On time-domain methods for milling stability analysis

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As a basic and advanced machining technique, the high-speed milling process plays an important role in realizing the goal of high performance manufacturing. From the viewpoint of machining dynamics, obtaining chatter-free machining parameters is a prerequisite to guaranteeing machining accuracy and improving machining efficiency. This paper gives an overview on recent progress in time domain semi-analytical methods for chatter stability analysis of milling processes. The state of art methods of milling stability prediction in milling processes and their applications are introduced in detail. The bottlenecks involved are analyzed, and potential solutions are discussed. Finally, a brief prospect on future works is presented.

milling dynamics, stability analysis, semi-analytical method

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High speed milling is widely utilized in manufacturing of complex surfaces for the fields of aerospace, ship, automotive, dies and molds etc., due to its advantage of obtaining high machining accuracy and high material removal rate with keeping low-amplitude cutting forces. Regarding to modeling of machining processes, the research topics on high speed milling can be categorized as: tool path planning, machining dynamics and machining physical modeling. Tool path planning is to plan the tool path according to the workpiece model, machining strategy and required machined error [1–5]. From the viewpoint of machining dynamics, the relative vibration between the workpiece and cutter in the milling process is the main reason of reducing product surface quality and limiting the production efficiency. To suppress the vibration impact by optimizing the machining parameters, the following two questions need to be addressed.

(1) Stability analysis based on the dynamics of milling processes. As far as chatter vibrations are concerned, there are four different mechanisms: regeneration [6], mode coupling [7], friction [8] and thermo-mechanics of chip for-

mation [9]. In milling processes, regenerative chatter is the most common form of self-excited vibration to reduce surface quality and machining efficiency [10,11]. The works on analysis of the stability of milling processes focus on calculating the stability boundary of the machining parameters based on the dynamic models characterizing the milling processes.

(2) Machining accuracy analysis on the basis of stability analysis. Due to forced vibrations in milling processes, chatter-free machining parameters cannot ensure high performance machining. Vibration-induced surface errors can be classified as surface location error (SLE) [12] and surface roughness [13]. It is essential to take the stability and dynamical errors into account in machining optimization models to achieve high performance milling.

1 Milling dynamics model

Without loss of generality, two degree-of-freedom (DOF) lumped parameter dynamics model of the cutter is shown in Figure 1. In this figure, $\phi_j(t)$ is the angular position of the j th tooth, Ω is the spindle speed (r min^{-1}), a_e is the radial depth

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Figure 1 Two DOF lumped parameter dynamics model of the milling process.

of cut, f is the feed direction of the workpiece with respect to the cutter, $F_{t,j}$ and $F_{n,j}$ are the tangential and normal cutting force components for the j th tooth, respectively.

The governing equation of the milling process shown in Figure 1 can be formulated as the following delayed differential equation (DDE) with the periodic coefficient [14–16]:

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{K}_c(t)[\mathbf{q}(t) - \mathbf{q}(t-T)], \quad (1)$$

where $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$ and $\mathbf{q}(t)$ represent the modal matrix, damping matrix, stiffness matrix and modal vector of the cutter, respectively. T is the time delay, equal to tooth passing period, i.e., $T = 60/(N\Omega)$, where N is the number of the cutter teeth. $\mathbf{K}_c(t)$ is the cutting force coefficient matrix (referring to [14]), and $\mathbf{K}_c(t) = \mathbf{K}_c(t+T)$, defined as

$$\mathbf{K}_c(t) = \begin{bmatrix} -a_p h_{xx}(t) & -a_p h_{xy}(t) \\ -a_p h_{yx}(t) & -a_p h_{yy}(t) \end{bmatrix}, \quad (2)$$

where a_p is the axial depth of cut, $h_{xx}(t)$, $h_{xy}(t)$, $h_{yx}(t)$ and $h_{yy}(t)$ are

$$h_{xx}(t) = \sum_{j=1}^N g(\phi_j(t)) \sin(\phi_j(t)) \left[K_t \cos(\phi_j(t)) + K_n \sin(\phi_j(t)) \right], \quad (3)$$

$$h_{xy}(t) = \sum_{j=1}^N g(\phi_j(t)) \cos(\phi_j(t)) \left[K_t \cos(\phi_j(t)) + K_n \sin(\phi_j(t)) \right], \quad (4)$$

$$h_{yx}(t) = \sum_{j=1}^N g(\phi_j(t)) \sin(\phi_j(t)) \left[-K_t \sin(\phi_j(t)) + K_n \cos(\phi_j(t)) \right], \quad (5)$$

$$h_{yy}(t) = \sum_{j=1}^N g(\phi_j(t)) \cos(\phi_j(t)) \left[-K_t \sin(\phi_j(t)) + K_n \cos(\phi_j(t)) \right]. \quad (6)$$

In eqs. (3)–(6), K_t and K_n are the tangential and normal linearized cutting force coefficients, respectively. $\phi_j(t)$ is defined by

$$\phi_j(t) = (2\pi\Omega/60)t + (j-1) \cdot 2\pi/N. \quad (7)$$

The window function $g(\phi_j(t))$ is defined as

$$g(\phi_j(t)) = \begin{cases} 1 & \text{if } \phi_{st} < \phi_j(t) \bmod 2\pi < \phi_{ex}, \\ 0 & \text{otherwise,} \end{cases} \quad (8)$$

where ϕ_{st} and ϕ_{ex} are the start and exit angles of the j th tooth, respectively. For down-milling, $\phi_{st} = \arccos(2a_e/D - 1)$, and $\phi_{ex} = \pi$; for up-milling, $\phi_{st} = 0$ and $\phi_{ex} = \arccos(1 - 2a_e/D)$, where a_e/D is the radial immersion ratio (the radial depth of cut/the cutter diameter).

Denoting by $\mathbf{p}(t) = \mathbf{M}\dot{\mathbf{q}}(t) + \mathbf{C}\mathbf{q}(t)/2$ and $\mathbf{x}(t) = \begin{bmatrix} \mathbf{q}(t) \\ \mathbf{p}(t) \end{bmatrix}$, eq. (1) can be rewritten as the state-space form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}(t)[\mathbf{x}(t) - \mathbf{x}(t-T)], \quad (9)$$

where

$$\mathbf{A} = \begin{bmatrix} -\mathbf{M}^{-1}\mathbf{C}/2 & \mathbf{M}^{-1} \\ \mathbf{C}\mathbf{M}^{-1}\mathbf{C}/4 - \mathbf{K} & -\mathbf{C}\mathbf{M}^{-1}/2 \end{bmatrix}, \quad \mathbf{B}(t) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{K}_c(t) & \mathbf{0} \end{bmatrix}.$$

Due to the complexity of the geometry of cutters and interface dynamics of cutter-holder-spindle, it is challenging to obtain the modal parameters $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ by using analytical or numerical methods. Nowadays, there are two methods for obtaining the frequency response functions (FRFs) of the cutter as follows. (1) Experimental modal testing method [14]. With the advantage of being easy to implement, the main disadvantage of this method is that the modal testing experiment should be conducted for each combination of cutter-holder-spindle. (2) Experimental-analytical method. Schmitz et al. [17,18] proposed the receptance coupling substructure analysis (RCSA) method. By using this method, the interface parameters of cutter-holder-spindle can be determined and the FRFs can be analytically calculated. Hence, the work burden of modal testing experiments can be reduced dramatically.

To determine the cutting force parameters K_t and K_n , the main methods can be categorized as follows. (1) The mechanistic based method [14,19]. For the chosen cutter and workpiece, the mechanistic model can be calibrated by using measured cutting forces from several cutting tests. (2) The orthogonal cutting data based method [20,21]. The experimental times could be reduced, since the cutting force

coefficients can be calculated based on the orthogonal cutting database.

As for the multi-mode effect of the cutter and the workpiece in the milling processes, refer to [16] and [22], respectively. As for the mode-coupling effect, refer to [23].

It is worth noting that the stability of milling processes can also be directly determined via analysis of process signals, e.g., the cutting force signals [24] or the motor current signals [25], acquired from experiments. The advantage of this method is that no dynamic model is required. However, the disadvantage is also obvious, i.e., a large number of experiments due to different machining parameters should be conducted to determine the stability boundary. The main scope of this paper is to review the methods for stability analysis and machining accuracy analysis based on eq. (9).

2 Model based stability analysis and machining error calculation

Compared with traditional periodic systems without time delay, the monodromy operator and phase space of eq. (9) are both of infinite dimensions, which results in the essential difficulty for stability analysis of this system. Many kinds of approximate methods have been proposed for solving this problem. Generally, they can be classified as two kinds: numerical methods and analytical (or semi-analytical) methods.

2.1 Stability analysis methods

(1) Numerical methods. The key idea of the numerical methods is to calculate the time-domain response of the DDE, then distinguish instable vibrations according to some stability criteria.

Sridhar et al. [26] proposed a numerical algorithm for the general milling process. Tlustý et al. [27,28] proposed a finite difference based simulation procedure for chatter stability analysis in milling. Smith and Tlustý [29] suggested the Peak-to-Peak (PTP) diagrams for time-domain milling stability analysis. Davies et al. [30,31] proposed a discrete map based method for stability prediction in low radial immersion milling. Campomanes and Altintas [32] proposed an improved time domain simulation method and the instability criterion by defining the ratio of the maximum dynamic uncut chip thickness to the maximum static uncut chip thickness. Li and Liu [33] presented a time domain simulation method for milling processes based on the Runge-Kutta algorithm.

The numerical methods can be used for the complex cases considering nonlinear cutting force models and time-varying radial immersion boundaries etc. The main disadvantage of this kind of methods is that the computational efficiency is generally much lower than analytical (or semi-analytical) methods [10].

(2) Analytical (semi-analytical) methods. The essential idea of analytical (or semi-analytical) methods is to approximate the original DDE, yielding finite monodromy operators to approximate the original infinite one. The main advantage of this kind of methods is to efficiently predict the chatter stability with acceptable approximate errors.

Minis and Yanushevsky [34] used the Nyquist criterion to analyze the milling stability based on the theory of periodic differential equations. Altintas and Budak [35,36] proposed the zeroth order approximation (ZOA) method. The analytical solution of the critical axial depth of cut for one fixed spindle speed is obtained by this method [35]. The main advantage of this method is its computational efficiency. However, it cannot predict the flip bifurcation in low radial immersion milling. Merdol and Altintas [37] then proposed the multi frequency solution to solve this problem. Since the iteration procedure for chatter frequencies is employed, there is no analytical solution. In these years, the team of Professor Altintas has applied the frequency domain method in the cases of ball-end milling, milling with variable pitch cutters, plunge milling, etc.

Bayly et al. [38] proposed the temporal finite element analysis (TFEA) method for single DOF milling model, then generalized it to two DOF case. This method assumes the form of the solution on the time elements, and then constructs the Floquet transition matrix based on the weighted residual method. Mann and Patel [39] presented the TFEA method in the state space form. Garg et al. [40] generalized the original TFEA for stability analysis of general time-delayed systems with parametric excitation, employing multi-grid techniques.

Inspurger and Stépán [41,42] proposed the zeroth order semi-discretization method (SDM) for milling stability analysis. By using the zeroth approximation of the time delay item and the time-periodic item on one time interval, this method simplifies the original DDE as a series of ordinary differential equations (ODEs). On this basis, the approximate Floquet transition matrix is constructed. To improve the calculation accuracy, Inspurger and Stépán then proposed the first order SDM [43]. In this procedure, the time delayed item is approximated by the linear polynomial. Long et al. utilized the SDM to analyze the milling stability considering the loss-of-contact effects between the workpiece and the tool [44] and the varying spindle speed effect [45]. Li et al. [46] proposed an improved SDM by using the Magnus-Gaussian series. Song et al. used the SDM to calculate the chatter stability considering the runout effect of the tool [47] and the feed effect on stability [48]. Wan et al. [49] proposed the unified SDM taking the multi-time delay effect into account.

Besides the aforementioned methods, Olgac and Hosek [50] proposed the root locus plot analysis method for prediction of chatter stability, and then generalized to variable pitch milling [51]. Ulsoy et al. [52,53] proposed the Lambert W function based method for machining stability anal-

ysis. Butcher et al. presented the Chebyshev polynomial based method [54] and Chebyshev collocation method [55] for milling stability prediction, respectively.

2.2 Machining error calculation methods

For calculation of surface location error, there exist the numerical method [12], the frequency domain method [56], the TFEA method [16,57,58] and the harmonic balance method [59]. It is worth noting that the TFEA method can be used for simultaneous prediction of milling stability and SLE [58]. It is much efficient than other methods, since the fixed point technique is employed to calculate the SLE.

There are some shortcomings of these widely utilized stability analysis methods in computational efficiency or generality, respectively. Since high speed milling technique has the characteristics of high speed, time-varying feed speed and radial immersion, the high-efficient and general calculation method is urgently demanded for simultaneous prediction of chatter stability and SLE.

3 Time-domain semi-analytical methods

The key idea of time-domain semi-analytical methods is to approximate the original DDE by using some reasonable simplifications, and then construct the corresponding Floquet transition matrices according to the simplified forms. At last, the stability of the system can be evaluated by using the Floquet theory. From the viewpoint of modern numerical analysis, there exist three kinds of numerical techniques to calculate the dynamic response of dynamical systems, i.e., the differential equation based methods, the variational method based methods and the integral equation based methods. The SDM and TFEA correspond to the first two kinds, respectively. Inspired by the numerical methods for integral equations with high precision, we proposed the full-discretization method (FDM) [61] and numerical integration method [62] in the framework of integral equations.

3.1 Differential equation based methods

In the SDM [42], the first step to analyze eq. (9) is to discretize the time period T as m small time intervals, i.e., $T = m\tau$, where m is one of the positive integers. On each interval $k\tau \leq t \leq (k+1)\tau$, ($k=0, \dots, m$), with the initial condition $\mathbf{x}_k = \mathbf{x}(k\tau)$, by discretizing the time delayed item $\mathbf{x}(t-T)$, performing zeroth order average of the delayed item and the periodic matrix $\mathbf{B}(t)$ on each time interval, respectively, eq. (9) can be simplified as the following ODE,

$$\dot{\mathbf{x}}_k(t) = (\mathbf{A} + \mathbf{B}_k) \mathbf{x}_k(t) - \frac{1}{2} \mathbf{B}_k [\mathbf{x}_{k-m+1} + \mathbf{x}_{k-m}], \quad (10)$$

where

$$\mathbf{B}_k = \frac{1}{\tau} \int_{t_k}^{t_{k+1}} \mathbf{B}(t) dt. \quad (11)$$

From eq. (10), $\mathbf{x}_{k+1} = \mathbf{x}(k\tau + \tau)$ can be expressed as

$$\mathbf{x}_{k+1} = e^{(\mathbf{A} + \mathbf{B}_k)\tau} \mathbf{x}_k - \frac{1}{2} [e^{(\mathbf{A} + \mathbf{B}_k)\tau} - \mathbf{I}] [\mathbf{A} + \mathbf{B}_k]^{-1} \mathbf{B}_k [\mathbf{x}_{k-m+1} + \mathbf{x}_{k-m}]. \quad (12)$$

Eq. (12) can be rewritten as

$$\mathbf{y}_{k+1} = \mathbf{S}_k \mathbf{y}_k, \quad (13)$$

where $\mathbf{y}_k = \text{col}(\mathbf{x}_k, \mathbf{x}_{k-1}, \dots, \mathbf{x}_{k-m})$,

$$\mathbf{S}_k = \begin{bmatrix} e^{(\mathbf{A} + \mathbf{B}_k)\tau} & \mathbf{0} & \dots & \mathbf{0} & -\frac{1}{2} [e^{(\mathbf{A} + \mathbf{B}_k)\tau} - \mathbf{I}] [\mathbf{A} + \mathbf{B}_k]^{-1} \mathbf{B}_k & -\frac{1}{2} [e^{(\mathbf{A} + \mathbf{B}_k)\tau} - \mathbf{I}] [\mathbf{A} + \mathbf{B}_k]^{-1} \mathbf{B}_k \\ \mathbf{I} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix}.$$

On this basis, the Floquet matrix can be constructed by using eq. (13), i.e.,

$$\Psi_1 = \mathbf{S}_{m-1} \mathbf{S}_{m-2} \dots \mathbf{S}_0. \quad (14)$$

According to eq. (14), the chatter stability can be predicted based on the Floquet theory, i.e.,

$$\max(|\lambda(\Psi_1)|) \begin{cases} < 1 & \text{stable,} \\ = 1 & \text{stable boundary,} \\ > 1 & \text{instable,} \end{cases} \quad (15)$$

where $\lambda(\cdot)$ denotes the spectral radius of a matrix.

3.2 Weighted residual method based methods

In the TFEA method [63], denote by t_0 the time cutter leaves the workpiece, t_f the duration of the free vibration after the cutter leaves the workpiece, t_c the duration of the forced vibration in cut. Firstly, the duration of forced vibration $[t_0 + t_f, t_0 + T]$ is equally divided into m parts, where $\tau = (T - t_f)/m$. Denote by the sampling points by $t_i = t_0 + t_f + (i-1)\tau$, where $i = 1, \dots, m+1$ and $t_1 = t_0 + t_f$.

On the interval $[t_i, t_{i+1}]$, $i = 1, \dots, m$, the assumed state function and time-delayed function are

$$\mathbf{x}_i(t) = \sum_{k=1}^3 \mathbf{a}_{ik}^n \zeta_k(\sigma), \quad (16)$$

$$\mathbf{x}_i(t-T) = \sum_{k=1}^3 \mathbf{a}_{ik}^{n-1} \zeta_k(\sigma), \quad (17)$$

where $\zeta_k(\sigma)$ is the Legendre polydomial, σ is the local time $0 \leq \sigma \leq \tau$, $\mathbf{a}_{ik}^n$ represents the k th state on the interval $[t_i, t_{i+1}]$ in the n th period.

Without loss of generality, let $m = 2$, and substitute eqs. (16), (17) into eq. (9), the following algebraic equation can be derived based on the weighed residual method:

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{N}_{11}^1 & \mathbf{N}_{12}^1 & \mathbf{N}_{13}^1 & \mathbf{0} & \mathbf{0} \\ \mathbf{N}_{21}^1 & \mathbf{N}_{22}^1 & \mathbf{N}_{23}^1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{N}_{11}^2 & \mathbf{N}_{12}^2 & \mathbf{N}_{13}^2 \\ \mathbf{0} & \mathbf{0} & \mathbf{N}_{21}^2 & \mathbf{N}_{22}^2 & \mathbf{N}_{23}^2 \end{bmatrix} \begin{bmatrix} \mathbf{a}_{11}^n \\ \mathbf{a}_{12}^n \\ \mathbf{a}_{21}^n \\ \mathbf{a}_{22}^n \\ \mathbf{a}_{23}^n \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \exp(\mathbf{A}t_f) \\ \mathbf{P}_{11}^1 & \mathbf{P}_{12}^1 & \mathbf{P}_{13}^1 & \mathbf{0} & \mathbf{0} \\ \mathbf{P}_{21}^1 & \mathbf{P}_{22}^1 & \mathbf{P}_{23}^1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{P}_{11}^2 & \mathbf{P}_{12}^2 & \mathbf{P}_{13}^2 \\ \mathbf{0} & \mathbf{0} & \mathbf{P}_{21}^2 & \mathbf{P}_{22}^2 & \mathbf{P}_{23}^2 \end{bmatrix} \begin{bmatrix} \mathbf{a}_{11}^{n-1} \\ \mathbf{a}_{12}^{n-1} \\ \mathbf{a}_{21}^{n-1} \\ \mathbf{a}_{22}^{n-1} \\ \mathbf{a}_{23}^{n-1} \end{bmatrix}, \quad (18)$$

where

$$\mathbf{N}_{pk}^i = \int_0^\tau (\mathbf{I} \dot{\zeta}_k(\sigma) - (\mathbf{A} + \mathbf{B}) \zeta_k(\sigma)) \psi_p(\sigma) d\sigma, \quad (19)$$

$$\mathbf{P}_{pk}^i = \int_0^\tau \mathbf{B} \zeta_k(\sigma) \psi_p(\sigma) d\sigma, \quad (20)$$

$\psi_p(\sigma)$ is the Legendre polynomial, i.e., $\psi_1(\sigma) = 1$ and $\psi_2(\sigma) = 2(\sigma/\tau) - 1$.

Eq. (18) can be expressed as the compact form $\mathbf{G}\mathbf{a}_n = \mathbf{H}\mathbf{a}_{n-1}$. Then the Floquet matrix is

$$\mathbf{\Psi}_2 = \mathbf{G}^{-1} \mathbf{H}. \quad (21)$$

In similar, the stability can be predicted by using the spectral radius of $\mathbf{\Psi}_2$ according to the Floquet theory.

3.3 Integral equation based methods

In eq. (9), regarding the item $\mathbf{B}(t)[\mathbf{x}(t) - \mathbf{x}(t-T)]$ as the non-homogenous item of the equation $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$, the solution of eq. (9) can be expressed as

$$\mathbf{x}(t) = \exp[\mathbf{A}(t-t_0)]\mathbf{x}(t_0) + \int_{t_0}^t \left\{ \exp[\mathbf{A}(t-\xi)] \mathbf{B}(\xi) [\mathbf{x}(\xi) - \mathbf{x}(\xi-T)] \right\} d\xi, \quad (22)$$

where $\mathbf{x}(t_0)$ is the state of the system at $t = t_0$.

(1) Full-discretization method (FDM). Similar to the SDM, the first step of the FDM [61] for eq. (22) is also equally dividing the period T as m parts, i.e., $T = m\tau$, where m is one of the integral numbers. On each time interval $[k\tau, (k+1)\tau]$, ($k = 0, \dots, m$), regarding $\mathbf{x}_k = \mathbf{x}(k\tau)$ as the initial condition, eq. (22) can be rewritten as ($0 \leq t \leq \tau$)

$$\begin{aligned} \mathbf{x}(k\tau + t) &= \exp(\mathbf{A}t)\mathbf{x}(k\tau) + \int_0^t \exp(\mathbf{A}\xi) \mathbf{B}(k\tau + t - \xi) \\ &\quad \times [\mathbf{x}(k\tau + t - \xi) - \mathbf{x}(k\tau + t - \xi - T)] d\xi. \end{aligned} \quad (23)$$

As for eq. (23), letting $t = \tau$, and performing linear interpolations for $\mathbf{B}(k\tau + \tau - \xi)$, $\mathbf{x}(k\tau + \tau - \xi)$ and $\mathbf{x}(k\tau + \tau - \xi - T)$ on the interval $[k\tau, (k+1)\tau]$, $\mathbf{x}_{k+1}$ can be derived as

$$\begin{aligned} \mathbf{x}_{k+1} &= [\mathbf{I} - \mathbf{F}_{k+1}]^{-1} (\mathbf{F}_0 + \mathbf{F}_{0,k}) \mathbf{x}_k \\ &\quad + [\mathbf{I} - \mathbf{F}_{k+1}]^{-1} \mathbf{F}_{m-1} \mathbf{x}_{k+1-m} + [\mathbf{I} - \mathbf{F}_{k+1}]^{-1} \mathbf{F}_m \mathbf{x}_{k-m}, \end{aligned} \quad (24)$$

where

$$\mathbf{F}_0 = \mathbf{\Phi}_0, \quad (25)$$

$$\mathbf{F}_{0,k} = (\mathbf{\Phi}_2/\tau) \mathbf{B}_0^{(k)} + (\mathbf{\Phi}_3/\tau) \mathbf{B}_1^{(k)}, \quad (26)$$

$$\mathbf{F}_{k+1} = (\mathbf{\Phi}_1 - \mathbf{\Phi}_2/\tau) \mathbf{B}_0^{(k)} + (\mathbf{\Phi}_2 - \mathbf{\Phi}_3/\tau) \mathbf{B}_1^{(k)}, \quad (27)$$

$$\mathbf{F}_{m-1} = -\mathbf{F}_{k+1}, \quad (28)$$

$$\mathbf{F}_m = -\mathbf{F}_{0,k}, \quad (29)$$

$$\begin{aligned} \mathbf{\Phi}_0 &= \exp(\mathbf{A}\tau), \mathbf{\Phi}_1 = \int_0^\tau \exp(\mathbf{A}\xi) d\xi, \mathbf{\Phi}_2 \\ &= \int_0^\tau \xi \exp(\mathbf{A}\xi) d\xi, \mathbf{\Phi}_3 = \int_0^\tau \xi^2 \exp(\mathbf{A}\xi) d\xi. \end{aligned} \quad (30)$$

Eq. (24) can be expressed as the compact form

$$\mathbf{y}_{k+1} = \mathbf{E}_k \mathbf{y}_k, \quad (31)$$

where $\mathbf{y}_k = \text{col}(\mathbf{x}_k, \mathbf{x}_{k-1}, \dots, \mathbf{x}_{k-m})$,

$$\mathbf{E}_k = \begin{bmatrix} [\mathbf{I} - \mathbf{F}_{k+1}]^{-1} (\mathbf{F}_0 + \mathbf{F}_{0,k}) & \mathbf{0} & \dots & \mathbf{0} & [\mathbf{I} - \mathbf{F}_{k+1}]^{-1} \mathbf{F}_{m-1} & [\mathbf{I} - \mathbf{F}_{k+1}]^{-1} \mathbf{F}_m \\ \mathbf{I} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix}. \quad (32)$$

Using eq. (31), the Floquet matrix can be constructed as

$$\mathbf{\Psi}_3 = \mathbf{E}_{m-1} \mathbf{E}_{m-2} \dots \mathbf{E}_0. \quad (33)$$

Based on the Floquet theory, the system stability can be determined by using the spectral radius of $\mathbf{\Psi}_3$. During the procedure of the FDM, the matrix exponential related functions in eq. (30) all only depend on spindle speed, yet independent of the depth of cut. On the other hand, in eq. (12) of the SDM, the matrix exponential $e^{(\mathbf{A} + \mathbf{B}_k)\tau}$ always depends on the spindle speed and depth of cut simultaneously. Hence, the procedure of sweeping the depth of cut and spindle speed plane to determine the stability diagram, the efficiency of the FDM is much higher than that of the SDM.

The higher order FDM [64] is proposed by employing the higher interpolation schemes for $\mathbf{B}(k\tau + \tau - \xi)$, $\mathbf{x}(k\tau + \tau - \xi)$ and $\mathbf{x}(k\tau + \tau - \xi - T)$. Furthermore, this method can be generalized for the case of systems with multiple time delays [65].

(2) Numerical integration method (NIM). Similar to the TFEA method, for the numerical integration method [62], denote by t_0 the time the cutter leaves the workpiece, t_f the duration of the free vibration after the cutter leaves the workpiece, and t_c the duration of the forced vibration. When the cutter leaves the workpiece, $\mathbf{B}(\xi)$ in eq. (22) equals to $\mathbf{0}$, and eq. (22) becomes $\mathbf{x}(t) = e^{A(t-t_0)} \mathbf{x}(t_0)$. At the end of the free vibration, i.e., at $t = t_0 + t_f$, the state item is expressed as

$$\mathbf{x}(t_0 + t_f) = \exp(\mathbf{A}t_f) \mathbf{x}(t_0). \quad (34)$$

As for the forced vibration duration $[t_0 + t_f, t_0 + T]$, divide it equally as m parts, i.e., $\tau = (T - t_f)/m$. Denote by $t_i = t_0 + t_f + (i-1)\tau$, $i = 1, \dots, m+1$ the sampling time points. Note that $\mathbf{x}(t_1) = \mathbf{x}(t_0 + t_f)$.

At t_i ($i = 2, \dots, m+1$), $\mathbf{x}(t_i)$ can be expressed as

$$\begin{aligned} \mathbf{x}(t_i) = & \exp[\mathbf{A}(t_i - t_{i-1})] \mathbf{x}(t_{i-1}) + \int_{t_{i-1}}^{t_i} \{ \exp[\mathbf{A}(t_i - \xi)] \\ & \times \mathbf{B}(\xi) [\mathbf{x}(\xi) - \mathbf{x}(\xi - T)] \} d\xi. \end{aligned} \quad (35)$$

which is a Volterra equation of the second kind [66]. Using the trapezium rule [67], eq. (35) can be simplified as

$$\begin{aligned} \mathbf{x}(t_i) = & \exp[\mathbf{A}(t_i - t_{i-1})] \mathbf{x}(t_{i-1}) \\ & + \frac{\tau}{2} \{ \exp[\mathbf{A}(t_i - t_{i-1})] \mathbf{B}(t_{i-1}) (\mathbf{x}(t_{i-1}) \\ & - \mathbf{x}(t_{i-1} - T)) + \mathbf{B}(t_i) (\mathbf{x}(t_i) - \mathbf{x}(t_i - T)) \}. \end{aligned} \quad (36)$$

Combining eqs. (34) and (36), the discrete map is derived as

$$\left(\mathbf{I} - \mathbf{C}_1 - \frac{\tau}{2} \mathbf{D}_1 \right) \begin{bmatrix} \mathbf{x}(t_1) \\ \vdots \\ \mathbf{x}(t_{m+1}) \end{bmatrix} = \left(-\frac{\tau}{2} \mathbf{D}_1 + \mathbf{E} \right) \begin{bmatrix} \mathbf{x}(t_1 - T) \\ \vdots \\ \mathbf{x}(t_{m+1} - T) \end{bmatrix}, \quad (37)$$

where

$$\mathbf{C}_1 = \begin{bmatrix} \mathbf{0} & & & \\ \exp(\mathbf{A}\tau) & \mathbf{0} & & \\ & \exp(\mathbf{A}\tau) & \mathbf{0} & \\ & & \ddots & \ddots \\ & & & \exp(\mathbf{A}\tau) & \mathbf{0} \end{bmatrix},$$

$$\mathbf{D}_1 = \begin{bmatrix} \mathbf{0} & & & \\ \exp(\mathbf{A}\tau) \mathbf{B}_1 & \mathbf{B}_2 & & \\ & \exp(\mathbf{A}\tau) \mathbf{B}_2 & \mathbf{B}_3 & \\ & & \ddots & \ddots \\ & & & \exp(\mathbf{A}\tau) \mathbf{B}_m & \mathbf{B}_{m+1} \end{bmatrix},$$

$$\mathbf{E} = \begin{bmatrix} \mathbf{0} & \cdots & \mathbf{0} & \exp(\mathbf{A}t_f) \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \end{bmatrix} \text{ and } \mathbf{B}_i = \mathbf{B}(t_i), \quad i = 1, \dots, m+1.$$

Then, on one tooth passing period, the Floquet matrix Ψ_4 is obtained

$$\Psi_4 = \mathbf{J}^{-1} \mathbf{L}, \quad (38)$$

$$\text{where } \mathbf{J} = \left(\mathbf{I} - \mathbf{C}_1 - \frac{\tau}{2} \mathbf{D}_1 \right), \quad \mathbf{L} = \left(-\frac{\tau}{2} \mathbf{D}_1 + \mathbf{E} \right).$$

In the same way, the chatter stability can be predicted by using the spectral radius of Ψ_4 according to the Floquet theory. On this basis, the spectral method with exponential convergence rate [68] and the variable-step integration method suitable for the multiple time delay systems [69] are proposed.

As shown in Table 1, compared eq. (14) via the SDM with eq. (33) via the FDM, and compared eq. (21) via the TFEA method and eq. (38) via the NIM, the integral equation theory provides the milling stability analysis as a general framework.

4 Applications of the integral equation based algorithms

Besides chatter stability analysis, FDM and NIM can also be applied for calculation of surface location error (SLE) and sensitivity analysis.

4.1 Simultaneous prediction of SLE

As stated in Section 3, the prerequisite of high performance milling is to predict simultaneously chatter stability and dynamic machining error [70]. The FDM in Section 3.3 can be generalized to predict SLE. To discuss the simultaneous prediction problem, eq. (1) should be modified as

Table 1 Comparison of floquet transition matrices

Time-domain methods		Floquet matrices	Equations
SDM [42]		$\Psi_1 = \mathbf{S}_{m-1} \mathbf{S}_{m-2} \cdots \mathbf{S}_0$	(14)
TFEA [39]		$\Psi_2 = \mathbf{G}^{-1} \mathbf{H}$	(21)
Integral equation	FDM [61]	$\Psi_3 = \mathbf{E}_{m-1} \mathbf{E}_{m-2} \cdots \mathbf{E}_0$	(33)
	NIM [62]	$\Psi_4 = \mathbf{J}^{-1} \mathbf{L}$	(38)

$$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = K_c(t)[q(t) - q(t-T)] + f_0(t), \quad (39)$$

where

$$f_0(t) = \sum_{j=1}^N a_p \cdot g(\phi_j(t)) \left\{ f_t \cdot \begin{bmatrix} -K_t s\phi_j c\phi_j - K_n s^2\phi_j \\ K_t s^2\phi_j - K_n s\phi_j c\phi_j \end{bmatrix} + \begin{bmatrix} -K_{te} c\phi_j - K_{ne} s\phi_j \\ K_{te} s\phi_j - K_{ne} c\phi_j \end{bmatrix} \right\}, \quad (40)$$

is the steady cutting force vector, $s\phi_j = \sin\phi_j$ and $c\phi_j = \cos\phi_j$.

Eq. (39) can be also rewritten in the state space form

$$\dot{x}(t) = Ax(t) + B(t)[x(t) - x(t-T)] + f(t), \quad (41)$$

where the matrices A and $B(t)$ are the same as those in

$$\text{eq. (9), } f(t) = \begin{bmatrix} 0 \\ f_0(t) \end{bmatrix}.$$

Similar to eq. (23), on the interval $[k\tau, (k+1)\tau]$, ($k = 0, \dots, m$), the solution of eq. (41) can be expressed as ($0 \leq t \leq \tau$)

$$\begin{aligned} x(k\tau + t) = & \exp(At)x(k\tau) + \int_0^t \left\{ \exp(A\xi)B(k\tau + t - \xi) \right. \\ & \times [x(k\tau + t - \xi) - x(k\tau + t - \xi - T)] \Big\} d\xi \\ & + \int_0^t \exp(A\xi)f(k\tau + t - \xi)d\xi. \end{aligned} \quad (42)$$

In eq. (42), letting $t = \tau$, and using linear interpolations for $B(k\tau + \tau - \xi)$, $x(k\tau + \tau - \xi)$, $x(k\tau + \tau - \xi - T)$ and $f(k\tau + \tau - \xi)$, respectively, yielding

$$\begin{aligned} x_{k+1} = & [I - F_{k+1}]^{-1} (F_0 + F_{0,k})x_k + [I - F_{k+1}]^{-1} F_{m-1}x_{k+1-m} \\ & + [I - F_{k+1}]^{-1} F_m x_{k-m} + P_k, \end{aligned} \quad (43)$$

where

$$P_k = \Phi_1 f_0^{(k)} + \Phi_2 f_1^{(k)}. \quad (44)$$

Using the vector $y_k = \text{col}(x_k, x_{k-1}, \dots, x_{k+1-m}, x_{k-m})$, eq. (43) is re-expressed as

$$y_{k+1} = E_k y_k + Q_k P_k, \quad (45)$$

where E_k is the same as eq. (32), and

$$Q_k = \text{col}([I - F_{k+1}]^{-1}, 0, \dots, 0). \quad (46)$$

The state transition relationship on one period can be expressed by the matrices E_k, Q_k, P_k ($k = 0, \dots, m-1$), i.e.,

$$y_m = \Psi_3 y_0 + P, \quad (47)$$

where Ψ_3 is as shown in eq. (33), and P is defined by

$$P = Q_{m-1}P_{m-1} + \sum_{i=0}^{m-2} (E_{m-1}E_{m-2} \cdots E_{i+1}Q_i P_i). \quad (48)$$

The steady-state coefficients of the ℓ -th period is calculated from eq. (47) as

$$y_\ell^* = (I - \Psi_3)^{-1} P. \quad (49)$$

Using y_ℓ^* in eq. (47), the SLE and the vibration induced roughness can be calculated easily [13].

4.2 Sensitivity analysis of the stability boundary

Kurdi et al. [72] discussed the sensitivity of the stability boundary by using the TFEA method and finite difference method. Based on the NIM, constructing the Floquet transition matrix, we derive the semi-analytical expressions of the Floquet matrix with respect to machining parameters, without employing the finite difference method [71].

From eq. (38), the derivative of the Floquet matrix Ψ_4 with respect to the axial depth of cut a_p is

$$\frac{\partial \Psi_4}{\partial a_p} = J^{-1} \left(-\frac{\tau}{2} D_1 + \frac{\tau}{2} D_1 J^{-1} L \right). \quad (50)$$

The derivative of Ψ_4 with respect to the spindle speed Ω is

$$\frac{\partial \Psi_4}{\partial \Omega} = J^{-1} \left(\frac{\partial L}{\partial \Omega} - \frac{\partial J}{\partial \Omega} J^{-1} L \right), \quad (51)$$

where

$$\begin{aligned} \frac{\partial L}{\partial \Omega} = & \left(-\frac{1}{2} \frac{d\tau}{d\Omega} \right) D_1 - \frac{\tau}{2} \frac{d\tau}{d\Omega} \begin{bmatrix} 0 & & & & \\ Ae^{A\tau} B_1 & 0 & & & \\ & Ae^{A\tau} B_2 & 0 & & \\ & & \ddots & \ddots & \\ & & & Ae^{A\tau} B_m & 0 \end{bmatrix} \\ & + \frac{dt_f}{d\Omega} \begin{bmatrix} 0 & \cdots & 0 & e^{A\tau_f} A \\ 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix}, \end{aligned} \quad (52)$$

$$\begin{aligned} \frac{\partial J}{\partial \Omega} = & \left(-\frac{1}{2} \frac{d\tau}{d\Omega} \right) D_1 - \frac{\tau}{2} \frac{d\tau}{d\Omega} \begin{bmatrix} 0 & & & & \\ Ae^{A\tau} B_1 & 0 & & & \\ & Ae^{A\tau} B_2 & 0 & & \\ & & \ddots & \ddots & \\ & & & Ae^{A\tau} B_m & 0 \end{bmatrix} \\ & - \frac{d\tau}{d\Omega} \begin{bmatrix} 0 & & & & \\ Ae^{A\tau} & 0 & & & \\ & Ae^{A\tau} & 0 & & \\ & & \ddots & \ddots & \\ & & & Ae^{A\tau} & 0 \end{bmatrix}. \end{aligned} \quad (53)$$

Denoting by ρ the ratio of t_c to T_1 , in eqs. (52), (53), the derivatives of the duration of the free vibration t_f and the

time step τ with respect to the spindle speed, respectively, are

$$\frac{dt_f}{d\Omega} = -\frac{60}{\Omega^2 N}(1-\rho), \quad (54)$$

$$\frac{d\tau}{d\Omega} = -\frac{60\rho}{\Omega^2 Nm}. \quad (55)$$

The derivative of one eigenvalue of the transition matrix Ψ_4 with respect to one variable z is [73]:

$$\frac{\partial \lambda}{\partial z} = \frac{\left\langle v, \frac{\partial \Psi_4}{\partial z} u \right\rangle}{\langle v, u \rangle}, \quad (56)$$

where v and u are the left and right eigenvectors of the eigenvalue λ , respectively, and the symbol $\langle \cdot, \cdot \rangle$ represents the inner product of two vectors. The derivative of the most largest module of the eigenvalue with respect to one variable z is

$$\frac{\partial |\lambda|_{\max}}{\partial z} = \frac{\lambda_{\max} \frac{\bar{\lambda}_{\max}}{\partial z} + \bar{\lambda}_{\max} \frac{\lambda_{\max}}{\partial z}}{2|\lambda|_{\max}}, \quad (57)$$

where the symbol $\bar{\lambda}$ denotes the conjugate of λ .

At last, similar to [72], for one fixed spindle speed Ω_0 and corresponding critical axial depth of cut $a_{\lim}$, the analytical expression of the sensitivity of the stability boundary is obtained

$$\left. \frac{da_p}{d\Omega} \right|_{\substack{\Omega=\Omega_0 \\ a_p=a_{\lim}}} = - \left. \frac{\partial |\lambda|_{\max}}{\partial \Omega} \right|_{\substack{\Omega=\Omega_0 \\ a_p=a_{\lim}}} \bigg/ \left. \frac{\partial |\lambda|_{\max}}{\partial a_p} \right|_{\substack{\Omega=\Omega_0 \\ a_p=a_{\lim}}}. \quad (58)$$

Eq. (58) serves as a basis for gradient-based optimization of machining parameters and uncertainty analysis of the machining process.

5 Conclusion and future work

The paper review the time-domain semi-analytical methods for milling process stability analysis, and states the similarities and differences of the SDM, TFEA method, FDM and NIM, with emphasis on the FDM and NIM in the framework of integral equations.

To break the conventional idea constructing the approximate algorithms in the differential equation framework, the FDM describes the dynamics of milling processes in the framework of integral equations. Based on this integral equation, the time-delay item, the state item, and the periodic coefficients are simultaneously linearly interpolated. The Floquet transition matrix is then semi-analytically constructed on a single period. The matrix exponential related

functions involved in the calculation procedure depend only on the spindle speed, independent of the depth of cut. Therefore the FDM has a high computational efficiency. This method has four advantages: (1) suitable for large/small radial depth of cut, large/small axial depth of cut, thin-wall milling and other operations; (2) simultaneous prediction of the surface position error (SLE); (3) it can be applied to the cases of considering the nonlinear cutting force models, varying-speed, etc.; (4) it can be applied to the multiple time delay milling systems, providing a basis for simultaneous optimization of cutter parameters and machining process parameters. On the basis of the FDM, the numerical integration method is developed by using the classical techniques of integral equations. Then, the derivatives of the Floquet transition matrix with respect to the processing parameters are obtained, which lays the foundation for the gradient-based optimization of the milling process parameters.

The further development of the milling process dynamics and stability analysis method should focus on the following aspects. (1) We should build up more accurate dynamic models of 5-axis milling processes, such as for flank milling dynamic model of the distribution parameters, models for the difficult-to-cut materials, low-speed cutting process damping models and a more comprehensive machine-tool-workpiece-fixture system dynamic model. (2) We should propose the active control strategy based on the dynamic model. To break through the conventional strategy selecting the machining parameters in the plane of the spindle speed and depth of cut select to avoid chatter vibrations, we should design the active control system, develop devices and sensors to monitor and control the machining processes, and enlarge the stable area of the system. (3) We should integrate the physical model of the machining process with the dynamic model. To break through the limitations of conventional milling dynamic model for prediction of the geometric accuracy of machined surfaces, we should fuse the process dynamics with the multi-physics model [74–77], developing efficient numerical methods to explore the macro/micro performance of machined surfaces with respect to process parameters. (4) The last one is to fuse the dynamics model with some advanced machine tool control strategies [78–88] for high performance control ensuring high performance machining.

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