The relation between Pauli and γ representation in QED₃ and induced Chern-Simons term

SHI Song¹, SUN WeiMin^{1,2,3} & ZONG HongShi^{1,2,3*}

¹Department of Physics, Nanjing University, Nanjing 210093, China;

²Joint Center of Particle, Nuclear Physics and Cosmology, Nanjing University and PMO, Nanjing 210093, China;

³State Key Laboratory of Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190, China

Received May 27, 2013; accepted June 28, 2013

In this letter, we propose to introduce a new Abelian gauge field B_μ corresponding to the so-called β symmetry into the normal Quantum Electrodynamics in 2+1 dimensions (QED₃) of γ representation. The resulting theory is shown to be equivalent to QED₃ containing two flavors of two-component fermions with mass of an opposite sign. We also show that B_μ field can generate a Chern-Simons term in perturbation theory. A comparison is made between the induced Chern-Simons term in the γ representation and that in Pauli representation.

QED₃, gauge field, Chern-Simons term

Citation: Shi S, Sun W M, Zong H S. The relation between Pauli and γ representation in QED₃ and induced Chern-Simons term. Chin Sci Bull, 2013, 58: 3735–3737, doi: 10.1007/s11434-013-6014-9

Quantum Electrodynamics in 2+1 dimensions (QED₃) has been extensively studied for many years. It has many features similar to Quantum Chromodynamics (QCD), such as dynamical chiral symmetry breaking (DCSB) in the massless fermion limit and confinement [1–4]. When studying DCSB in QED₃, one has to use four-component fermions with γ matrices [5–14], since the model with two-component fermions with Pauli matrices does not have chiral properties due to the property of Pauli matrix algebra. On the other hand, Chern-Simons theory has induced many significative studies in QED₃ [15–18]. Besides introducing Chern-Simons term into the normal QED₃ Lagrangian, it was also found that in QED₃ with two-component fermions a Chern-Simons like term can also be induced from radiative corrections [19–25] (Appendix A). However, this result does not hold for QED₃ with four-component fermions due to the property of γ matrix algebra. We note that γ matrices are defined on the basis of Pauli matrices. Except the algebraic connection, one can hardly find any connection between QED₃ with four-component fermions and that with two-component fermions. In this letter, we propose to introduce a new Abelian gauge

field B_μ into the normal QED₃ theory to establish such a connection, and also prove the equivalence between the resulting theory and QED₃ containing two flavors of two-component fermions with mass of an opposite sign. After introducing B_μ field into the theory, we find that a Chern-Simons like term can be induced in QED₃ with four-component fermions.

The Lagrangian of QED₃ with one flavor of four-component fermion is

$$\mathcal{L} = i\bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi - e\bar{\psi}\not{A}\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}. \quad (1)$$

In this expression, the γ matrices are defined as

$$\gamma^\mu = \begin{pmatrix} \sigma^\mu & 0 \\ 0 & -\sigma^\mu \end{pmatrix}, \quad \sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\sigma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Now we introduce a new matrix β

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

and it can be readily checked that β commutes with γ_μ as

*Corresponding author (email: zonghs@chenwang.nju.edu.cn)

$$\beta\gamma_\mu = \gamma_\mu\beta. \quad (2)$$

Therefore, the Lagrangian (1) has a global symmetry corresponding to the transformation $\psi' = e^{i\theta}\psi$. We call it β symmetry. Now we try to promote this β symmetry to be a local symmetry. However, we note that the original Lagrangian (1) does not have this local β symmetry, because one cannot give A_μ an appropriate transformation rule to guarantee this. In order to make our theory have both local β symmetry and the original gauge symmetry, we introduce a new Abelian gauge field B_μ into the Lagrangian, so that

$$\mathcal{L} = i\bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi - e\bar{\psi}A\psi - e\bar{\psi}B\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{4}G^{\mu\nu}G_{\mu\nu}, \quad (3)$$

where $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$. Under the normal local gauge transformation, the fields transform as

$$\begin{aligned} \psi'(x) &= e^{i\theta(x)}\psi(x), \quad \bar{\psi}'(x) = \bar{\psi}(x)e^{-i\theta(x)}, \\ A'_\mu(x) &= A_\mu(x) - \frac{1}{e}\partial_\mu\theta(x), \quad B'_\mu(x) = B_\mu(x), \end{aligned}$$

while under the local β transformation, the fields transform as

$$\begin{aligned} \psi'(x) &= e^{i\beta\theta(x)}\psi(x), \quad \bar{\psi}'(x) = \bar{\psi}(x)e^{-i\beta\theta(x)}, \\ A'_\mu(x) &= A_\mu(x), \quad B'_\mu(x) = B_\mu(x) - \frac{1}{e}\partial_\mu\theta(x). \end{aligned}$$

It can be seen that the Lagrangian (3) has both normal gauge symmetry and local β symmetry.

Then we give some definition as $\psi = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}$, $\bar{\psi}^1 = \psi^{1\dagger}\sigma^0$

and $\bar{\psi}^2 = \psi^{2\dagger}\sigma^0$. So $\bar{\psi} = (\bar{\psi}^1 \quad \bar{\psi}^2)$. We also define $A_\mu^1 = A_\mu + B_\mu$, $A_\mu^2 = A_\mu - B_\mu$, $F_{\mu\nu}^1 = \partial_\mu A_\nu^1 - \partial_\nu A_\mu^1$ and $F_{\mu\nu}^2 = \partial_\mu A_\nu^2 - \partial_\nu A_\mu^2$. Putting these definition into Lagrangian (3), we can rewrite it as

$$\begin{aligned} \mathcal{L} &= i\bar{\psi}^1\sigma^\mu\partial_\mu\psi^1 + i\bar{\psi}^2\sigma^\mu\partial_\mu\psi^2 - m\bar{\psi}^1\psi^1 + m\bar{\psi}^2\psi^2 - e\bar{\psi}^1\sigma^\mu A_\mu^1\psi^1 \\ &\quad - e\bar{\psi}^2\sigma^\mu A_\mu^2\psi^2 - \frac{1}{4}F_{\mu\nu}^1F^{1\mu\nu} - \frac{1}{4}F_{\mu\nu}^2F^{2\mu\nu}. \end{aligned} \quad (4)$$

The Lagrangian (4) resembles the Lagrangian of a two-flavor QED₃ theory with Pauli representation. So here we shall give a simple discussion about two-flavor QED₃. The Lagrangian for two-flavor QED₃ is as follows:

$$\mathcal{L} = i\bar{\psi}\sigma^\mu\otimes I\partial_\mu\psi - \bar{\psi}I\otimes M\psi - \sum_i e\bar{\psi}\sigma^\mu\otimes T^i A_\mu^i\psi - \sum_i \frac{1}{4}F_{\mu\nu}^iF^{i\mu\nu}. \quad (5)$$

In the Lagrangian (5), M represents the mass matrix $\text{diag}(m_1, m_2)$. When $m_1 = m_2$, this Lagrangian has a $U(1) \otimes SU(2)$ symmetry, while for $m_1 \neq m_2$ the symmetry group will be $e^{i\theta_1} \otimes e^{i\sigma^0\theta_2}$. In Lagrangian (4), $m_1 = m$ and $m_2 = -m$, so there should be two gauge fields. Note that in Lagrangian (5), $\bar{\psi}$ is defined as $\bar{\psi} = (\bar{\psi}^1 \quad \bar{\psi}^2)$, while in Lagrangian (3) $\bar{\psi}$ is defined as $(\bar{\psi}^1 \quad -\bar{\psi}^2)$. Inserting $\beta\beta = I_4$ into Lagrangian (5), we will obtain Lagrangian (3). This proves that after adding another appropriate gauge field, the QED₃ in γ representation is

equivalent to a two-flavor QED₃ in Pauli representation. After proving this equivalence, we can easily find out the physical meaning of the conserved charges corresponding to the above two gauge symmetries. For the normal gauge symmetry, the conserved current is

$$j^\mu = \bar{\psi}\gamma^\mu\psi. \quad (6)$$

The corresponding conserved charge is

$$Q = \int d^2x j^0 = \int d^2x \psi^{1\dagger}\psi^1 + \int d^2x \psi^{2\dagger}\psi^2 = N_1 + N_2. \quad (7)$$

For the β symmetry, the conserved current is

$$j_\beta^\mu = \bar{\psi}\beta\gamma^\mu\psi. \quad (8)$$

The corresponding conserved charge is

$$Q_\beta = \int d^2x j_\beta^0 = \int d^2x \psi^{1\dagger}\psi^1 - \int d^2x \psi^{2\dagger}\psi^2 = N_1 - N_2. \quad (9)$$

From eqs. (7) and (9), it is easily seen that the conservation of Q and Q_β is equivalent to the conservation of two types of fermion number N_1 and N_2 .

Normally, the induced Chern-Simons term appears in QED₃ of Pauli representation because of the identity $\text{Tr}[\sigma^\mu\sigma^\nu\sigma^\lambda] = -2i\varepsilon^{\mu\nu\lambda}$. For γ matrices the corresponding identity is $\text{Tr}[\gamma^\mu\gamma^\nu\gamma^\lambda] = 0$. Therefore, it appears that a Chern-Simons term will not be induced in QED₃ of γ representation. However, one can show that a Chern-Simons term can be induced from the modified Lagrangian (3). For example, let us consider the lowest order term of the correlation function of A and B in the perturbation theory. Its expression is

$$\begin{aligned} S_{AB}^{(2)} &= e^2 \int d^3x d^3y \bar{\psi}(x)A(x)\psi(x)\bar{\psi}(y)B(y)\psi(y) \\ &= -e^2 \int d^3x d^3y A_\mu(x)B_\nu(y) \text{Tr}[\beta\gamma^\mu S(x-y)\gamma^\nu S(y-x)] \\ &= -me^2 \int d^3k_1 \delta(k_1 + k_2) \varepsilon^{\mu\nu\lambda} A_\mu(k_1)B_\nu(-k_1)k_{1\lambda} \\ &\quad \times \int d^3k \Pi(k, k_1). \end{aligned} \quad (10)$$

This is a radiative Chern-Simons term just as in the Pauli representation case.

Now we will study a massless QED₃ theory with a Pauli-Villars regularization mass term in the four-component representation

$$\mathcal{L} = i\bar{\psi}\not{\partial}\psi - e\bar{\psi}A\psi + g\bar{\psi}\beta B\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{4}G^{\mu\nu}G_{\mu\nu} - M\bar{\psi}\psi. \quad (11)$$

Then, similar to eq. (10), we can have an induced Chern-Simons-like term from the single fermion loop as follows:

$$\begin{aligned} S_{CS} &= -8iegM\varepsilon^{\mu\nu\lambda} \int dp A_\mu(p)B_\nu(-p)p_\lambda \int dk \Pi(k, p) \\ &= -i\frac{2}{\pi}eg\varepsilon^{\mu\nu\lambda} \int dp A_\mu(p)B_\nu(-p)p_\lambda \frac{M}{|\vec{p}|} \\ &\quad \times \arcsin\left(\frac{|\vec{p}|}{\sqrt{p^2 + M^2}}\right). \end{aligned} \quad (12)$$

Note that in the limit $M \rightarrow \infty$, we have

$$\frac{M}{|\vec{p}|} \arcsin\left(\frac{|\vec{p}|}{\sqrt{p^2 + M^2}}\right) \sim \frac{1}{2} \frac{M}{|M|} + O\left(\frac{p^2}{M^2}\right). \quad (13)$$

Using this asymptotic relation in eq. (12), we find that in the limit $M \rightarrow \infty$ the induced Chern-Simons-like term becomes

$$S_{\text{CS}} = -\frac{i}{\pi} \frac{M}{|M|} e g \varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu B_\lambda. \quad (14)$$

Now we define a parity transformation of QED₃ in γ representation as

$$\tilde{x}^{0,2} = x^{0,2}, \quad \tilde{x}^1 = -x^1.$$

To ensure parity invariance of the fermion kinetic term, one requires

$$\psi(\tilde{x}) = \gamma^1 \psi(x). \quad (15)$$

To ensure parity invariance of the interaction terms, one takes the following transformation law of the gauge fields A_μ and B_μ under the parity transformation

$$\begin{aligned} A_{0,2}(\tilde{x}) &= A_{0,2}(x), & A_1(\tilde{x}) &= -A_1(x), \\ B_{0,2}(\tilde{x}) &= B_{0,2}(x), & B_1(\tilde{x}) &= -B_1(x). \end{aligned} \quad (16)$$

From the above relations, we can see that only the mass term breaks parity invariance. So, massless QED₃ is invariant under parity. But after we have considered Pauli-Villars regularization, it can cause an induced and mass-independent Chern-Simons-like term (eq. (14)) which gives a parity anomaly in four-component representation.

In this letter we propose to introduce a new Abelian gauge field B_μ corresponding to the so-called β symmetry into the normal QED₃ of γ representation. The resulting theory is shown to be equivalent to QED₃ containing two flavors of two-component fermions with mass of an opposite sign.

Comparing eq. (10) with eq. (A3), one sees that the ordinary Chern-Simons term contains one $U(1)$ gauge field, whereas the Chern-Simons term in eq. (10) contains two different $U(1)$ gauge fields. Hence, we believe that these two representations are not quite the same.

We also show that B_μ field can generate a Chern-Simons term in perturbation theory. There are still some open problems left, e.g., whether this equivalence holds at a nonperturbative level, whether it holds at finite temperature, whether

the β symmetry holds nonperturbatively, and what is the character of the induced Chern-Simons term at finite temperature and density. We shall study these problems in our future works.

This work was supported in part by the National Natural Science Foundation of China (10935001 and 11075075) and the Research Fund for the Doctoral Program of Higher Education (2012009111002).

- 1 Roberts C D, Williams A G. Prog Part Nucl Phys, 1994, 33: 477–575, and references therein
- 2 Roberts C D, Schmidt S M. Prog Part Nucl Phys, 2000, 45: 1–103, and references therein
- 3 Zhao A M, Zong H S, Sun W M. Sci China Phys Mech Astron, 2012, 55: 2049–2056
- 4 Zhu H X, Wang H J, Pan D F, et al. Sci China Phys Mech Astron, 2012, 55: 2425–2433
- 5 Pisarski R D. Phys Rev D, 1984, 29: 2423–2426
- 6 Appelquist T W, Bowick M, Karabali D, et al. Phys Rev D, 1986, 33: 3704–3713
- 7 Appelquist T, Nash D, Wijewardhana L C R. Phys Rev Lett, 1988, 60: 2575–2578
- 8 Atkinson D, Johnson P W, Maris P. Phys Rev D, 1990, 42: 602–609
- 9 Gusynin V P, Hams A H, Reenders M. Phys Rev D, 1996, 53: 2227–2235
- 10 Fischer C S, Alkofer R, Dahm T, et al. Phys Rev D, 2004, 70: 073007
- 11 Li W, Liu G Z. Phys Rev D, 2010, 81: 045006
- 12 Feng H T, He M, Sun W M, et al. Phys Lett B, 2010, 688: 178–184
- 13 Feng H T, Yin P L, Sun W M, et al. Sci China Phys Mech Astron, 2013, 56: 1116–1119
- 14 Feng H T, Wang B, Sun W M, et al. Eur Phys J C, 2013, 73: 2444
- 15 Chern S S, Simons J. Ann Math, 1974, 99: 48–69
- 16 Deser S, Jackiw R, Templeton S. Ann Phys, 1982, 140: 372–411
- 17 Dunne G V. Les Houches-Ecole d'Ete de Physique Theorique, 1999, 69: 177–263
- 18 Kondo K I, Maris P. Phys Rev D, 1995, 52: 1212–1233
- 19 Babu K S, Das A, Panigrahi P. Phys Rev D, 1987, 36: 3725–3730
- 20 Aitchison I J R, Fosco C D. Phys Rev D, 1998, 57: 1171–1177
- 21 Kao Y C, Yang M F. Phys Rev D, 1993, 47: 730–733
- 22 Redlich A N. Phys Rev D, 1984, 29: 2366–2374
- 23 Aitchison I J R, Fosco C D. Phys Rev D, 1993, 48: 5895–5904
- 24 Fosco C D, Rossini G L, Schaposnik F A. Phys Rev Lett, 1997, 79: 1980–1983
- 25 Fosco C D, Rossini G L, Schaposnik F A. Phys Rev D, 1997, 56: 6547–6555

Open Access This article is distributed under the terms of the Creative Commons Attribution License which permits any use, distribution, and reproduction in any medium, provided the original author(s) and source are credited.

Supporting Information

Appendix A Induced Chern-Simons term in Pauli representation of QED₃

The supporting information is available online at csb.scichina.com and www.springerlink.com. The supporting materials are published as submitted, without typesetting or editing. The responsibility for scientific accuracy and content remains entirely with the authors.