Editorial

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The mathematical theory of the Navier–Stokes equations poses—since the pioneering work of Jean Leray in 1934—a famous open problem for instationary solutions in three dimensions: It is unknown whether the so-called weak solution which does exist globally in time is regular or smooth for all times if the initial data is regular and large. Analogously, it is open whether a regular solution which can be constructed on a sufficiently small time interval does exist globally in time. Due to its utmost importance for the theory of nonlinear partial differential equations and applications in fluid mechanics this open problem is one of the seven Millennium Prize Problems of *Clay Mathematics Institute*, formulated in 2000.

It is well-known that, given a domain $\Omega \subset \mathbb{R}^3$, a time interval $(0, T)$, an external force field $f \in L^2$ on $\Omega \times (0, T)$ and an initial value $u_0 \in L^2$, there exists at least one weak solution in the energy class (the Leray–Hopf class)

$$u \in L^\infty(0, T; L^2(\Omega)) \cap L^2_{\text{loc}}(0, T; H^1_0(\Omega))$$

of the instationary Navier–Stokes system. However, it is open whether u is a strong solution, e.g. in the sense that $u \in L^\infty(0, T; H^1_0(\Omega)) \cap L^2(0, T; H^2(\Omega))$ or even $u \in C^\infty(\overline{\Omega} \times (0, T))$ under suitable assumptions on u_0 , f and $\partial\Omega$. Up to now, this

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result can be proved only under additional assumptions on u (or p or on other quantities). The classical result on *conditional regularity* is due to James Serrin (1962/1963), requiring that

$$u \in L^s(0, T; L^q(\Omega)), \quad \frac{2}{s} + \frac{3}{q} = 1, \quad s > 2, \quad q > 3.$$

Since then Serrin's condition has been generalized to the limit cases $s = 2, q = \infty$ and, more recently, $s = \infty, q = 3$, as well as to related conditions on ∇u , on specific components of either u or ∇u .

Of special interest is the vorticity $\omega = \operatorname{curl} u$. The nonlinear term $u \cdot \nabla u$ can be written in the form $u \cdot \nabla u = \omega \times u + \nabla(\frac{1}{2}|u|^2)$ so that $\frac{1}{2}|u|^2$ may be considered as part of the pressure p . Actually, in most results on the Navier–Stokes system the term $u \cdot \nabla u$ is directly estimated ignoring this special decomposition. On the other hand, ω satisfies the *vorticity transport equation*

$$\partial_t \omega - \nu \Delta \omega + u \cdot \nabla \omega - \omega \cdot \nabla u = \operatorname{curl} f,$$

where the term $\omega \cdot \nabla u$ prevents the application of a maximum principle and may lead to the phenomenon of *vortex stretching*, a local increase of $|\omega|$ when certain geometric conditions on ω and u are satisfied.

Additional problems occur in the analysis of viscous fluid flow around rotating obstacles. Assume that a compact obstacle $K \subset \mathbb{R}^3$ is rotating around a fixed axis of rotation $w = (0, 0, w_3)$ with angular velocity $|w| = w_3$. If K is not axially symmetric with respect to w , the domain $\Omega(t)$ occupied by the fluid is changing in time. Then the change to a coordinate system attached to the rotating body yields the modified Navier–Stokes equation

$$\partial_t u - \nu \Delta u + u \cdot \nabla u - (w \times x) \cdot \nabla u + w \times u + \nabla p = f. \quad (1)$$

The additional term $w \times u$ represents the Coriolis force, whereas the term $(w \times x) \cdot \nabla u$ is increasing as $|(x_1, x_2)| \rightarrow \infty$ and is not subordinate to the Laplacian. Actually, this latter term adds a hyperbolic effect to the parabolic Navier–Stokes system, and the semi-group generated by the Stokes-type operator $A_w u := P[-\nu \Delta u - (w \times x) \cdot \nabla u + w \times u]$, where P denotes the Helmholtz projection, is no longer analytic.

Recent progress on the above-mentioned and also other related topics was intensively discussed during the conference

Vorticité, Rotation et Symétrie—Stabilité des Écoulements

held at the *Centre International de Rencontres Mathématiques* (CIRM) in Luminy (Marseille), May 19–23, 2008. Several talks concerned the open problem of regularity of weak solutions to the Navier–Stokes system in three spatial dimensions and discussed e.g. criteria based on vorticity. Moreover, the set of initial values to guarantee the existence of regular solutions in general or with other specific properties was analyzed. Further talks have shown that the influence of the boundary itself and of

the choice of boundary conditions is crucial to prove local and/or global regularity of weak solutions.

The second main topic concerned the flow around a single or several rotating obstacles—even including translational motion. Special emphasis was put on the interaction of the fluid with rigid bodies which are allowed to move freely with the flow and may approach each other or the wall in finite or only infinite time. These results depend on the rheology of the fluid and on the shape of the bodies.

Furthermore, questions related to compressible and heat-conductive fluid flow were an important part of the conference, too. In particular, the incompressible limit of a compressible fluid flow was analyzed from different points of view. Finally, the conference also included lectures on fluid flows in unbounded domains—this topic deserves particular attention because the analysis in unbounded domains requires special tools such as weighted estimates or the careful choice of function spaces adapted to the considered problem and—for numerical computations—approximation of the domain by an appropriate bounded domain.

Besides 28 short communications given by specialists on mathematical fluid mechanics from all over the world, the conference was inspired by nine invited lecturers:

C. Bardos (Paris)
 H. Beirão da Veiga (Pisa), represented by L. Berselli (Pisa)
 Y. Chemin (Paris)
 H. J. Choe (Seoul)
 E. Feireisl (Prague)
 M. Hieber (Darmstadt)
 H. Kozono (Sendai)
 H. Sohr (Paderborn)
 I. Straškraba (Prague)

The organizers would like to thank all speakers of the conference and the staff at the conference site in Luminy—without them the conference would not have been possible. Last but not least, we are greatly indebted to the Managing Editor of the *Annali dell'Università di Ferrara*, Prof. Dr. Mariarosaria Padula, for giving us the possibility to publish the proceedings of the conference in a special volume of this journal.

We note that in addition to the papers published in this volume, the following articles published in vol. 55, no. 1 (2009) of this journal, are also related to the conference:

Pierre-Etienne Druet and Joachim Naumann: On the existence of weak solutions to a stationary one-equation RANS model with unbounded eddy viscosities
 Reinhard Farwig, Hermann Sohr and Werner Varnhorn: On optimal initial value conditions for local strong solutions of the Navier–Stokes equations

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