A note for the mechanism of high-frequency oscillation instability resulted from absorbing boundary conditions^{*}

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Abstract

In this paper the explanation of the mechanism of high-frequency oscillation instability resulted from absorbing boundary conditions is further improved. And we analytically prove the proposition that for one dimensional discrete model of elastic wave motion, the module of reflection factor will be greater than 1 in high frequency band when artificial wave velocity is greater than 1.5 times the ratio of discrete space step to discrete time step. Based on the proof, the frequency band in which instability occurs is discussed in detail, showing such high-frequency waves are meaningless for the numerical simulation of wave motion.

Key words: absorbing boundary conditions; high-frequency oscillation instability; wave equation; numerical simulation

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Introduction

The mechanism of high-frequency oscillation instability resulted from the absorbing boundary conditions is clarified by LIAO and LIU (1992), LIU (1989) as follows: the high frequency oscillation instability results from reflection and amplification of high-frequency waves, which are meaningless for numerical simulation of wave motion by finite elements or finite differences, and from repeated amplifications of multiple reflection of the waves within the finite computational region. Thus numerical simulation of wave motion can be realized by eliminating the instability occurred in the high frequency band through some approaches. Such principle has already been applied in anti-seismic analysis of large scale structure (CHEN *et al*, 1998). However, the clarification of the mechanism is based on a proposition that for the one dimensional discrete model of elastic wave motion the module of reflection factor will exceed 1 within a high frequency band when artificial wave velocity is greater than 1.5 times the ratio of discrete space step to discrete time step, which has not been proved analytically. In this paper the proposition will be proved. And the high frequency band which the oscillation instability occurs in will be discussed.

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1 Previous researches

The longitudinal wave propagation in a semi-infinite elastic rod is considered, lumped-mass finite element technique in the space domain and central difference in the time domain are used for discretization. Then the reflection factor R_j is given by (LIAO and LIU, 1992)

$$R_j = - \left\{ \frac{1 - d_l(\omega) \exp \left[-i2 \left(\frac{\omega}{\omega_u} \right) \arcsin \Delta \tau \right]}{1 - d_r(\omega) \exp \left[-i2 \left(\frac{\omega}{\omega_u} \right) \arcsin \Delta \tau \right]} \right\}^N \quad (1)$$

$$d_l(\omega) = t_1 + t_2 a(\omega) + t_3 a(\omega)^2; d_r(\omega) = t_1 + t_2 a(\omega)^{-1} + t_3 a(\omega)^{-2} \quad (2)$$

$$t_1 = \frac{(2-s)(1-s)}{2}, t_2 = s(2-s), t_3 = \frac{s(s-1)}{2}, s = \frac{c_a \Delta t}{\Delta x} \quad (3)$$

$$a(\omega) = \exp \left\{ 2i \arcsin \frac{\sin \left[\left(\frac{\omega}{\omega_u} \right) \arcsin \Delta \tau \right]}{\Delta \tau} \right\} \quad 0 \leq \omega \leq \omega_u \quad (4)$$

$$\omega_u = \frac{2}{\pi} \omega_N \arcsin \Delta \tau \quad (5)$$

$$\omega_N = \frac{\pi}{\Delta t} \quad (6)$$

$$\Delta \tau = \frac{c_0 \Delta t}{\Delta x} \quad (7)$$

$$\Delta \tau \leq 1 \quad (8)$$

where the integer N is called the transmitting order, Δt is the discrete space step, Δx is the discrete time step, c_0 is the wave velocity in the continuum, c_a is the artificial wave velocity, ω is circular frequency, ω_u is cut-off circular frequency, and ω_N is the Nyquist circular frequency. The CFL stability constraint is given by the inequality (8). LIAO and LIU (1992) give a proposition that for traveling waves within the band of $0 \leq \omega \leq \omega_u$, if the artificial wave velocity c_a satisfies

$$\frac{c_a}{c_0} > \frac{1.5}{\Delta \tau} \quad (9)$$

Then

$$|R_j| > 1 \quad (10)$$

The meaning of the proposition is that, the high frequency oscillation instability results from amplification of high-frequency waves, which are meaningless for numerical simulation of wave motion by finite elements or finite differences, reflected from the transmitting boundary, and from repeated amplifications of multiple reflections of the waves within the finite computational region. However, that proposition is only verified by numerical test. In this paper, for traveling waves

within the band of $0 \leq \omega \leq \omega_u$, we will prove that the inequality (9) is necessary and sufficient conditions for tenableness of inequality (10)

2 Necessary and sufficient conditions for $|R_J| > 1$

Equations (5~7) yield

$$2 \left(\frac{\omega}{\omega_u} \right) \arcsin \Delta \tau = \omega \Delta t$$

Recalling equation (1), the inequality of (10) becomes

$$\begin{aligned} & \sqrt{\{\operatorname{Re}[\exp(i\omega\Delta t) - d_1(\omega)]\}^2 + \{\operatorname{Im}[\exp(i\omega\Delta t) - d_1(\omega)]\}^2} \\ & > \sqrt{\{\operatorname{Re}[\exp(i\omega\Delta t) - d_R(\omega)]\}^2 + \{\operatorname{Im}[\exp(i\omega\Delta t) - d_R(\omega)]\}^2} \end{aligned} \quad (11)$$

We simplify equation (11) at first. Using equation (4), since $0 \leq \omega \leq \omega_u$, we have $|a(\omega)| = 1$. So that

$$a^{-1}(\omega) = a^*(\omega) \quad (12)$$

Where $a^*(\omega)$ is the complex conjugate of $a(\omega)$. Substituting equation (12) into equation (2), we find that

$$d_R = d_1^* \quad (13)$$

Thus

$$\operatorname{Re}[\exp(i\omega\Delta t) - d_1(\omega)] = \operatorname{Re}[\exp(i\omega\Delta t) - d_R(\omega)] \quad (14)$$

Combining equations (13), (14) and (11), we find

$$|\operatorname{Im}[\exp(i\omega\Delta t) - d_1(\omega)]| > |\operatorname{Im}[\exp(i\omega\Delta t) - d_1^*(\omega)]| \quad (15)$$

Since $\sin(\omega\Delta t) \geq 0$ under the condition that $0 \leq \omega \leq \omega_u$, the necessary and sufficient conditions for inequality (15) is

$$\operatorname{Im}[d_1(\omega)] < 0 \quad (16)$$

$$\operatorname{Im}[d_1(\omega)] = t_2 \operatorname{Im}[a(\omega)] + 2t_3 \operatorname{Re}[a(\omega)] \operatorname{Im}[a(\omega)] \quad (17)$$

$$\operatorname{Im}[a(\omega)] = \sin \left\{ 2 \arcsin \frac{\sin \left[\left(\frac{\omega}{\omega_u} \right) \arcsin \Delta \tau \right]}{\Delta \tau} \right\} \quad (18)$$

$$\operatorname{Re}[a(\omega)] = \cos \left\{ 2 \arcsin \frac{\sin \left[\left(\frac{\omega}{\omega_u} \right) \arcsin \Delta \tau \right]}{\Delta \tau} \right\} \quad (19)$$

For $\omega=0$ or $\omega=\omega_u$,

$$\operatorname{Re}[a(0)] = 1, \operatorname{Im}[a(0)] = 0, \operatorname{Re}[a(\omega_u)] = -1, \operatorname{Im}[a(\omega_u)] = 0$$

Hence

$$\text{Im}[d_1(0)] = \text{Im}[d_1(\omega_u)] = 0 \quad (20)$$

Form which, it is easily verified that equation (16) is not correct when taking $\omega=0$ or $\omega=\omega_u$.

Assuming $0 \leq \omega \leq \omega_u$, $\text{Im}[a(\omega)] > 0$. Equations (16) and (17) yield

$$t_2 + 2t_3 \text{Re}[a(\omega)] < 0$$

It follows from the expression for t_2 and t_3 that

$$\text{Re}[a(\omega)] < \frac{s-2}{s-1} \quad (21)$$

It is not difficult to verify that

$$\frac{d \text{Re}[a(\omega)]}{d\omega} < 0, \quad 0 < \omega < \omega_u$$

Also

$$\text{Re}[a(0)] = 1, \text{Re}[a(\omega_u)] = -1$$

Thus $\text{Re}[a(\omega)]$ is monotone decreasing function, where $0 < \omega < \omega_u$. For given s , if ω satisfy the inequality (21), we have $\omega^* < \omega < \omega_u$, where ω^* is a solution of following equation

$$\text{Re}[a(\omega)] = \frac{s-2}{s-1} \quad (22)$$

Here we define ω^* as demarcation circular frequency.

According the change law of $\text{Re}[a(\omega)]$ on the interval of $0 < \omega < \omega_u$, the right-side of equation (22) should satisfies that

$$-1 < \frac{s-2}{s-1} < 1$$

Then

$$s > 1.5 \quad (23)$$

Noting $s = c_a \Delta t / \Delta x = c_a \Delta \tau / c_0$, we obtain the inequality (9).

3 Frequency band in which the high frequency oscillation instability occurs

Assuming s satisfies the inequality (23) and substituting equation (19) into (22), we find

$$\frac{\omega^*}{\omega_u} = \frac{1}{\arcsin \Delta \tau} \left\{ \arcsin \left[\Delta \tau \sin \left(\frac{1}{2} \arccos \frac{s-2}{s-1} \right) \right] \right\} \quad (24)$$

Since $\omega^* = 2\pi c_0 / \lambda^*$, where λ^* is the wave length, we have

$$\frac{\Delta x}{\lambda^*} = \frac{1}{\pi \Delta \tau} \left\{ \arcsin \left[\Delta \tau \sin \left(\frac{1}{2} \arccos \frac{s-2}{s-1} \right) \right] \right\} \quad (25)$$

For $\Delta \tau = 1.0, 0.9, 0.8, 0.5$ and $\Delta \tau \rightarrow 0$, Figure 1 shows ω^* / ω_u as a function of s , Figure 2 shows $\Delta x / \lambda^*$ as a function of s . It can be seen from Figure 1 that ω^* / ω_u decreases while s increases, which means that the instability band of $\omega^* \leq \omega \leq \omega_u$ becomes wider. Assuming $s \leq 2$ and $\Delta \tau \leq 1$, we have

$1/2 \leq \omega^*/\omega_h \leq 1$. It means that the oscillation instability occurs in the frequency band that is meaningless for numerical simulation of wave motion. Because, for waves in the meaningful frequency band, their circular frequencies satisfy $0 < \omega < (1/3 \sim 1/4)\omega_h$ (LIAO, 2002), the numerical simulation of wave motion can be realized by eliminating the instability occurred in the high frequency band through some approaches. And it is showed from Figure 2 that the wavelength of the waves in the frequency band in which oscillation instability occurs contains only five elements. But for the waves that are meaningful for numerical simulation, their wavelengths contains 6~8 elements at least (LIAO, 2002).

Figure 1 ω^*/ω_h versus s at $\Delta\tau=1.0, 0.9, 0.8, 0.5$ and $\Delta\tau \rightarrow 0$

Figure 2 $\Delta x/\lambda^*$ versus s at $\Delta\tau=1.0, 0.9, 0.8, 0.5$ and $\Delta\tau \rightarrow 0$

4 Discussion and conclusions

In this paper the proposition presented in LIAO and LIU (1992) is proved. And the high frequency band which the oscillatory instability occurs in is illustrated. It should be noticed that the conclusions made in this paper are based on analysis on one dimensional model in frequency domain, it cannot be applied to multi-dimensional case directly. Also the analysis here does not include the impact of rounding error on stability. It is why in this paper we cannot explain the zero-frequency drift instability resulted from absorbing boundary conditions. Those will be discussed in incoming report.

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