

## Characterization of four-qubit states via Bell inequalities

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A set of Bell inequalities classifying the quantum entanglement of four-qubit states is presented. These inequalities involve only two measurement settings per observer and can characterize fully separable, bi-separable and tri-separable quantum states. In addition, a quadratic inequality of the Bell operators for four-qubit systems is derived.

### Bell inequalities, separability, Bell operators

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The Bell inequality [1] provided the first possibility to distinguish experimentally between quantum-mechanical predictions and those of local realistic models. Derivations of new and stronger Bell inequalities are one of the most important and challenging subjects in quantum information processing. Since Bell's work, there were many important generalizations such as [2–11] and references therein.

The Bell inequalities presented in [12] involve only two measurement settings per observer and can detect perfectly the quantum entanglement of the generalized GHZ states. By using the idea in constructing Bell operators [12], a set of new Bell inequalities are given in [13], which gives rise to a finer classification of the entanglement for three-qubit systems.

The entanglement of four-qubit systems has been treated in terms of Bell inequalities of Mermin-Klyshko type. In [14] the quantum nonlocality of some four-qubit states, the GHZ state, W state, cluster state and the state proposed in [15], has been investigated, towards the optimal violations of the Bell inequality for these states. The classification of entanglement has been also studied in such as [16–19] with linear inequalities for qubit systems and [20] with non-linear inequalities for detecting bi-separable states in arbitrary dimensional quantum systems.

In this work, we study the quantum entanglement of four-

qubit systems by using the idea in constructing Bell operators in [12]. We generalize the results of three-qubit systems in [13] to four-qubit systems. It has been shown that the standard Werner-Wolf-Żukowski-Brukner (WWZB) inequalities cannot detect the entanglement of the generalized Greenberger-Horne-Zeilinger (GHZ) states given by  $|\psi\rangle = \cos\alpha|0, \dots, 0\rangle + \sin\alpha|1, \dots, 1\rangle$  with  $0 \leq \alpha \leq \pi/4$  [21, 22]. However, the Bell operators constructed in the way provided in [12] can detect the entanglement of the generalized GHZ state wholly. Our Bell operators are constructed by using the idea in [12]. The resulted Bell inequalities can distinguish fully separable, bi-separable and tri-separable states of a four-qubit system. Moreover, these linear Bell inequalities involve only two measurement settings per observer. Analytical formulas of the average values of the Bell operators for four-qubit systems are also derived. And a quadratic inequality of the Bell operators for all four-qubit systems has been presented. Explicit geometrical pictures show the relations between the different types of quantum entanglement and the violations of the inequalities.

We fix some notations used in this work. We use the Dirac's symbols throughout this paper. If a quantum system is in one of a number of states  $|\psi\rangle_i$ , where  $i$  is an index, with respective probabilities  $p_i$ , then  $\{p_i, |\psi\rangle_i\}$  is called an ensemble of pure states, and the associated density operator for the system is defined by  $\rho = \sum_i p_i \rho_i$ . The average value of the

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observable  $M$  is written  $\langle M \rangle = \langle \psi | M | \psi \rangle$ .

### 1 Classification of four qubits with Bell inequalities

Consider  $N$  parties and allow each of them to choose independently between two dichotomic observables  $A_j, B_j$  for the  $j$ -th observer, where  $A_j = \vec{a}_j \cdot \vec{\sigma}^j$  and  $B_j = \vec{b}_j \cdot \vec{\sigma}^j$ , with  $\vec{\sigma}^j = (\sigma_1^{(j)}, \sigma_2^{(j)}, \sigma_3^{(j)})$  the Pauli matrices on the  $j$ -th qubit, and  $\vec{a}_j = (a_j^{(1)}, a_j^{(2)}, a_j^{(3)})$ ,  $\vec{b}_j = (b_j^{(1)}, b_j^{(2)}, b_j^{(3)})$  the real unit vectors. The quantum mechanical Bell operator on the  $N - 1$  qubits except for the  $i$ -th qubit is defined as [12]

$$D_N^{(i)} = B_{N-1}^{(i)} \otimes \frac{1}{2}(A_i + B_i) + I_{N-1} \otimes \frac{1}{2}(A_i - B_i), \quad i = 1, \dots, N,$$

where  $B_{N-1}^{(i)}$  is the Bell operator of WWZB inequalities on the  $N - 1$  qubits except for the  $i$ -th qubit,

$$B_{N-1}^{(i)} = \frac{1}{2^{N-1}} \sum_{S_1, \dots, S_{N-1} = -1, 1} S(S_1, \dots, S_{N-1}) \sum_{K_1, \dots, K_{N-1} = 1, 2} S_1^{K_1-1} \dots S_{N-1}^{K_{N-1}-1} \otimes_{j=1}^{N-1} O_j(K_j),$$

where  $I_{N-1}$  denotes the corresponding identity matrix. For  $S(S_1, \dots, S_{N-1}) = \sqrt{2} \cos(\frac{\pi}{4}(S_1 + \dots + S_{N-1} - N) - \frac{\pi}{4})$ ,  $O_j(1) = A_j$ ,  $O_j(2) = B_j$ , one recovers the Mermin-Ardehali-Belinskii-Klyshko (MABK) inequalities [2].

In the following we study the characterization of entanglement for four-qubit systems,  $N = 4$ .

**Theorem 1.** For fully separable states  $\rho$ , we have

$$|\langle D_4^{(i)} \rangle| \leq 1, \quad i = 1, 2, 3, 4. \tag{1}$$

**Proof.** A general pure four-qubit state can be written as  $|\psi\rangle = \sum_{i,j,k,l=0}^1 a_{ijkl} |ijkl\rangle$  with normalization  $\sum_{i,j,k,l=0}^1 |a_{ijkl}|^2 = 1$ . A mixed four-qubit state can be expressed as  $\rho = \sum_{\alpha} p_{\alpha} \rho_{\alpha}$ , where  $0 < p_{\alpha} \leq 1$ ,  $\sum_{\alpha} p_{\alpha} = 1$ ,  $\rho_{\alpha} = |\psi_{\alpha}\rangle\langle\psi_{\alpha}|$  are pure states. Due to the linear property of the average values,

$$|\langle D_4^{(i)} \rangle| = |\text{tr}(\sum_{\alpha} p_{\alpha} \rho_{\alpha} D_4^{(i)})| = |\sum_{\alpha} p_{\alpha} \text{tr}(\rho_{\alpha} D_4^{(i)})| \leq \sum_{\alpha} |p_{\alpha} \text{tr}(\rho_{\alpha} D_4^{(i)})| \leq \sum_{\alpha} |\text{tr}(\rho_{\alpha} D_4^{(i)})|,$$

it is sufficient to consider the pure states. The fully separable pure states can be transformed into the form  $|\psi\rangle = |0000\rangle$  in suitable bases. Therefore it is direct to verify that

$$|\langle D_4^{(1)} \rangle| = \frac{1}{4}((a_1^{(3)} + b_1^{(3)})(-a_2^{(3)} a_3^{(3)} a_4^{(3)} + a_2^{(3)} b_3^{(3)} b_4^{(3)} + b_2^{(3)} a_3^{(3)} b_4^{(3)} + b_2^{(3)} b_3^{(3)} a_4^{(3)}) + \frac{1}{2}(a_1^{(3)} - b_1^{(3)}) \leq 1.$$

Similarly one can prove that  $|\langle D_4^{(i)} \rangle| \leq 1$  for  $i = 2, 3, 4$ .  $\square$

Next we consider the cases of tri-separable states. We denote  $\rho_{ij-k-l}$  a tri-separable state of the form  $\rho_{ij} \otimes \rho_k \otimes \rho_l$ , in

which qubits  $i$  and  $j$  are entangled, while qubits  $k$  and  $l$  are separable,  $i \neq j \neq k \neq l = 1, 2, 3, 4$ .

**Theorem 2.** For any tri-separable states  $\rho_{ij-k-l}$ ,  $i \neq j \neq k \neq l = 1, 2, 3, 4$ , we have

$$|\langle D_4^{(i)} \rangle| = |\langle D_4^{(j)} \rangle| \leq 1, \quad |\langle D_4^{(k)} \rangle| = |\langle D_4^{(l)} \rangle| \leq \frac{3}{2}. \tag{2}$$

**Proof.** We consider the case of  $\rho_{12-3-4}$ . Every pure state in  $\rho_{12-3-4}$  can be written in a Schmidt form,  $|\psi\rangle = (\cos \alpha |01\rangle - \sin \alpha |10\rangle) \otimes |0\rangle \otimes |0\rangle$ . Therefore

$$\begin{aligned} |\langle D_4^{(1)} \rangle_{|\psi\rangle}| &= \left| \frac{1}{2} \left[ - (a_1^{(3)} a_2^{(3)} + a_2^{(3)} b_1^{(3)}) \right. \right. \\ &\quad \left. \left. - \left( \sum_{k=1}^2 a_1^{(k)} a_2^{(k)} + a_2^{(k)} b_1^{(k)} \right) \sin 2\alpha \right] \right. \\ &\quad \left. \times \frac{1}{2} (-a_3^{(3)} a_4^{(3)} + b_3^{(3)} b_4^{(3)}) \right. \\ &\quad \left. + \frac{1}{2} \left[ - (a_1^{(3)} b_2^{(3)} + b_1^{(3)} b_2^{(3)}) \right. \right. \\ &\quad \left. \left. - \left( \sum_{k=1}^2 a_1^{(k)} b_2^{(k)} + b_1^{(k)} b_2^{(k)} \right) \sin 2\alpha \right] \right. \\ &\quad \left. \times \frac{1}{2} (a_3^{(3)} b_4^{(3)} + b_3^{(3)} a_4^{(3)}) \right. \\ &\quad \left. + \frac{1}{2} (a_1^{(3)} - b_1^{(3)}) \cos 2\alpha \right| \leq 1. \end{aligned}$$

Similarly one can prove that  $|\langle D_4^{(2)} \rangle_{|\psi\rangle}| \leq 1$ . For the Bell operator  $D_4^{(3)}$ , we have

$$\begin{aligned} |\langle D_4^{(3)} \rangle_{|\psi\rangle}| &= \left| \frac{1}{4} \left\{ \left[ - (a_1^{(3)} a_2^{(3)} + b_1^{(3)} b_2^{(3)}) \right. \right. \right. \\ &\quad \left. \left. - \left( \sum_{k=1}^2 -a_1^{(k)} a_2^{(k)} + b_1^{(k)} b_2^{(k)} \right) \sin 2\alpha \right] [(a_3^{(3)} + b_3^{(3)}) a_4^{(3)}] \right. \\ &\quad \left. + \left[ - (a_1^{(3)} b_2^{(3)} + b_1^{(3)} a_2^{(3)}) \right. \right. \\ &\quad \left. \left. - \left( \sum_{k=1}^2 a_1^{(k)} b_2^{(k)} + b_1^{(k)} a_2^{(k)} \right) \sin 2\alpha \right] [(a_3^{(3)} + b_3^{(3)}) b_4^{(3)}] \right\} \right. \\ &\quad \left. + \frac{1}{2} (a_3^{(3)} - b_3^{(3)}) \right| \leq \frac{3}{2}. \end{aligned}$$

In a similar way, we have  $|\langle D_4^{(4)} \rangle_{|\psi\rangle}| \leq \frac{3}{2}$ . The cases of  $\rho_{1-2-34}$ ,  $\rho_{13-2-4}$ ,  $\rho_{1-3-24}$ ,  $\rho_{14-2-3}$  and  $\rho_{1-4-23}$  can be similarly proved. According to the linear property of average values, for all tri-separable states we have (2).  $\square$

Finally we consider the cases of bi-separable states. There are two classes of bi-separable ones. (i) Two entangled qubits  $i$  and  $j$  are separable from other entangled qubits  $k$  and  $l$ . For example, we denote  $\rho_{12-34}$  the bi-separable state of the form  $\rho_{12} \otimes \rho_{34}$ , where the qubits 12 and 34 are entangled respectively. (ii) A qubit  $i$  is separable from the rest genuine tripartite entangled qubits  $j, k$  and  $l$ . For instance,  $\rho_{1-234}$  denotes a bi-separable state of the form  $\rho_1 \otimes \rho_{234}$ , where qubits 234 are genuine entangled.

**Theorem 3.** For all bi-separable states  $\rho$ , we have

$$|\langle D_4^{(i)} \rangle_\rho| \leq \frac{3}{2}, \quad i = 1, 2, 3, 4, \quad (3)$$

for  $\rho$  in class i), and

$$|\langle D_4^{(i)} \rangle_\rho| \leq \sqrt{3}, \quad i = 1, 2, 3, 4, \quad (4)$$

for  $\rho$  in class ii).

**Proof.** We first consider bi-separable states in class i). A pure state in  $\rho_{12-34}$  has the following general form,  $|\psi\rangle = (\cos\alpha|01\rangle - \sin\alpha|10\rangle) \otimes (\cos\beta|01\rangle - \sin\beta|10\rangle)$ . Hence

$$\begin{aligned} |\langle D_4^{(1)} \rangle_{|\psi\rangle}| &= \left| \frac{1}{4} \left\{ \left[ - (a_1^{(3)} a_2^{(3)} + a_2^{(3)} b_1^{(3)}) \right. \right. \right. \\ &\quad \left. \left. - \left( \sum_{k=1}^2 a_1^{(k)} a_2^{(k)} + a_2^{(k)} b_1^{(k)} \right) \sin 2\alpha \right] \right. \\ &\quad \times \left[ - ( - a_3^{(3)} a_4^{(3)} + b_3^{(3)} b_4^{(3)}) \right. \\ &\quad \left. - \left( \sum_{k=1}^2 - a_3^{(k)} a_4^{(k)} + b_3^{(k)} b_4^{(k)} \right) \sin 2\beta \right] \\ &\quad \left. + \left[ - (a_1^{(3)} b_2^{(3)} + b_1^{(3)} b_2^{(3)}) \right. \right. \\ &\quad \left. - \left( \sum_{k=1}^2 a_1^{(k)} b_2^{(k)} + b_1^{(k)} b_2^{(k)} \right) \sin 2\alpha \right] \right. \\ &\quad \times \left[ - (a_3^{(3)} b_4^{(3)} + a_4^{(3)} b_3^{(3)}) \right. \\ &\quad \left. - \left( \sum_{k=1}^2 a_3^{(k)} b_4^{(k)} + a_4^{(k)} b_3^{(k)} \right) \sin 2\beta \right] \left. \right\} \\ &\quad + \frac{1}{2} (a_1^{(3)} - b_1^{(3)}) \cos 2\alpha \Big| \leq \frac{3}{2}. \end{aligned}$$

Similarly one can get  $|\langle D_4^{(i)} \rangle_{|\psi\rangle}| \leq \frac{3}{2}$  for  $i = 2, 3, 4$ . For the cases of  $\rho_{13-24}$  and  $\rho_{14-23}$ , we can also similarly have  $|\langle D_4^{(i)} \rangle_{|\psi\rangle}| \leq \frac{3}{2}$  for  $i = 1, 2, 3, 4$ .

For bi-separable states in class ii), we consider the case of  $\rho_{1-234}$ . There are two inequivalent kinds of genuine three-qubit entangled states, the GHZ-type and W-type [23]. For simplicity in the following we denote  $c_x = \cos x$  and  $s_x = \sin x$ . The GHZ-type state can be written as

$$|\psi_{\text{GHZ}}\rangle = \sqrt{K}(c_\delta|0\rangle|0\rangle|0\rangle + s_\delta e^{i\varphi}|\phi_A\rangle|\phi_B\rangle|\phi_C\rangle),$$

where  $|\phi_A\rangle = c_\alpha|0\rangle + s_\alpha|1\rangle$ ,  $|\phi_B\rangle = c_\beta|0\rangle + s_\beta|1\rangle$ ,  $|\phi_C\rangle = c_\gamma|0\rangle + s_\gamma|1\rangle$ ,  $\delta \in (0, \frac{\pi}{4}]$ ,  $\alpha, \beta, \gamma \in (0, \frac{\pi}{2}]$ ,  $\varphi \in [0, 2\pi)$  and  $K = (1 + 2c_\delta s_\delta c_\alpha c_\beta c_\gamma c_\varphi)^{-1} \in (\frac{1}{2}, \infty)$  is a normalization factor. The W-type state can be written as

$$|\psi_W\rangle = \sqrt{a}|001\rangle + \sqrt{b}|010\rangle + \sqrt{c}|100\rangle + \sqrt{d}|000\rangle,$$

where  $a, b, c > 0$  and  $d = 1 - (a + b + c) \geq 0$ . Therefore every pure state in  $\rho_{1-234}$  via a suitable choice of bases can be written as  $|\psi_{0\text{-GHZ}}\rangle = |0\rangle \otimes |\psi_{\text{GHZ}}\rangle$ , or  $|\psi_{0\text{-W}}\rangle = |0\rangle \otimes |\psi_W\rangle$ .

We calculate here the value  $|\langle A_2 A_3 A_4 \rangle_{|\psi_{\text{GHZ}}\rangle}|$ . The other items have similar expressions.

$$\begin{aligned} |\langle A_2 A_3 A_4 \rangle_{|\psi_{\text{GHZ}}\rangle}| &= \\ &K \left[ a_2^{(3)} a_3^{(3)} a_4^{(3)} (c_\delta^2 + s_\delta^2 \cdot \cos 2\alpha \cdot \cos 2\beta \cdot \cos 2\gamma) \right. \\ &+ a_2^{(1)} a_3^{(3)} a_4^{(3)} (\sin 2\delta \cdot c_\varphi c_\alpha c_\beta c_\gamma + s_\delta^2 \cdot \sin 2\alpha \cdot \cos 2\beta \cdot \cos 2\gamma) \\ &+ a_2^{(3)} a_3^{(1)} a_4^{(3)} (\sin 2\delta \cdot c_\varphi c_\alpha s_\beta c_\gamma + s_\delta^2 \cdot \cos 2\alpha \cdot \sin 2\beta \cdot \cos 2\gamma) \\ &+ a_2^{(3)} a_3^{(3)} a_4^{(1)} (\sin 2\delta \cdot c_\varphi c_\alpha c_\beta s_\gamma + s_\delta^2 \cdot \cos 2\alpha \cdot \cos 2\beta \cdot \sin 2\gamma) \\ &+ a_2^{(1)} a_3^{(1)} a_4^{(3)} (\sin 2\delta \cdot c_\varphi s_\alpha s_\beta s_\gamma + s_\delta^2 \cdot \sin 2\alpha \cdot \sin 2\beta \cdot \cos 2\gamma) \\ &- a_2^{(2)} a_3^{(2)} a_4^{(3)} \sin 2\delta \cdot c_\varphi s_\alpha s_\beta c_\gamma \\ &+ a_2^{(1)} a_3^{(3)} a_4^{(1)} (\sin 2\delta \cdot c_\varphi c_\alpha c_\beta s_\gamma + s_\delta^2 \cdot \sin 2\alpha \cdot \cos 2\beta \cdot \sin 2\gamma) \\ &- a_2^{(2)} a_3^{(3)} a_4^{(2)} \sin 2\delta \cdot c_\varphi s_\alpha c_\beta s_\gamma \\ &+ a_2^{(3)} a_3^{(1)} a_4^{(1)} (\sin 2\delta \cdot c_\varphi c_\alpha s_\beta s_\gamma + s_\delta^2 \cdot \cos 2\alpha \cdot \sin 2\beta \cdot \sin 2\gamma) \\ &- a_2^{(3)} a_3^{(2)} a_4^{(2)} \sin 2\delta \cdot c_\varphi c_\alpha s_\beta s_\gamma \\ &+ a_2^{(1)} a_3^{(1)} a_4^{(1)} s_\delta^2 \cdot \sin 2\alpha \cdot \sin 2\beta \cdot \sin 2\gamma \\ &+ (a_2^{(1)} a_3^{(1)} a_4^{(1)} - a_2^{(2)} a_3^{(2)} a_4^{(1)} - a_2^{(2)} a_3^{(1)} a_4^{(2)} - a_2^{(1)} a_3^{(2)} a_4^{(2)}) \sin 2\delta \\ &\quad \times c_\varphi s_\alpha s_\beta s_\gamma \Big|. \end{aligned}$$

$|\langle A_2 A_3 A_4 \rangle_{|\psi_{\text{GHZ}}\rangle}|$  attains its maximum at  $\alpha = \beta = \gamma = \frac{\pi}{2}$ ,  $\varphi = 0$  and  $\delta = \frac{\pi}{4}$  according to the value of  $K$  and the property of the trigonometric functions. Hence we have

$$\begin{aligned} |\langle D_4^{(1)} \rangle_{|\psi_{0\text{-GHZ}}\rangle}| &\leq \frac{1}{4} (a_1^{(3)} + b_1^{(3)}) [a_2^{(1)} (a_3^{(1)} a_4^{(1)} - a_3^{(2)} a_4^{(2)} - b_3^{(1)} b_4^{(1)} + b_3^{(2)} b_4^{(2)}) \\ &\quad + a_2^{(2)} (-a_3^{(1)} a_4^{(2)} - a_3^{(2)} a_4^{(1)} + b_3^{(1)} b_4^{(2)} + b_3^{(2)} b_4^{(1)}) \\ &\quad + b_2^{(1)} (-a_3^{(1)} b_4^{(1)} + a_3^{(2)} b_4^{(2)} - b_3^{(1)} a_4^{(1)} + b_3^{(2)} a_4^{(2)}) \\ &\quad + b_2^{(2)} (a_3^{(1)} b_4^{(2)} + a_3^{(2)} b_4^{(1)} + b_3^{(1)} a_4^{(2)} + b_3^{(2)} a_4^{(1)})] \\ &\quad + \frac{1}{2} (a_1^{(3)} - b_1^{(3)}) \leq 1. \end{aligned}$$

Using the similar method above, we have also  $|\langle D_4^{(i)} \rangle_{|\psi_{0\text{-GHZ}}\rangle}| \leq 1$  for  $i = 2, 3, 4$ .

Next we compute  $|\langle A_2 A_3 A_4 \rangle_{|\psi_W\rangle}|$ .

$$\begin{aligned} |\langle A_2 A_3 A_4 \rangle_{|\psi_W\rangle}| &= |(d - a - b - c) a_2^{(3)} a_3^{(3)} a_4^{(3)} + 2\sqrt{cda} a_2^{(1)} a_3^{(3)} a_4^{(3)} \\ &\quad + 2\sqrt{bda} a_2^{(3)} a_3^{(1)} a_4^{(3)} + 2\sqrt{ada} a_2^{(3)} a_3^{(3)} a_4^{(1)} + 2\sqrt{bc} (a_2^{(1)} a_3^{(1)} a_4^{(3)} \\ &\quad + a_2^{(2)} a_3^{(2)} a_4^{(3)}) + 2\sqrt{ac} (a_2^{(1)} a_3^{(3)} a_4^{(1)} + a_2^{(2)} a_3^{(3)} a_4^{(2)}) \\ &\quad + 2\sqrt{ab} (a_2^{(3)} a_3^{(1)} a_4^{(1)} + a_2^{(3)} a_3^{(2)} a_4^{(2)})| \\ &\leq \frac{1}{2} |a_2^{(3)} (-a_3^{(3)} a_4^{(3)} + a_3^{(1)} a_4^{(1)} + a_3^{(2)} a_4^{(2)} + a_3^{(1)} a_4^{(3)} + a_3^{(3)} a_4^{(1)}) \\ &\quad + a_2^{(1)} (a_3^{(3)} a_4^{(3)} + a_3^{(1)} a_4^{(3)} + a_3^{(3)} a_4^{(1)}) + a_2^{(2)} (a_3^{(2)} a_4^{(2)} + a_3^{(3)} a_4^{(2)})| \\ &\leq \frac{1}{2} |a_2^{(3)} + a_2^{(1)} + a_2^{(2)}| \leq \frac{\sqrt{3}}{2}, \end{aligned}$$

where we have used the Cauchy-Schwarz inequality and the relation  $\sqrt[4]{abcd} \leq \frac{a+b+c+d}{4}$ , for which the equality holds if and only if  $a = b = c = d$ . Therefore we have

$$|\langle D_4^{(1)} \rangle_{|\psi_{0-w}\rangle}| \leq \frac{1}{4} \times 2 \times \left(\frac{\sqrt{3}}{2} \times 4\right) = \sqrt{3}.$$

Similarly one can obtain  $|\langle D_4^{(i)} \rangle_{|\psi_{0-w}\rangle}| \leq \sqrt{3}$  for  $i = 2, 3, 4$ .

The cases for  $\rho_{2-134}$ ,  $\rho_{3-124}$  and  $\rho_{4-123}$  can be similarly proved. □

## 2 The quadratic inequality of Bell operator for four qubits

We derive now an analytical quadratic inequality of the Bell operator for four qubits. The four-qubit states  $\rho$  can be written as [24],

$$\begin{aligned} \rho = & \frac{1}{16} (I \otimes I \otimes I \otimes I + \sum_{i_1=1}^3 Q_{i_1}^{(1)} \sigma_{i_1}^{(1)} \otimes I \otimes I \otimes I \\ & + \sum_{i_2=1}^3 Q_{i_2}^{(2)} I \otimes \sigma_{i_2}^{(2)} \otimes I \otimes I + \sum_{i_3=1}^3 Q_{i_3}^{(3)} I \otimes I \otimes \sigma_{i_3}^{(3)} \otimes I \\ & + \sum_{i_4=1}^4 Q_{i_4}^{(4)} I \otimes I \otimes I \otimes \sigma_{i_4}^{(4)} + \dots \\ & + \sum_{i_1, i_2, i_3, i_4=1}^3 Q_{i_1 i_2 i_3 i_4}^{(1234)} \sigma_{i_1}^{(1)} \otimes \sigma_{i_2}^{(2)} \otimes \sigma_{i_3}^{(3)} \otimes \sigma_{i_4}^{(4)}). \end{aligned} \tag{5}$$

Set

$$\begin{aligned} \vec{\alpha} = & (Q_1^{(1)}, Q_2^{(1)}, Q_3^{(1)}), \quad \vec{\beta} = (Q_1^{(2)}, Q_2^{(2)}, Q_3^{(2)}), \\ \vec{\gamma} = & (Q_1^{(3)}, Q_2^{(3)}, Q_3^{(3)}), \quad \vec{\varepsilon} = (Q_1^{(4)}, Q_2^{(4)}, Q_3^{(4)}), \\ \vec{S} = & (Q_{111}^{(124)}, Q_{112}^{(124)}, Q_{113}^{(124)}, \dots, Q_{331}^{(124)}, Q_{332}^{(124)}, Q_{333}^{(124)}), \\ \vec{T} = & (Q_{111}^{(123)}, Q_{112}^{(123)}, Q_{113}^{(123)}, \dots, Q_{331}^{(123)}, Q_{332}^{(123)}, Q_{333}^{(123)}), \\ \vec{U} = & (Q_{111}^{(234)}, Q_{112}^{(234)}, Q_{113}^{(234)}, \dots, Q_{331}^{(234)}, Q_{332}^{(234)}, Q_{333}^{(234)}), \\ \vec{V} = & (Q_{111}^{(134)}, Q_{112}^{(134)}, Q_{113}^{(134)}, \dots, Q_{331}^{(134)}, Q_{332}^{(134)}, Q_{333}^{(134)}), \\ \vec{Q} = & (Q_{1111}^{(1234)}, Q_{1112}^{(1234)}, Q_{1113}^{(1234)}, \dots, Q_{3331}^{(1234)}, Q_{3332}^{(1234)}, Q_{3333}^{(1234)}). \end{aligned}$$

We have the following lemma.

**Lemma.** For four qubits pure states, we have the following equality,

$$|\vec{\alpha}|^2 + |\vec{\beta}|^2 + |\vec{\gamma}|^2 + |\vec{\varepsilon}|^2 + |\vec{S}|^2 + |\vec{T}|^2 + |\vec{U}|^2 + |\vec{V}|^2 + |\vec{Q}|^2 = 9. \tag{6}$$

**Proof.** A four-qubit pure state  $|\psi\rangle$  can be also written as [25],

$$|\psi\rangle = l_0|0000\rangle + l_1|0011\rangle + l_2|0101\rangle + l_3|0110\rangle + l_4|1100\rangle$$

$$+ l_5|1001\rangle + l_6|1010\rangle + l_7|1011\rangle + l_8|0111\rangle + l_9|1101\rangle + l_{10}|1110\rangle + l_{11}|1111\rangle,$$

where  $l_i$  with  $i = 7, 8, 9, 10, 11$ , are non-negative real numbers and  $|l_0| \geq |l_i|$  for  $i = 1, 2, \dots, 11$ . Comparing  $\rho = |\psi\rangle\langle\psi|$  with (5), we have the relation (6) by straightforward calculation.

Here  $|\vec{\alpha}|^2, |\vec{\beta}|^2, |\vec{\gamma}|^2, |\vec{\varepsilon}|^2, |\vec{S}|^2, |\vec{T}|^2, |\vec{U}|^2, |\vec{V}|^2, |\vec{Q}|^2$  are all invariants under local unitary transformations, and equality (6) holds for all pure states. The minimum of  $|\vec{Q}|$  is attained for fully separable states and the maximum of  $|\vec{Q}|$  is obtained for maximally entangled states.

**Theorem 4.** Any four-qubit mixed state  $\rho$  satisfies the following inequality,

$$\omega = \langle D_4^{(1)} \rangle_\rho^2 + \langle D_4^{(2)} \rangle_\rho^2 + \langle D_4^{(3)} \rangle_\rho^2 + \langle D_4^{(4)} \rangle_\rho^2 \leq 4. \tag{7}$$

**Proof.** Due to that the quadratic function is a convex function,

$$\begin{aligned} \langle D_4^{(i)} \rangle_\rho^2 = & [\text{tr}(\sum p_\alpha \rho_\alpha D_4^{(i)})]^2 = [\sum p_\alpha \text{tr}(\rho_\alpha D_4^{(i)})]^2 \\ \leq & \sum p_\alpha [\text{tr}(\rho_\alpha D_4^{(i)})]^2 \leq \sum [\text{tr}(\rho_\alpha D_4^{(i)})]^2, \end{aligned}$$

it is sufficient to consider only pure states. Set

$$\begin{aligned} C_i = & \frac{1}{2}(A_i + B_i), \quad D_i = \frac{1}{2}(B_i - A_i), \\ \vec{s}_i = & \frac{1}{2}(\vec{b}_i + \vec{a}_i), \quad \vec{t}_i = \frac{1}{2}(\vec{b}_i - \vec{a}_i), \end{aligned} \tag{8}$$

We have  $|\vec{s}_i|^2 + |\vec{t}_i|^2 = 1, \vec{s}_i \cdot \vec{t}_i = 0$ , and

$$\begin{aligned} \langle D_4^{(1)} \rangle_\rho^2 = & \{[\vec{s}_1 \otimes \vec{s}_2 \otimes (\vec{s}_3 \otimes \vec{t}_4 - \vec{t}_3 \otimes \vec{t}_4 + \vec{s}_3 \otimes \vec{s}_4 + \vec{t}_3 \otimes \vec{s}_4) \\ & + \vec{s}_1 \otimes \vec{t}_2 \otimes (\vec{s}_3 \otimes \vec{s}_4 - \vec{t}_3 \otimes \vec{s}_4 - \vec{t}_3 \otimes \vec{t}_4 - \vec{s}_3 \otimes \vec{t}_4)] \\ & \cdot \vec{Q} - \vec{t}_1 \cdot \vec{\alpha}\}^2, \\ \langle D_4^{(2)} \rangle_\rho^2 = & \{[\vec{t}_1 \otimes \vec{s}_2 \otimes (\vec{s}_3 \otimes \vec{s}_4 - \vec{t}_3 \otimes \vec{s}_4 - \vec{s}_3 \otimes \vec{t}_4 - \vec{t}_3 \otimes \vec{t}_4) \\ & + \vec{s}_1 \otimes \vec{s}_2 \otimes (\vec{s}_3 \otimes \vec{t}_4 - \vec{t}_3 \otimes \vec{t}_4 + \vec{s}_3 \otimes \vec{s}_4 \\ & + \vec{t}_3 \otimes \vec{s}_4)] \cdot \vec{Q} - \vec{t}_2 \cdot \vec{\beta}\}^2, \\ \langle D_4^{(3)} \rangle_\rho^2 = & \{[\vec{t}_1 \otimes \vec{s}_2 \otimes (\vec{s}_3 \otimes \vec{s}_4 - \vec{t}_3 \otimes \vec{t}_4) + \vec{s}_1 \otimes \vec{t}_2 \otimes (\vec{s}_3 \otimes \vec{s}_4 \\ & - \vec{s}_3 \otimes \vec{t}_4) + \vec{s}_1 \otimes \vec{s}_2 \otimes (\vec{s}_3 \otimes \vec{s}_4 + \vec{s}_3 \otimes \vec{t}_4) \\ & - \vec{t}_1 \otimes \vec{t}_2 \otimes (\vec{s}_3 \otimes \vec{s}_4 - \vec{s}_3 \otimes \vec{t}_4)] \cdot \vec{Q} - \vec{t}_3 \cdot \vec{\gamma}\}^2, \\ \langle D_4^{(4)} \rangle_\rho^2 = & \{[\vec{t}_1 \otimes \vec{s}_2 \otimes (\vec{s}_3 \otimes \vec{s}_4 - \vec{t}_3 \otimes \vec{s}_4) + \vec{s}_1 \otimes \vec{t}_2 \otimes (\vec{s}_3 \otimes \vec{s}_4 \\ & - \vec{t}_3 \otimes \vec{s}_4) + \vec{s}_1 \otimes \vec{s}_2 \otimes (\vec{s}_3 \otimes \vec{s}_4 + \vec{t}_3 \otimes \vec{s}_4) \\ & - \vec{t}_1 \otimes \vec{t}_2 \otimes (\vec{s}_3 \otimes \vec{s}_4 - \vec{t}_3 \otimes \vec{s}_4)] \cdot \vec{Q} - \vec{t}_4 \cdot \vec{\varepsilon}\}^2, \end{aligned}$$

where  $\vec{s} \otimes \vec{t} \otimes \vec{p} \otimes \vec{q} \cdot \vec{Q}$  denotes  $\sum_{ijkl} s_i t_j p_k q_l Q_{ijkl}$ .  $\omega$  attains its maximum at either  $|\vec{Q}| = 1$  or  $|\vec{Q}| = 3$ . For the case of  $|\vec{Q}| = 1, |\psi\rangle$  is fully separable and the inequality is satisfied by using Theorem 1. For the case of  $|\vec{Q}| = 3, |\psi\rangle$  is maximally entangled. Without losing generality, we consider the maximally entangled state  $|\psi\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$ . Since

$Q_{1111} = Q_{2222} = Q_{3333} = 1, Q_{1122} = Q_{1212} = Q_{2112} = Q_{2121} = Q_{1221} = Q_{2211} = -1$ , the rest  $Q_{ijkh} = 0$  and  $|\vec{s}_i|^2 + |\vec{t}_i|^2 = 1, \vec{s}_i \cdot \vec{t}_i = 0$ , thus to attain the maximum of  $\omega$ , the third components of  $\vec{s}_i$  and  $\vec{t}_i$  should be zero, and either  $|\vec{s}_i| = |\vec{t}_i|$  or one of the  $|\vec{s}_i|$  and  $|\vec{t}_i|$  is zero and the other one is 1. Let  $i \neq j \neq k \neq l \in \{1, 2, 3, 4\}$ , we can obtain the following classifications:

- (1)  $|\vec{s}_i| = 1 \left(0 \text{ or } \frac{1}{\sqrt{2}}\right), |\vec{s}_j| = |\vec{s}_k| = |\vec{s}_l| = 1 \left(0 \text{ or } \frac{1}{\sqrt{2}}\right)$ ,
- (2)  $|\vec{s}_i| = |\vec{s}_j| = 0, |\vec{s}_k| = |\vec{s}_l| = \frac{1}{\sqrt{2}}$ ,
- (3)  $|\vec{s}_i| = |\vec{s}_j| = 1, |\vec{s}_k| = |\vec{s}_l| = 0 \left(\text{ or } \frac{1}{\sqrt{2}}\right)$ ,
- (4)  $|\vec{s}_i| = |\vec{s}_j| = 1, |\vec{s}_k| = 0, |\vec{s}_l| = \frac{1}{\sqrt{2}}$ ,
- (5)  $|\vec{s}_i| = |\vec{s}_j| = 0, |\vec{s}_k| = 1, |\vec{s}_l| = \frac{1}{\sqrt{2}}$ ,
- (6)  $|\vec{s}_i| = |\vec{s}_j| = \frac{1}{\sqrt{2}}, |\vec{s}_k| = 0, |\vec{s}_l| = 1$ .

For the case  $|\vec{s}_1| = |\vec{s}_2| = |\vec{s}_3| = |\vec{s}_4| = 1$ , we have

$$\omega = 4(\vec{s}_1 \otimes \vec{s}_2 \otimes \vec{s}_3 \otimes \vec{s}_4 \cdot \vec{Q})^2 = 4\langle \psi | C_1 C_2 C_3 C_4 | \psi \rangle^2 \leq 4.$$

For the case  $|\vec{s}_1| = |\vec{s}_2| = |\vec{s}_3| = 1$  and  $|\vec{s}_4| = \frac{1}{\sqrt{2}}$ , we get

$$\begin{aligned} \omega &= 3(\vec{s}_1 \otimes \vec{s}_2 \otimes \vec{s}_3 \otimes (\vec{t}_4 + \vec{s}_4) \cdot \vec{Q})^2 \\ &\quad + (\vec{s}_1 \otimes \vec{s}_2 \otimes \vec{s}_3 \otimes \vec{s}_4 \cdot \vec{Q})^2 \\ &= 3\langle \psi | C_1 C_2 C_3 (D_4 + C_4) | \psi \rangle^2 \\ &\quad + \langle \psi | C_1 C_2 C_3 C_4 | \psi \rangle^2 \leq 4. \end{aligned}$$

For the case  $|\vec{s}_1| = 1$  and  $|\vec{s}_2| = |\vec{s}_3| = |\vec{s}_4| = \frac{1}{\sqrt{2}}$ , using the orthogonal relation of  $\vec{s}_i$  and  $\vec{t}_i$ , we can express  $\vec{s}_i$  and  $\vec{t}_i$  as

$$\begin{aligned} \vec{s}_1 &= (\sin \theta_1 \cos \varphi_1, \sin \theta_1 \sin \varphi_1, \cos \theta_1), \\ \vec{s}_i &= \frac{1}{\sqrt{2}}(\sin \theta_i \cos \varphi_i, \sin \theta_i \sin \varphi_i, \cos \theta_i), \\ \vec{t}_i &= \frac{1}{\sqrt{2}}(\cos \theta_i \cos \varphi_i, \cos \theta_i \sin \varphi_i, -\sin \theta_i), \end{aligned}$$

where  $i = 2, 3, 4$ . After straightforward calculations, we have

$$\begin{aligned} \omega &= \left[ \frac{1}{2\sqrt{2}} \sin(\theta_1 + \varphi_1 + \theta_2 + \varphi_2 + \theta_3 + \varphi_3 + \theta_4 + \varphi_4) \right. \\ &\quad \left. - \cos(\theta_1 + \varphi_1 + \theta_2 + \varphi_2 + \theta_3 + \varphi_3 + \theta_4 + \varphi_4) \right] \\ &\quad + \frac{4}{2\sqrt{2}} \cos \theta_1 \cos \theta_4 [\cos(\theta_2 + \theta_3) - \sin(\theta_2 + \theta_3)]^2 \\ &\leq 4. \end{aligned}$$

For the case  $|\vec{s}_1| = |\vec{s}_2| = |\vec{s}_3| = |\vec{s}_4| = \frac{1}{\sqrt{2}}$ , one can set

$$\begin{aligned} \vec{s}_i &= \frac{1}{\sqrt{2}}(\sin \theta_i \cos \varphi_i, \sin \theta_i \sin \varphi_i, \cos \theta_i), \\ \vec{t}_i &= \frac{1}{\sqrt{2}}(\cos \theta_i \cos \varphi_i, \cos \theta_i \sin \varphi_i, -\sin \theta_i) \end{aligned}$$

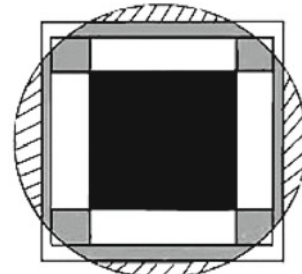
for  $i = 1, 2, 3, 4$ , and obtain  $\omega \leq 4$ . For other cases inequality (7) can be similarly proved.  $\square$

The inequalities in Theorems 1 to 4 give rise to an explicit geometric picture. Taking the average of  $D_4^{(1)}, D_4^{(2)}, D_4^{(3)}$  and  $D_4^{(4)}$  as the four coordinates of a four dimensional space, from Theorem 4 we have that  $\langle D_4^{(i)} \rangle, i = 1, \dots, 4$ , constitute a 3-dimensional sphere. All fully separable states are confined in the center, in a 4-dimensional rectangular with size  $1 \times 1 \times 1 \times 1$ , see Figure 1. While from the Theorem 2, the tri-separable state, for example,  $\rho_{12-3-4}$ , is in a 4-dimensional rectangular with the size  $1 \times 1 \times 3/2 \times 3/2$ . From the Theorem 3, the bi-separable state in class  $i$ ), for example  $\rho_{12-34}$ , is in a 4-dimensional rectangular with size  $3/2 \times 3/2 \times 3/2 \times 3/2$ . For the bi-separable state in class  $ii$ ), for example  $\rho_{1-234}$ , we have a 4-dimensional rectangular with size  $\sqrt{3} \times \sqrt{3} \times \sqrt{3} \times \sqrt{3}$ .

### 3 Conclusions and discussion

We have investigated the classification of four-qubit entanglement in terms of Bell inequalities that involving only two measurement settings per observer. And a quadratic inequality of Bell operator for four-qubit systems has been obtained. The Bell inequalities satisfied by fully separable, bi-separable and tri-separable states of four-qubit systems are analytically derived. Our approach and some of the obtained formulas can be directly generalized to multipartite qubit systems.

However, our inequalities are not both sufficient and necessary for separability of general four-qubit states. The separability problem in terms of Bell inequalities has been solved only for two-qubit case, any pure entangled two-qubit states violate the CHSH inequality [26], as well as the three-qubit case where Chen et al. [27] showed that all pure entangled three-qubit states violate a Bell inequality. For mixed four-qubit systems, the separability problem remains open. Endrejat et al. [28] discussed the relations between optimization operators and combination of the global entanglement measures. One may conjecture that to make the inequalities sufficient conditions for separability of a four-qubit mixed state,



**Figure 1** Projection of the state space onto the plane constituted by  $\langle D_4^{(1)} \rangle$  and  $\langle D_4^{(3)} \rangle$ . The fully separable states are in the black region. The tri-separable states  $\rho_{12-3-4}$  and  $\rho_{14-2-3}$  are located inside the white areas respectively between black and gray areas. The bi-separable states  $\rho_{12-34}, \rho_{14-23}, \rho_{1-234}$  and  $\rho_{3-124}$  are inside the gray area. The genuine entangled states are located in the designated slash regions.

in addition to our Bell operators, some new Bell operators are needed.

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