Single boson realizations of the deformed angular momentum algebra of Witten's types

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Received October 19, 2012; accepted December 23, 2012

By generalizing the Holstein-Primakoff realization and the Dyson realization of the Lie algebra $SU(2)$, various realizations of the deformed angular momentum algebra $\mathcal{R}_{q,s}^{c_0,c_1,c_2}$, a five-parameter deformed $SU(2)$ by combining Witten's two deformation schemes, are investigated in terms of the single boson and the single inversion boson respectively. For each kind, the unitary realization, the non-unitary realization, and their connection by the corresponding similarity transformation are respectively discussed.

deformed algebra, Lie algebra $SU(2)$, boson, inversion boson

Citation: Huang H J, Ruan D. Single boson realizations of the deformed angular momentum algebra of Witten's types. *Chin Sci Bull*, 2013, 58: 2039–2043, doi: 10.1007/s11434-013-5872-5

Lie algebras have played an exciting and significant role in the development of various branches of physics and many other areas of science and technology [1–11]. In 1983, Kulish et al. [12] showed that the algebra that governs the XXZ-Heisenberg spin model was a deformation of the Lie algebra $SU(2)$, called nowadays $SU_q(2)$. Since then, there are many works devoted to various types of nonlinear algebras, i.e. some specific deformations of the usual Lie algebras obtained by introducing deformation parameters, due to their interesting mathematical structure and possible applications in a broad range of physical phenomena [13–27]. The nonlinear algebra to be discussed in this paper is a kind of multi-parameter deformed angular momentum algebra, which generalizes Witten's first and second deformation schemes of $SU(2)$ [14]. Here we denote it by $\mathcal{R}_{q,s}^{c_0,c_1,c_2}$, whose three elements J_μ ($\mu = 3, -, +$) satisfy the following commutation relations:

$$\begin{aligned} [J_3, J_-]_q &= -C_0 J_-, \\ [J_3, J_+]_q &= \frac{C_0}{q} J_+, \\ [J_+, J_-]_s &= D_0 + D_1 J_3 + D_2 J_3^2, \end{aligned} \quad (1)$$

where $[X, Y]_q \equiv XY - qYX$ is a q -deformed commutator, q and s are positive real numbers, C_i ($i = 0, 1, 2$) are real numbers, and three expansion coefficients D_i ($i = 0, 1, 2$) take respectively

$$\begin{aligned} D_0 &= C_0(C_1 - C_0 C_2), \quad D_1 = C_1(1 - q) + 2C_0 C_2 q, \\ D_2 &= C_2(1 - q^2) \end{aligned}$$

in order that the Casimir invariant of the type considered by Polychronakos [15] and Roček [16] exists in the limit of $s \rightarrow 1$. When $q = p^2$, $s = 1$, $C_0 = p$, $C_1 = 1/(1 + p^2)$, and $C_2 = 1/[p(1 + p^2)]$, eq. (1) reduces to Witten's first deformation, when $q = r^2$, $s = r^4$, $C_0 = r$, $C_1 = 2r^2/(1 - r^2)$, and $C_2 = 0$, eq. (1) to Witten's second deformation. Furthermore, different from the quantum group $SU_q(2)$ [13, 19] and the polynomial angular momentum algebras [21–24], both the deformed commutators and the power series of J_3 appear in the algebraic structure (1). When $q = s = C_0 = C_1 = C_2 = 1$ (or $q = s = C_0 = 1$ and $C_1 = C_2 = -1$), $\mathcal{R}_{1,1}^{1,1,1}$ (or $\mathcal{R}_{1,1}^{1,-1,-1}$) becomes $SU(2)$ (or its non-compact type $SU(1,1)$). If let $q = s = C_0 = C_1 = 1$ and $C_2 = 0$, then $\mathcal{R}_{1,1}^{1,1,0}$ becomes the standard Heisenberg-Weyl algebra [28]. Hence, $\mathcal{R}_{q,s}^{c_0,c_1,c_2}$ can be viewed as a type of five-parameter deformation of $SU(2)$ (or $SU(1,1)$) or Heisenberg-Weyl algebra.

In this paper, we study various single boson realizations

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and single inversion boson realizations of $\mathcal{R}_{q,s}^{c_0, c_1, c_2}$, which are analogous to the Holstein-Primakoff realization and the Dyson realization of SU(2) and SU(1,1). This paper is arranged as follows. In Section 1, some results of boson operators and inversion boson operators are briefly reviewed. In Sections 2 and 3, the single boson realizations and single inversion boson realizations of $\mathcal{R}_{q,s}^{c_0, c_1, c_2}$ are respectively studied in detail, which include unitary realizations, non-unitary realizations, and their connections. A simple discussion is given in the final section.

1 Preliminaries

Let a^+ and a (a is adjoint to a^+ , i.e., $a = (a^+)^\dagger$, $a^+ = (a)^\dagger$) be creation and annihilation boson operators respectively, which, together with the particle number operator $\hat{n} \equiv a^+a$, satisfy commutation relations [28]

$$[a, a^+] = 1, \quad [\hat{n}, a^+] = a^+, \quad [\hat{n}, a] = -a. \quad (2)$$

The complete set of basis vectors of Fock space

$$\mathcal{F} = \{|n\rangle | n = 0, 1, 2, \dots\} \quad (3)$$

may be constructed from the vacuum state $|0\rangle$ using the definition

$$|n\rangle = \frac{(a^+)^n}{\sqrt{n!}} |0\rangle. \quad (4)$$

In fact, these vectors are normalized eigenvectors of $\hat{n}$ belonging to eigenvalue n , i.e.

$$\hat{n} |n\rangle = n |n\rangle, \quad (5)$$

and satisfy

$$a^+ |n\rangle = \sqrt{n+1} |n+1\rangle, \quad a |n\rangle = \sqrt{n} |n-1\rangle. \quad (6)$$

Though the boson operators, a and a^+ , do not possess any inversion in a strict sense because of their singular feature, the generalized inversion of these boson operators, denoted by a^{-1} and $(a^+)^{-1}$ respectively, may be defined by their action on the basis vectors of Fock space [29–31]:

$$\begin{aligned} a^{-1} |n\rangle &= \frac{1}{\sqrt{n+1}} |n+1\rangle, \\ (a^+)^{-1} |n\rangle &= (1 - \delta_{n,0}) \frac{1}{\sqrt{n}} |n-1\rangle. \end{aligned} \quad (7)$$

In fact, a^{-1} is only the right inversion of a since it satisfies

$$aa^{-1} = I.$$

However, $a^{-1}a$ is not unity (I) but is given by

$$a^{-1}a = I - |0\rangle\langle 0|,$$

where $|0\rangle\langle 0|$ is the projection operator on vacuum. Similarly, $(a^+)^{-1}$ is only the left inversion of a^+ . As is seen, a^{-1} behaves as the creation operator, while $(a^+)^{-1}$ as the annihilation operator.

By direct calculations, it is easy to obtain the following commutation relations:

$$\begin{aligned} [(a^+)^{-1}, a^+] &= |0\rangle\langle 0|, \\ [\hat{n}, (a^+)^{-1}] &= -(a^+)^{-1}, \\ [\hat{n}, a^{-1}] &= a^{-1}. \end{aligned} \quad (8)$$

2 Boson realizations of $\mathcal{R}_{q,s}^{c_0, c_1, c_2}$

The single boson realizations of $\mathcal{R}_{q,s}^{c_0, c_1, c_2}$ may be chosen in the following form:

$$\begin{aligned} B^{(k)}(J_3) &= h^{(k)}(\hat{n}), \\ B^{(k)}(J_+) &= f^{(k)}(\hat{n})a^k, \\ B^{(k)}(J_-) &= (a^+)^k g^{(k)}(\hat{n}), \end{aligned} \quad (9)$$

where k is an integer, $h^{(k)}(\hat{n})$, $f^{(k)}(\hat{n})$, and $g^{(k)}(\hat{n})$, being real operator functions of $\hat{n}$ only, need determining by the commutation relations (1). The action of $B^{(k)}(J_\pm)$ on the basis vector $|n\rangle$ of the Fock space $\mathcal{F}$ leads to $|n \mp k\rangle \sim B^{(k)}(J_\pm)|n\rangle$.

The boson realizations $B^{(k)}(J_\mu)$ ($\mu = 3, \pm$) are unitary if they satisfy

$$\begin{aligned} [B^{(k)}(J_3)]^\dagger &= B^{(k)}(J_3), \\ [B^{(k)}(J_\pm)]^\dagger &= B^{(k)}(J_\mp). \end{aligned} \quad (10)$$

We call eq. (10) the unitary conditions of the boson realization.

For the case of $k = 1$. Using the first or second equation of eq. (1), we may obtain the single-variable difference equation satisfied by $h^{(1)}(\hat{n})$

$$h^{(1)}(\hat{n}) - \frac{1}{q} h^{(1)}(\hat{n} + 1) = \frac{C_0}{q}, \quad (11)$$

with the help of the relations

$$\begin{aligned} (a^+)^k f(\hat{n}) &= f(\hat{n} - k)(a^+)^k, \\ a^k f(\hat{n}) &= f(\hat{n} + k)a^k. \end{aligned} \quad (12)$$

The solution of eq. (11) reads

$$h^{(1)}(\hat{n}) = \frac{C_0}{q-1} (1 - q^{\hat{n}-\alpha}), \quad (13)$$

where α is an arbitrary real number. It is worth mentioning that $h^{(1)}(\hat{n})$ is independent of the parameters s , C_1 , and C_2 .

The third equation of eq. (1) requires that $f^{(1)}(\hat{n})$ and $g^{(1)}(\hat{n})$ satisfy the following difference equation:

$$\begin{aligned} (\hat{n} + 1)f^{(1)}(\hat{n})g^{(1)}(\hat{n}) - s\hat{n}f^{(1)}(\hat{n} - 1)g^{(1)}(\hat{n} - 1) \\ = C_0(C_1 - C_0C_2) + [C_1(1 - q) + 2C_0C_2q]h^{(1)}(\hat{n}) \\ + C_2(1 - q^2)[h^{(1)}(\hat{n})]^2. \end{aligned} \quad (14)$$

Note that the functions $f^{(1)}(\hat{n})$ and $g^{(1)}(\hat{n})$ do not appear separately but only appear as their product $f^{(1)}(\hat{n})g^{(1)}(\hat{n})$.

Inserting eq. (13) into eq. (14) and solving it, we obtain

$$f^{(1)}(\hat{n})g^{(1)}(\hat{n}) = \frac{C_0 q^{-2\alpha}}{\hat{n}+1} \left[\left(C_1 + \frac{2C_0 C_2}{q-1} \right) q^\alpha D(1, \hat{n}+1) - \frac{C_0 C_2 (q+1)}{q-1} D(2, \hat{n}+1) \right], \quad (15)$$

where $D(x, y)$ is defined as

$$D(x, y) = \frac{q^{xy} - s^y}{q^x - s}. \quad (16)$$

This solution shows that we may have some freedom in the choice of the functions $f^{(1)}(\hat{n})$ and $g^{(1)}(\hat{n})$.

(i) The unitary boson realization. The second equation of eq. (10) requires that $f^{(1)}(\hat{n}) = g^{(1)}(\hat{n})$, then solving eq. (15), we may obtain from eq. (9) a kind of unitary single boson realization

$$\begin{aligned} \check{B}^{(1)}(J_3) &= \frac{C_0}{q-1} (1 - q^{\hat{n}-\alpha}), \\ \check{B}^{(1)}(J_+) &= \sqrt{\frac{q^{-2\alpha} C_0}{\hat{n}+1} \left[\left(C_1 + \frac{2C_0 C_2}{q-1} \right) q^\alpha D(1, \hat{n}+1) - \frac{C_0 C_2 (q+1)}{q-1} D(2, \hat{n}+1) \right]} a, \\ \check{B}^{(1)}(J_-) &= a^\dagger \sqrt{\frac{q^{-2\alpha} C_0}{\hat{n}+1} \left[\left(C_1 + \frac{2C_0 C_2}{q-1} \right) q^\alpha D(1, \hat{n}+1) - \frac{C_0 C_2 (q+1)}{q-1} D(2, \hat{n}+1) \right]}. \end{aligned} \quad (17)$$

In order to obtain the real boson realization, the values of n in the matrix elements of $\check{B}^{(1)}(J_\pm)$ in the Fock space $\mathcal{F} = \{|n\rangle | n = 0, 1, 2, \dots\}$ need limiting for the given $\{q, s, C_0, C_1, C_2\}$. When $q = s = C_0 = C_1 = C_2 = 1$, eq. (17) becomes the Holstein-Primakoff realization of SU(2) [32]. Hence, we call $\check{B}^{(1)}(J_\mu)$ the Holstein-Primakoff-like realization of $\mathcal{R}_{q,s}^{C_0, C_1, C_2}$.

(ii) The non-unitary boson realization. If the unitary conditions (10) need not satisfying, it follows from eq. (15) that the convenient choice, for example, $g(\hat{n}) = 1$, may immediately give rise to a kind of non-unitary single boson realization:

$$\begin{aligned} \bar{B}^{(1)}(J_3) &= \frac{C_0}{q-1} (1 - q^{\hat{n}-\alpha}), \\ \bar{B}^{(1)}(J_+) &= \frac{C_0 q^{-2\alpha}}{\hat{n}+1} \left[\left(C_1 + \frac{2C_0 C_2}{q-1} \right) q^\alpha D(1, \hat{n}+1) - \frac{C_0 C_2 (q+1)}{q-1} D(2, \hat{n}+1) \right] a, \\ \bar{B}^{(1)}(J_-) &= a^\dagger. \end{aligned} \quad (18)$$

When $q = s = C_0 = C_1 = C_2 = 1$, eq. (18) becomes the standard Dyson realization of SU(2) [33]. Hence, we call $\bar{B}^{(1)}(J_\mu)$ the Dyson-like realization of $\mathcal{R}_{q,s}^{C_0, C_1, C_2}$. Different from the Holstein-Primakoff-like realization (17), no square-root symbol appears in eq. (18) so that the Dyson-like realization may not only avoid the convergence questions associated

with the expansion of square-root symbol but also make the values of n in $\mathcal{F} = \{|n\rangle | n = 0, 1, 2, \dots\}$ unlimited.

(iii) Unitarization of the non-unitary realization. It is not difficult to find that the non-unitary realization $\bar{B}^{(k)}(J_\mu)$ may be related to the unitary realization $\check{B}^{(k)}(J_\mu)$ by a similarity transformation S , i.e.

$$S \bar{B}^{(k)}(J_\mu) S^{-1} = \check{B}^{(k)}(J_\mu). \quad (19)$$

In general, S is an operator function with respect to $\{a, a^\dagger, \hat{n}\}$. Using the unitary condition $\check{B}^{(k)}(J_+) = (\check{B}^{(k)}(J_-))^\dagger$, we obtain from eq. (19)

$$U^{-1} (\bar{B}^{(k)}(J_-))^\dagger U = \bar{B}^{(k)}(J_+), \quad (20)$$

where $U \equiv S^\dagger S$. Note that $\bar{B}^{(k)}(J_3)$ is already Hermitian, so we call eq. (20) the unitarization equation of the Dyson-like realization.

Now let us calculate the explicit expression of S_1 , which transforms the Dyson-like realization $B^{(1)}(J_\mu)$ to the Holstein-Primakoff-like realization $\check{B}^{(1)}(J_\mu)$. Owing to the fact that $\bar{B}^{(k)}(J_3) = \check{B}^{(k)}(J_3)$ for the given integer k , eq. (19) implies that S_1 commutes with J_3 and is at most the function of $\hat{n}$, i.e. $S_1(\hat{n})$. Thus, calculating the matrix element of eq. (20) between the basis vectors $\langle n-1|$ and $|n\rangle$ and using eq. (18), we may derive the equation satisfied by $\langle n|S_1(\hat{n})|n\rangle \equiv S_1(n)$, i.e.

$$\begin{aligned} n \langle n|S_1(\hat{n})|n\rangle^2 &= C_0 q^{-2\alpha} \left[\left(C_1 + \frac{2C_0 C_2}{q-1} \right) q^\alpha D(1, \hat{n}) - \frac{C_0 C_2 (q+1)}{q-1} D(2, \hat{n}) \right] \langle n-1|S_1(\hat{n})|n-1\rangle^2. \end{aligned} \quad (21)$$

Solving eq. (21) with the initial condition $\langle 0|S_1(\hat{n})|0\rangle = \kappa_0$ (κ_0 is a real number) gives

$$\begin{aligned} S_1(n) &= \kappa_0 \left[\frac{-C_0}{(q-1)(q-s)(q^2-s)} \right]^{n/2} \\ &\cdot \prod_{l=0}^{n-1} \frac{1}{\sqrt{l+1}} \left\{ C_1 q^{-\alpha} (q-1)(q^2-s)(q^{l+1}-s^{l+1}) \right. \\ &\quad \left. + C_0 C_2 q^{-\alpha} (q-s)[2(q+s)(q^{l+1}-s^{l+1}) - q^{-\alpha} (q+1)(q^{2(l+1)}-s^{l+1})] \right\}^{1/2}. \end{aligned} \quad (22)$$

Using eq. (5), $S_1(n)$ may be written as the operator function $S_1(\hat{n})$ by replacing directly $n \rightarrow \hat{n}$ in eq. (22).

3 Inversion boson realizations of $\mathcal{R}_{q,s}^{C_0, C_1, C_2}$

For the case of $k = -1$. Substitution of eq. (9) in eq. (1) yields the following two equations satisfied by $h^{(-1)}(\hat{n})$ and $f^{(-1)}(\hat{n})g^{(-1)}(\hat{n})$, respectively:

$$h^{(-1)}(\hat{n}) - \frac{1}{q} h^{(-1)}(\hat{n}-1) = \frac{C_0}{q}, \quad (23)$$

and

$$\frac{1}{\hat{n}} f^{(-1)}(\hat{n})g^{(-1)}(\hat{n}) - \frac{s}{\hat{n}+1} f^{(-1)}(\hat{n}+1)g^{(-1)}(\hat{n}+1)$$

$$= C_0(C_1 - C_0C_2) + [C_1(1 - q) + 2C_0C_2q]h^{(-1)}(\hat{n}) + C_2(1 - q^2)[h^{(-1)}(\hat{n})]^2. \quad (24)$$

Their solutions are respectively

$$h^{(-1)}(\hat{n}) = \frac{C_0}{q-1} (1 - q^{-\hat{n}-1}), \quad (25)$$

$$f^{(-1)}(\hat{n})g^{(-1)}(\hat{n}) = \hat{n}C_0 \left[\left(C_1 + \frac{2C_0C_2}{q-1} \right) D(1, -\hat{n}) - \frac{C_0C_2(q+1)}{q-1} D(2, -\hat{n}) \right], \quad (26)$$

where $D(x, y)$ is given by eq. (16).

It follows from eq. (26) that we obtain the explicit expressions for the unitary single inversion boson realization by taking $f^{(-1)}(\hat{n}) = g^{(-1)}(\hat{n})$, i.e.

$$\begin{aligned} \check{B}^{(-1)}(J_3) &= \frac{C_0}{q-1} (1 - q^{-\hat{n}-1}), \\ \check{B}^{(-1)}(J_+) &= \sqrt{\hat{n}C_0 \left[\left(C_1 + \frac{2C_0C_2}{q-1} \right) D(1, -\hat{n}) - \frac{C_0C_2(q+1)}{q-1} D(2, -\hat{n}) \right]} a^{-1}, \\ \check{B}^{(-1)}(J_-) &= (a^+)^{-1} \sqrt{\hat{n}C_0 \left[\left(C_1 + \frac{2C_0C_2}{q-1} \right) D(1, -\hat{n}) - \frac{C_0C_2(q+1)}{q-1} D(2, -\hat{n}) \right]}, \end{aligned} \quad (27)$$

and the non-unitary one by taking $g^{(-1)}(\hat{n}) = 1$, i.e.

$$\begin{aligned} \bar{B}^{(-1)}(J_3) &= \frac{C_0}{q-1} (1 - q^{-\hat{n}-1}), \\ \bar{B}^{(-1)}(J_+) &= \hat{n}C_0 \left[\left(C_1 + \frac{2C_0C_2}{q-1} \right) D(1, -\hat{n}) - \frac{C_0C_2(q+1)}{q-1} D(2, -\hat{n}) \right] a^{-1}, \\ \bar{B}^{(-1)}(J_-) &= (a^+)^{-1}. \end{aligned} \quad (28)$$

The above two kinds of inversion boson realizations are unfamiliar to us. Now let us see the corresponding results for a simple case $\mathcal{R}_{1,1}^{c_0, c_1, c_2}$.

Taking $q = s = 1$ in eq. (1), introducing the parameter $\lambda = C_1/C_2$, and rescaling $J_{\pm}$, the commutation relations of $\mathcal{R}_{1,1}^{c_0, c_1, c_2}$ read

$$\begin{aligned} [J_3, J_-] &= -C_0J_-, \\ [J_3, J_+] &= C_0J_+, \\ [J_+, J_-] &= (\lambda - 1) + 2J_3. \end{aligned} \quad (29)$$

In fact $\mathcal{R}_{1,1}^{c_0, c_1, c_2}$ is a two-parameter deformed algebra. Thus, from eq. (27) we may immediately obtain the single inversion boson realization of $\mathcal{R}_{1,1}^{c_0, c_1, c_2}$:

$$\begin{aligned} \check{B}_{\mathcal{R}_{1,1}}(J_3) &= C_0(\hat{n} + 1), \\ \check{B}_{\mathcal{R}_{1,1}}(J_+) &= \hat{n} \sqrt{-C_0(\lambda + C_0\hat{n})} a^{-1}, \\ \check{B}_{\mathcal{R}_{1,1}}(J_-) &= (a^+)^{-1} \hat{n} \sqrt{-C_0(\lambda + C_0\hat{n})}. \end{aligned} \quad (30)$$

For the realization (30), the acting space of $\check{B}_{\mathcal{R}_{1,1}}(J_{\mu})$ ($\mu = 3, -, +$) may be certain subspace of the Fock space $\mathcal{F}$, in which n needs limiting in order that the values of the square roots appeared in the matrix elements $\langle n \pm 1 | \check{B}_{\mathcal{R}_{1,1}}(J_{\pm}) | n \rangle$ must be greater than or equal to zero, i.e. n has to satisfy the constraint equation $-C_0(\lambda + C_0n) \geq 0$. If $C_0 < 0$, then n may take $0, 1, 2, \dots, \lceil \lambda/|C_0| \rceil$ (where $\lceil x \rceil$ means integer part of x). If $C_0 > 0$, then n may take $0, 1, 2, \dots, \lceil -\lambda/C_0 \rceil$. Their corresponding acting spaces are finite-dimensional. Especially, when $C_0 = \lambda = 1$, eq. (29) becomes the commutation relations of $SU(2)$, then the square roots in the matrix elements $\langle n \pm 1 | \check{B}_{SU(2)}(J_{\pm}) | n \rangle$ are pure imaginary or null.

Unitarization of the non-unitary realization. Calculating the matrix element of eq. (20) between the basis vectors $\langle n+1 |$ and $| n \rangle$ and using eq. (28), we obtain the equation satisfied by $\langle n | S_2(\hat{n}) | n \rangle \equiv S_2(n)$ of the similarity transformation $S_2(\hat{n})$, which transforms the non-unitary realization $\bar{B}^{(-1)}(J_{\mu})$ to the unitary realization $\check{B}^{(-1)}(J_{\mu})$:

$$\begin{aligned} \langle n | S_2(\hat{n}) | n \rangle^2 &= (n+1)C_0 \left[\left(C_1 + \frac{2C_0C_2}{q-1} \right) D(1, -n-1) - \frac{C_0C_2(q+1)}{q-1} D(2, -n-1) \right] \langle n+1 | S_2(\hat{n}) | n+1 \rangle^2. \end{aligned} \quad (31)$$

Its solution is

$$\begin{aligned} S_2(n) &= \varepsilon_0 \left[\frac{-C_0}{(q-1)(q-s)(q^2-s)} \right]^{-n/2} \\ &\cdot \prod_{l=0}^{n-1} \frac{q^{l+1} s^{(l+1)/2}}{\sqrt{l+1}} \left\{ -C_1(q-1)(q^2-s)q^{l+1}(q^{l+1}-s^{l+1}) \right. \\ &\quad + C_0C_2[(q-1)(q+s)q^{2(l+1)} + (q+1)(q-s)s^{l+1} \\ &\quad \left. - 2(q^2-s)q^{l+1}s^{l+1}] \right\}^{-1/2}, \end{aligned} \quad (32)$$

with the initial condition being $\langle 0 | S_2(\hat{n}) | 0 \rangle = \varepsilon_0$ (ε_0 is a real number).

4 Conclusion

In this paper we have obtained the single boson and the single inversion boson realizations of the five-parameter deformed algebra $\mathcal{R}_{q,s}^{c_0, c_1, c_2}$ by generalizing the Holstein-Primakoff realization and the Dyson realization of $SU(2)$. Hence, using these results, the representations of $\mathcal{R}_{q,s}^{c_0, c_1, c_2}$ in the Fock space $\mathcal{F}$ may be directly obtained using eqs. (5)–(7), which are not given here. Comparing the boson realization for $\alpha = 0$ (see eqs. (13) and (15)) with the inversion boson realization (see eqs. (25) and (26)), there are simple replacements: $\hat{n} + 1 \rightarrow -\hat{n}$ in the functions $h(\hat{n})$ and $D(x, \hat{n})$, and $\hat{n} + 1 \rightarrow 1/\hat{n}$ elsewhere. We have revealed the fact that the non-unitary realizations and the unitary ones may be related by the similarity transformations, which have been obtained by solving the corresponding unitarization equations.

This work was supported by the National Basic Research Program of China (2009CB929402).

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