_a , given by equation (6) is always greater than or equal to zero.

Figure 2 Simplified slip-weakening law. D_c is the critical slip weakening distance, σ_0 and σ_1 are initial stress and final stress in the fault, respectively. $\bar{\sigma}$ is the average stress of σ_0 and σ_1 , σ_f is the average stress of $\sigma(u)$ and σ_f in the total rupture and slip process. According to the energy distribution principle, the area of $\triangle ACD$, $\triangle ABC$ and rectangle $BEFD$ are seismic radiation energy E_S , fracture energy E_G , and heat, respectively.

With a comparison between equations (1) and (2), we know that the apparent stress, σ_a , in equation (1) perhaps negative in some times comes from an ambiguous description of the average stress $\bar{\sigma}$. In equation (1), the averaged stress is given by $\bar{\sigma} = (\sigma_0 + \sigma_1)/2$ in which σ_1 is a dynamic fractional stress and equals to the stress corresponding to final slip Δu^1 , but the integral upper limit D given in equation (1) is an instantaneous accumulation slip. As a result, when $D \leq D'$, $\int_0^D \sigma(u) du \geq \int_0^D \bar{\sigma}(\Delta u^1) du$, resultant to $\sigma_a \leq 0$. In order to solve the negative apparent stress σ_a appeared in equation (1), we change the integral upper limit of time t_1 to instantaneous time t ($0 \leq t \leq t_1$) according to equation (3). In this case, corresponding to time t , the integral upper limit of slip in equation (5) changes to D , and correspondingly, $\Delta u^1 = D$, $\sigma_1 = \sigma(D)$, $\bar{\sigma} = [\sigma_0 + \sigma(D)]/2$, respectively. After such modification, $\bar{\sigma} = [\sigma_0 + \sigma(D)]/2$ becomes an instantaneous averaged stress, which depends on the time or slip process. The radiated seismic energy can be calculated by:

$$\begin{aligned} E_S(D) &= \int_{x_0} dS \int_0^D \frac{1}{2} [\sigma_0 + \sigma(D)] du \\ &\quad - \int_{x_0} dS \int_0^D \sigma(u) du. \quad (7) \end{aligned}$$

From equation (7), we can see that, when $D \leq D_c$, we always have $\sigma(D) \geq \sigma_1$ and $\bar{\sigma} = [\sigma_0 + \sigma(D)]/2 \geq \sigma(D)$.

The integral of the first and second terms in the right side of equation (7) are shown in Figure 3. From Figure 3, we can see that, when $D \leq D_c$, $E_S=0$, while $D \geq D_c$, $E_S(D)$ increases linearly with slip D . When $\sigma(D)=\sigma_1$, the averaged stress of $\bar{\sigma}=[\sigma_0 + \sigma_1]/2$ becomes a constant. In this way, σ_a is always a constant. Therefore, equation (1) should be rewritten as

$$\sigma_a(D) = \frac{1}{D\Sigma_0} \int_{\Sigma_0} dS \int_0^D \left\{ \frac{1}{2} [\sigma_0 + \sigma(D)] - \sigma(u) \right\} du. \quad (8)$$

Figure 4 shows that the apparent stress in equation (8) increases with slip. Compared with Figure 1, the apparent stress σ_a in equation (8) is always greater than or equal to zero, and reaches to the maximum value in

Figure 3 Representation of equation (7) when $D < D_c$ (a) and $D > D_c$ (b). When $D < D_c$, the first term is the area of rectangle $BOFG$, while the second term is the area of trapezoid $AOF E$. Due to the area of $\triangle ABC$ equals to the area of $\triangle CEG$, the area of $BOFG$ and area of $AOF E$ are the same. In this case, E_S is always zero all the time; When $D > D_c$, the first term is the area of rectangle $BOFI$ too, while the second term is the area of $AOFEG$. Because the area of $\triangle ABC$ is equal to the area of $\triangle CGH$, the area of $BOFI$ is always larger than the area of $AOFEG$; resultant a net difference of area of $HGEI$. In this case, E_S is just the area of $HGEI$.

Figure 4 Time-dependent/Slip-dependent variations of apparent stress of σ_a with slip increase under a slip-weakening law. (a) A simplified slip-weakening model. When $D < D_c$, $\sigma_a=0$, and when $D \geq D_c$, The seismic energy is radiated from fault plane associated with fault rupture motion. In this case, σ_a becomes a constant on the fault. The instantaneous stresses are shown as by the dashed line; (b) A general slip-weakening model $\sigma(u)$ and the variation of σ_a showed as the solid line.

$D=D_c$. It is remarkable that, the radiated seismic energy from equation (8) is always conserved with $E_S=(\Delta u^1-D_c)(\sigma_0-\sigma_1)\Sigma_0/2$. From this study, we suggest that, in earthquake source parameter inversion, we should not only use apparent stress σ_a as a constraint, but also need to take the radiated seismic energy E_S as an additional con-

straint condition in order to improve the derived precision of D_c , the critical slip-weakening distance.

3 Conclusions

With the slip-weakening model, Madariaga (1977) first proposed the variation of apparent stress σ_a with

slip. In recent source parameter inversion, this formula has been used as a constrained condition to derive rupture speed variation and the critical slip distance distribution pattern on the fault for the 1992 M_w 7.3 Landers earthquake, California (Pulido and Irikura, 2000). In the description of apparent stress σ_a , Pulido and Irikura (2000) referred to that the apparent stress could be less than zero in which the negative value was considered as the dissipation energy and incorporated to the fraction energy (Madariaga, 1977). According to earthquake energy partition principle, we proposed an alternative way to take account of the apparent stress expression in order to avoid such a non-physical phenomenon. Although, the former description might have no influence to the solution of critical slip-weakening distance D_c , but it could often overestimate the radiated seismic energy and artificially add to certain uncertainty in source parameter estimation. Through a consideration of the average stress varying instantaneously with time or slip during fault rupture process, we have derived an alternative description of time-dependent/slip-dependent apparent stress formula based on the slip-weakening model without breaching physical principle. In addition, a new way used to estimate the critical slip-weakening distance

D_c and radiated seismic energy E_s simultaneously is proposed based on our study.

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