Quantum information and many body physics with cold atoms

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We review our recent theoretical advances in quantum information and many body physics with cold atoms in various external potential, such as harmonic potential, kagome optical lattice, triangular optical lattice, and honeycomb lattice. The many body physics of cold atom in harmonic potential is investigated in the frame of mean-field Gross-Pitaevskii equation. Then the quantum phase transition and strongly correlated effect of cold atoms in triangular optical lattice, and the interacting Dirac fermions on honeycomb lattice, are investigated by using cluster dynamical mean-field theory and continuous time quantum Monte Carlo method. We also study the quantum spin Hall effect in the kagome optical lattice.

ultracold atom, Bose-Einstein condensate, soliton, optical lattice, quantum phase transition

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Observation of Bose-Einstein condensates (BECs) in gases of weakly interacting alkali-metal atoms has stimulated intensive studies of the nonlinear matter waves, where both bright and dark solitons have been observed [1–3] and the gap solitons are realized in a optical lattice experimentally [4, 5]. Moreover, experimental realization of BECs, in which two (or more) internal states or different atoms can be populated, has stimulated great interests in vector solitons. In real experiments, both amplitude and sign of the scattering length can be modified by utilizing the Feshbach resonance. This technique provides a very promising method for the manipulation of atomic matter waves and the nonlinear excitations in BEC by tuning the interatomic interaction.

With the development of the experiments, a new developing technology called optical lattices presents a highly controllable and clean system for studying strongly correlated system, in which the relevant parameter can be adjusted independently [6–11]. Optical lattices with different geometrical property can be set up by adjusting the propagation direc-

tions of laser beams, such as triangular [12], honeycomb [13] and Kagomé [14–17] optical lattice. The optical lattice system has gradually become a promising platform to simulate and study a lot of quantum phenomena in condensed-matter physics, because almost all parameters of the system can be well controlled.

In this article, we review quantum information and many body physics with cold atoms. In Section 1, we consider the many-body physics of BEC in harmonic potential. Section 2 is devoted to investigate the quantum phase transition and strongly correlated effect of cold atoms in triangular optical lattice, and the interacting Dirac fermions on honeycomb lattice. In Section 3, we discuss the quantum spin Hall effect of cold atom in the kagome optical lattice. Finally, we summarize our results.

1 Dynamic of ultracold atom in harmonic potential

At the mean-field level, the dynamics of a trapped condensate

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at zero temperature is govern by a nonlinear wave equation known as the Gross-Pitaevskii equation [18, 19]. In the case of cigar-shaped BECs, where $\omega_y = \omega_z = \omega_\perp$ and $\omega_x/\omega_\perp \ll 1$, it is reasonable to reduce the GP equation into an effective 1D dimensionless nonlinear Schrödinger equation:

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi}{\partial x^2} + \frac{a_s(t)}{a_B}|\psi|^2\psi + \frac{\omega_\perp^2}{2\omega_\perp^2}x^2\psi, \quad (1)$$

where the time t and coordinate x are measured, respectively, in units of $\omega_\perp^{-1}$ and $a_\perp$, with $a_\perp \equiv \sqrt{\hbar/m\omega_\perp}$; ψ is measured in units of $1/(\sqrt{2\pi a_\perp^2 a_B})$, with a_B as the Bohr radius.

From a general mathematical and nonlinear physical point of view, eq. (1) is an example of a classically nonintegrable field theory, which needs to be treated numerically. To obtain the exact soliton solutions of eq. (1), we take advantage of the method of similar transformation, which is widely used to solve the general nonlinear Schrödinger equation with time- and space-modulated coefficients. To this end, we first assume the order parameter is written in terms of amplitude and phases as $\psi = \phi \exp(i\alpha)$, with ϕ positively defined as real function. Then separating real and imaginary parts, we obtain a set of coupled nonlinear equations. The results show that if the integrability relation $-\frac{1}{a_s(t)}\frac{d^2 a_s(t)}{dt^2} + \frac{2}{a_s^2(t)}(\frac{da_s(t)}{dt})^2 + \frac{\omega_\perp^2(t)}{\omega_\perp^2} = 0$ is satisfied [20], then the nonlinear GP equation (1) can be reduced to the standard nonlinear Schrödinger equation and exactly solved, with the following general solution:

$$\psi(x, t) = \exp\left[-\int_{t_0}^t \Gamma(t)dt\right] \phi(X, T) \exp[iT(t)x^2], \quad (2)$$

where ϕ is the solutions for the standard nonlinear Schrödinger equation. Actually, the above integrability relation is the well-known Riccati equation. We can solve it not only for $\frac{\omega_\perp^2(t)}{\omega_\perp^2} = \text{const}$, but also for various type of $\Omega^2(t)$, such as $a + b \sin(\lambda t)$, $a + b \cosh(t)$, $ae^{\lambda t}$ (a, b, λ are arbitrary constants) and so on. So we build a bridge between the general GP equation with the standard one, and many interesting phenomena such as fusion, fission, warp, oscillation, elastic collision can be observed within this model.

Now we are now in a position to extend this method to two-component BECs. Considering a two-component BECs each of mass m trapped in a quasi-one-dimensional (1D) harmonic potential, the evolution of this system is govern by a pair of coupled dimensionless GP equations:

$$\begin{aligned} i\frac{\partial\psi_1}{\partial t} &= \left[-\frac{\partial^2}{\partial x^2} + V_1 + g_{11}|\psi_1|^2 + g_{12}|\psi_2|^2\right]\psi_1, \\ i\frac{\partial\psi_2}{\partial t} &= \left[-\frac{\partial^2}{\partial x^2} + V_2 + g_{22}|\psi_2|^2 + g_{21}|\psi_1|^2\right]\psi_2, \end{aligned} \quad (3)$$

where ψ_i denotes the macroscopic wave functions of the i th component, with the normalization conditions: $\int_{-\infty}^{\infty} |\psi_1|^2 dx = 1$ and $\int_{-\infty}^{\infty} |\psi_2|^2 dx = N_2/N_1$. Here we do not allow for the

components to transform into each other and $N_1 = N_2$, so the number of atoms is conserved for both components. We define variables $\sigma_1 = g_{12}^2 - g_{11}g_{22}$, $\sigma_2 = g_{12} - g_{11}$, and $\sigma_3 = g_{12} - g_{22}$ for the facilitation of expression.

To begin with, we consider a simpler case with the interactions take equal values, then eq. (3) can be simply written in the same form with eq. (1). In this case, we retrieve the integrability condition. Next we turn to eq. (3) with a more general situation, with unequal interaction parameters $g_{11}(t) \neq g_{22}(t) \neq g_{12}(t)$.

Expressing the order parameters in terms of their modulus and phases, i.e. $\psi_i(x, t) = \sqrt{n_i} \exp(i\Delta_i)$, and then separating real and imaginary parts, we obtain a set of coupled nonlinear equations for n_i and Δ_i . By setting $\sqrt{n_i} = A_0(t) + A_i(t)\phi(\xi)$, $\xi = p_1(t)x + p_2(t)$, and $\Delta_i = k_i(t) + \Gamma_i(t)x + \Lambda_i(t)x^2$ and using the auxiliary equation $\phi'^2 = c_0 + c_2\phi^2 + c_4\phi^4$, we derive a set of over-determined partial differential equations with respect to $p_1(t)$, $p_2(t)$, $k_i(t)$, $\Gamma_i(t)$, $\Lambda_i(t)$, $A_0(t)$, $A_1(t)$, and $A_2(t)$. Finally, solving these equations, we can obtain the following constrains for the existence of the exact vector-soliton solutions:

$$\begin{aligned} g_{11}A_1^2(t) + g_{12}A_2^2(t) &= g_{22}A_2^2(t) + g_{12}A_1^2(t), \\ \frac{-\partial^2 p_1(t)}{\partial t^2} p_1(t) + 2\left(\frac{\partial p_1(t)}{\partial t}\right)^2 + 4p_1^2(t)\lambda^2 &= 0, \\ p_1^2(t) - g_{11}p_1(t) - g_{12}\frac{\sigma_2}{\sigma_3}p_1(t) &= 0, \end{aligned} \quad (4)$$

where λ is the trapping frequency.

In the case of all the interaction parameters $g_{ij} < 0$, eq. (3) has bright-bright vector solitons:

$$\begin{aligned} \psi_{1B} &= C_1 \sqrt{-a(t)} \text{sech}(\xi(x, t)) \exp(iv_{1B}(x, t)), \\ \psi_{2B} &= C_2 \sqrt{-\sigma a(t)} \text{sech}(\xi(x, t)) \exp(iv_{2B}(x, t)), \end{aligned} \quad (5)$$

where $a(t) = \frac{\sigma_1}{\sigma_3} < 0$, $\sigma = \frac{\sigma_2}{\sigma_3} > 0$.

When $a(t) > 0$ and $\sigma(t) > 0$, eq. (3) has dark-dark vector solitons:

$$\begin{aligned} \psi_{1D} &= D_1 \sqrt{a(t)} \tanh(\xi(x, t)) \exp(iv_{1D}(x, t)), \\ \psi_{2D} &= D_2 \sqrt{\sigma a(t)} \tanh(\xi(x, t)) \exp(iv_{2D}(x, t)). \end{aligned} \quad (6)$$

Based on the above results, we can conclude the following conditions for the existence of the exact vector-soliton solutions:

$$\begin{aligned} \Delta_1 > 0, \Delta_2 < 0 &: \text{bright-bright (BB)}, \\ \Delta_1 < 0, \Delta_2 > 0 &: \text{dark-dark (DD)}, \end{aligned} \quad (7)$$

with $\Delta_1 \equiv -\sigma_3/\sigma_1$ and $\Delta_2 \equiv \sigma_2/\sigma_1$. As long as eqs. (4) and (7) are satisfied, we can obtain the exact vector-soliton solutions, either bright-bright or dark-dark, for an arbitrary periodic time dependence of the scattering length, since we can choose an appropriate λ to satisfy the above equations. It is relevant to mention that all interaction parameters g_{ii} and g_{ij} can be functions of time t in our cases [21].

One step further, we investigate the formation of various types of vector solitons in two-species Bose-Einstein condensates with arbitrary scattering lengths, and find that by tuning the interaction parameter via Feshbach resonance, transformation between different types of vector solitons is possible.

Figure 1 illustrates the conversion between different types of vector-solitons via Feshbach resonance. The initial condition is a zero-velocity bright-dark vector-soliton. The value of b_{12} is then decreased and the system evolves to two bright-bright vector-soliton pairs. Then, we restore the initial value of b_{12} before the two bright-bright vector-solitons merge into one. As seen in Figure 1, such a controlled tuning of b_{12} can also recover the initial bright-dark vector-soliton. This represents a remarkable example of the control over the BECs dynamics, afforded by the tunability of inter-species interaction strength [22].

Another interesting problem is the space-modulated nonlinearities in harmonic potential. To do so, we also make use of the similarity transformations that connect problems with time- and space-modulated nonlinearities with simpler ones that have an homogeneous nonlinearity. We consider the exact spatially localized solutions for which $\lim_{|x| \rightarrow \infty} \psi_{1,2}(x, t) = 0$, taking the similarity transformation: $\psi_i(x, t) = \beta_i(x, t)e^{i\alpha_i(x, t)}U[X(x, t)]$, $i = 1, 2$, with $U(X)$, $V(X)$ satisfying

$$\begin{aligned} U_{XX} + g_{11}U^3 + g_{12}UV^2 &= 0, \\ V_{XX} + g_{22}V^3 + g_{12}VU^2 &= 0. \end{aligned} \quad (8)$$

Then we have a set of partial differential equations (PDEs). By solving this set of PDEs, we can obtain the constraint conditions for the nonlinear and external harmonic potential coefficients. In this case, different matter wave structures such as breathing solitons, quasibreathing solitons, resonant solitons, and moving solitons can be obtained. For a more detailed discussion, see [23].

Figure 1 (Color online) Conversion between bright-bright and bright-dark vector solitons. The left and right plots of the upper panel represent the longitudinal density profiles of the first and second species, respectively. The lower panel shows the variation of b_{12} . Reprinted with permission from [22]. Copyright 2009 The American Physical Society.

So far we focus on the quasi-one dimensional BECs. Now we focus our attention on the quasi-two-dimensional (2D) Bose-Einstein condensates with tunable nonlinearity in a harmonic potential. The 2D dimensionless form of the GP equation can be written as

$$i\frac{\partial\psi}{\partial t} = -(1/2)\nabla^2\psi + g(t)|\psi|^2\psi + (1/2)\Omega^2r^2\psi, \quad (9)$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 = \partial^2/\partial r^2 + 1/r \times \partial/\partial r + \partial^2/\partial \theta^2$, $\Omega = \omega_r/\omega_z$, $g(t) = a(t)/a_0$.

In general, there is no effective method to solve eq. (9), especially when the last term is time-dependent, $V(t)$. Thus we develop a method which can transform the general form of eq. (9) to the one with constant nonlinearity and in the absence of external potential. Use a transformation

$$\psi(r, t) = Q(R(r, t), T(r, t))e^{ia(r, t)+c(t)}, \quad (10)$$

where $R(r, t)$, $T(t)$, $a(r, t)$, and $c(t)$ are assumed to be real function and the transformation parameters read $R(r, t) = \alpha(t)r$, $T(t) = \int \alpha^2(t')dt' + C_1$, $c(t) = \frac{1}{2} \ln \frac{\alpha^2(t)}{C}$, $a(r, t) = -\frac{1}{2\alpha(t)} \frac{d\alpha(t)}{dt} r^2$.

The condition that such transformations exist is same as the integrability relation mentioned above. Under this condition, all solutions of the standard GP equation can be recast into the corresponding solutions of eq. (9). Furthermore, it is worth while to note that the transformation method can be used to solve the general equation:

$$\begin{aligned} i\frac{du(r, t)}{dt} + \varepsilon f(t)\left(\frac{\partial^2 u(r, t)}{\partial r^2} + \frac{1}{r} \frac{\partial u(r, t)}{\partial r}\right) \\ + \delta g(t)|u(r, t)|^2 u(r, t) + V(t)r^2 u(r, t) = 0, \end{aligned} \quad (11)$$

when $g(t)$ is proportional to $f(t)$. Eq. (11) completely describe the dynamics and modulation of both electric field in optical systems and macroscopic order parameter in atomic BECs in quasi-2D with circular symmetry [24].

We also consider a BECs confined in a harmonic trap, but with spatially modulated nonlinearity. In this case, the GP equation reads as

$$i\psi_t = -\frac{1}{2}(\psi_{xx} + \psi_{yy}) + \frac{1}{2}\omega^2(x^2 + y^2)\psi + g(x, y)|\psi|^2\psi, \quad (12)$$

where $g(x, y) = 4\pi a_s(x, y)$ can be spatially inhomogeneous by magnetically tuning the Feshbach resonances.

Now we consider the spatially localized stationary solution $\psi(x, y, t) = \phi(x, y)e^{-i\mu t}$ of eq. (12) with $\phi(x, y)$ being a real function for $\lim_{|x|, |y| \rightarrow \infty} \phi(x, y) = 0$. Solving the stationary equation by similarity transformation, we obtain a families of exact localized nonlinear wave solutions for eq. (12) as

$$\psi_n = \frac{(n+1)K(k)\eta}{\sqrt{\nu}} \text{cn}(\theta, k)e^{-i\mu t}, \quad n = 0, 2, 4, \dots \quad (13)$$

$$\psi_n = \frac{(n+1)K(k)\eta}{\sqrt{2\nu}} \text{sd}(\theta, k)e^{-i\mu t}, \quad n = 1, 3, 5, \dots \quad (14)$$

where $k = \sqrt{2}/2$ is the modulus of elliptic function, ν is a positive real constant, $K(k) = \int_0^{\pi/2} [1 - k^2 \sin^2 \zeta]^{-1/2} d\zeta$ is elliptic integral of the first kind, θ, η and g are determined by

$$\begin{aligned}\theta &= (n+1)K(k)\text{erf}[\sqrt{2\omega}(x+y)/2], \\ \eta &= e^{\omega xy} \text{KummerU}[-\mu/(2\omega), 1/2, \omega(x-y)^2/2], \\ g(x, y) &= -2\omega\nu/(\pi\eta^2)e^{-\omega(x+y)^2}.\end{aligned}\quad (15)$$

Here $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\tau^2} d\tau$ is error function, and $\text{KummerU}(a, c, s)$ is Kummer function of the second kind which is a solution of ordinary differential equation $s\Lambda''(s) + (c-s)\Lambda'(s) - a\Lambda(s) = 0$. It is easy to see that when $|x|, |y| \rightarrow \infty$ we have $\psi_n \rightarrow 0$ for solutions ψ_n in eqs. (13)–(14) with eq. (15), thus they are localized bound state solutions.

From eq. (15), we observe that the number of zero points of function η is equal to that of function $\text{KummerU}[-\mu/(2\omega), 1/2, \omega(x-y)^2/2]$. We assume the number of zero points in η along line $y = -x$ is l . Then we find that integer n is associated with the energy levels of the atoms and integers n, l determine the topological properties of atom packets, so n and l are named the principal quantum number and secondary quantum number in quantum mechanics. In addition, the three free parameters ω, μ and ν are positive, so the dimensionless interaction function $g(x, y)$ is negative, which indicates an attractive interaction between atoms.

Figure 2 shows the density distribution of the quasi-2D BECs with spatially modulated nonlinearities in harmonic potential for $l = 0$. It is easy to see that the matter wave functions (eq. (13)) satisfy $\psi_n(-x, -y) = \psi_n(x, y)$, so they are even parity and are invariant under space inversion. Figure 2(a)–(c) demonstrate the density profiles of the even parity wave functions (eq. (13) with eq. (15)) for $n = 0, 2, 4$,

Figure 2 (Color online) The density distributions of the quasi-2D BECs for different principle quantum numbers n in the case of the secondary quantum number $l = 0$. The parameters ω and ν are 0.02 and 0.1, respectively. (a)–(c) show the density profiles of eq. (13) for $n = 0, 2$ and 4 , respectively. (d)–(f) demonstrate the density profiles of eq. (14) for $n = 1, 3$ and 5 , respectively. Reprinted with permission from [25]. Copyright 2010 The American Physical Society.

which correspond to a low energy state and two highly excited states. The matter wave functions (eq. (14)) satisfy $\psi_n(-x, -y) = -\psi_n(x, y)$, which denotes that they are odd parity. Figure 2(d)–(f) demonstrates the density profiles of the odd parity wave functions (eq. (14) with eq. (15)) for $n = 1, 3, 5$, which correspond to three highly excited states. It is observed that when the secondary quantum number $l = 0$, the number of nodes along line $y = x$ for each quantum state is equal to the corresponding principal quantum number n . And the number of density packets increases one by one along line $y = x$ when the n increases. This is similar to the quantum properties in the linear harmonic oscillator.

Next when the principal quantum number n is fixed, we tune the secondary quantum number l to observe the novel quantum phenomenon in quasi-2D BECs. In Figure 3 we demonstrate the density distributions of quasi-2D BECs for different secondary quantum number. Figure 3(a)–(d) shows the density profiles of the even parity wave function (eq. (13) with eq. (15)) for $n = 0$, and $l = 0, 1, 2$ and 3 , respectively. It is seen that the number of nodes for the density packets along line $y = -x$ is equal to the corresponding secondary quantum number l which describes the topological patterns of the atom packets, and the number of density packets increases one by one when l increases. Figure 3(e)–(h) shows the density profiles of the odd parity wave function (eq. (14) with eq. (15)) for $n = 1$ and $l = 0, 1, 2, 3$. We see that the number of density packets increases pair by pair when l increases. The number of density packets for each quantum state is equal to $(n+1) \times (l+1)$, and all the density packets are symmetrical with respect to lines $y = \pm x$, as shown in Figures 2 and 3 [25].

Figure 3 (Color online) The density distributions of the quasi-2D BECs in harmonic potential for different secondary quantum number l . (a)–(d) show the density profiles of the even parity wave function eq. (13) for principle quantum number $n = 0$. (e)–(h) show the density profiles of the odd parity wave function eq. (14) for $n = 1$, corresponding to $l = 0, 1, 2, 3$. The other parameters are the same as that of Figure 2. Reprinted with permission from [25]. Copyright 2010 The American Physical Society.

2 Quantum phase transition of cold atom in optical lattice

In this section, we consider the cold atom in triangular optical and honeycomb lattice. Using cluster dynamical mean-field theory (CDMFT) and continuous time quantum Monte Carlo method, we investigate the quantum phase transition and strongly correlated effect of cold atoms in triangular optical lattice, and the interacting Dirac fermions on honeycomb lattice.

As an artificial frustrated system, the triangular optical lattice can be set up by three laser beams, such as the Yb fiber laser at wavelength $\lambda = 1064$ nm, with a $2\pi/3$ angle between each other, as illustrated in Figure 4(a). The potential of optical lattice is given by

$$V(x, y) = V_0 \left(3 + 4 \cos\left(\frac{k_x x}{2}\right) \cos\left(\frac{\sqrt{3} k_y y}{2}\right) + 2 \cos(\sqrt{3} k_y y) \right), \quad (16)$$

where V_0 is the barrier height of standing wave formed by laser beams in the x - y plane, k_x and k_y are the two components of wave vector $k = 2\pi/\lambda$ along x and y directions. In experiments, V_0 is always given in units of recoil energy $E_r = \hbar^2 k^2 / 2m$. The landscape of potential of triangular optical lattice in the x - y plane is shown in Figure 4(b), where the dark blue parts in the figure indicate the minimum lattice

Figure 4 (Color online) (a) Sketch of experimental setup to form a triangular optical lattice. Each arrow depicts a laser beam; the sphere in center of the figure depicts a fermionic quantum gas, such as ^{40}K . (b) Landscape of potential $V(x, y)$. (c) The contour lines of triangular optical lattice. The dark blue circles indicate the minimum lattice potential. The dash red lines show the geometry of this triangular optical lattice by connecting the minimum lattice potential. Reprinted with permission from [26]. Copyright 2010 The American Physical Society.

potential. Figure 4(c) shows the contour lines of the triangular optical lattice. By connecting the center of the circular which indicates the minimum lattice potential, we may get the geometry of this triangular optical lattice, as shown by the red dash lines.

The Hamiltonian of the interacting fermionic atoms trapped in this artificial frustrated system is written as

$$H = -t \sum_{\langle ij \rangle \sigma} c_{i\sigma}^\dagger c_{j\sigma} + U \sum_i n_{i\uparrow} n_{i\downarrow}, \quad (17)$$

where $c_{i\sigma}^\dagger$ and $c_{i\sigma}$ denote the creation and the annihilation operator of the fermionic atom on lattice site i respectively, $n_{i\sigma} = c_{i\sigma}^\dagger c_{i\sigma}$ represents the density operator of fermionic atom. And $t = (4/\sqrt{\pi}) E_r (V_0/E_r)^{3/4} \exp(-2(V_0/E_r)^{1/2})$ is the kinetic energy, which can be adjusted by the lattice depth V_0 . $U = \sqrt{8/\pi} k a_s E_r (V_0/E_r)^{3/4}$ is the on-site interaction determined by the s-wave scattering length a_s , which can be adjusted by Feshbach resonance.

The phase diagram of cold atoms trapped in triangular optical lattice is shown in Figure 5. The translation between the Fermi liquid and the pseudogap shows a reentrant behavior. For a fixed interaction weaker than $U/t = 7.2$, when the temperature decreases, the system translates from Fermi liquid to pseudogap. When the temperature is lower than the critical temperature distributing on the red solid line, the system translates from pseudogap to Fermi liquid. There is a Kondo peak region, which is signed by the blue solid line and the pink dashed line. If temperature is lower than $T/t = 2.0$, a Kondo peak emerges before the appearing of the pseudogap when the interaction increases. When the interaction is stronger than the critical interaction of Mott transition distributing on the purple line, the system translates from pseudogap to insulator confirmed by an opened gap. See [26] for a detailed discussion.

Next we study the Mott transition of the interacting Dirac

Figure 5 (Color online) The phase diagram of Fermi atoms in triangular optical lattice, where the square plots with solid line (green, red, blue) indicate the transition line of the Fermi liquid and the pseudogap, the circular plots with solid line (purple) indicate the Mott transition line, the triangular plots with dash line (pink) mark the Kondo peak appearing region. Reprinted with permission from [26]. Copyright 2010 The American Physical Society.

fermions on honeycomb lattice by CDMFT method [27]. The experimental setup of hexagonal optical lattice is illustrated in Figure 6(a). We can use Yb fiber laser at wavelength $\lambda = 1064$ nm to get three trapping potentials. Each of the potential can be formed by two blue-detuned laser beams at a 70.4° angle, as shown in Figure 6(a). Then we can get the trapping potentials which can be written in form $V(x, y) = V_0 \sum_{j=1,2,3} \sin^2(k_L(x \cos \theta_j + y \sin \theta_j) + \frac{\pi}{2})$, where $\theta_1 = \frac{\pi}{6}$, $\theta_2 = -\frac{\pi}{6}$, $\theta_3 = \frac{\pi}{2}$, k_L is the optical wave vector projected onto the x - y plane, V_0 is the depth of the lattice potential [13]. The contour lines of this hexagonal optical lattice is shown in Figure 6(b), the blue lines indicate the minimum lattice potential.

In Figure 7, we present the phase diagram of the interacting Dirac fermions on honeycomb lattice. The diagram is separated into semimetallic and insulating phases by the second-order Mott transition line. The critical value U_c decreases as temperature T decreases, and a finite $U_c/t \sim 3.3$ for zero temperature phase transition is suggested, which is quite agree with the large scale quantum Monte carlo result $U_c/t \sim 3.5$ [28]. Comparing to the DMFT result $U_c/t \sim 10$ [29], the Mott transition critical value U_c within CDMFT is much smaller, and more reasonable according to the previous studies [30–32].

Figure 6 (Color online) (a) Sketch map of the experimental setup to form hexagonal optical lattice. (b) The contourlines of hexagonal optical lattice, the blue lines indicate the minimum lattice potential.

Figure 7 (Color online) Phase diagram of the interacting Dirac fermions on honeycomb lattice. The line separating the semimetal and Mott insulator marks a second-order phase transition. Reprinted with permission from [27]. Copyright 2010 The American Physical Society.

3 Quantum Hall effect of cold atom in Kagomé optical lattice

In this section, we consider how to simulate and detect the 2-dimensional (2D) quantum spin Hall (QSH) insulator in a Kagomé optical lattice with a trimer and a nearest-neighbor spin-orbit (SO) coupling term, where the lattice SO coupling can be designed by use of the laser-induced-gauge-field method [33, 34].

Let us consider the tight-binding model for two-component fermionic atoms on the Kagomé optical lattice, which consists of three triangular sublattices A, B and C (Figure 8). The Hamiltonian is given by

$$H_0 = t_0 \sum_{\langle i,j \rangle, \alpha} (c_{i\alpha}^\dagger c_{j\alpha} + \text{H.c.}) + i\lambda_{\text{SO}} \sum_{\langle i,j \rangle} v_{ij} c_i^\dagger \sigma_z c_j + \sum_{\langle i,j \rangle, \sigma} \kappa_{ij} c_{i\sigma}^\dagger c_{j\sigma}, \quad (18)$$

where $t_{ij}=t_0$ is the hopping amplitude of spin-independent part between the nearest neighbor link $\langle i, j \rangle$, $c_{i\alpha}^\dagger$ ($c_{i\alpha}$) is the creation (annihilation) operator of an electron with spin α (up or down) on lattice site i . The second term represents the nearest neighbor SO coupling interaction, where λ_{SO} is the SO coupling constant, σ_z is the Pauli matrix and $c_i^\dagger = (a_{i\uparrow}^\dagger, a_{i\downarrow}^\dagger)$. v_{ij} equals to +1 (−1) when the electrons hop along (against) the arrowed direction in Figure 8(a). The third term describes lattice trimerization interaction which breaks inversion symmetry of Kagomé lattice. κ_{ij} equals to $+\kappa$ ($-\kappa$) corresponding to hopping amplitude along the thick (thin) bonds in Figure 8(b).

We take above SO coupling and trimerized interaction as perturbation, which means $\lambda_{\text{SO}} \ll t$ and $\kappa \ll t$. Although both of perturbations can bring gaps at Dirac points independently, these two gaps have different topological nature. The former is non-trivial and quantum spin Hall effect will occur if Fermi energy level locates in the gap; the latter is a trivial gap. We can understand it through the edge state effects in Figure 9. From Figure 9(a), we can see that there is a pair of chiral gapless edge states for every band gap when the SO coupling dominates. This means that the system is in topological insulator phases at 1/3- and 2/3-filling. When only trimer term

Figure 8 (Color online) (a) Schematic picture of the nearest neighbor intrinsic SO coupling in 2D Kagomé lattice. The up-spin atoms hop along (against) the arrowed direction with amplitude $i\lambda_{SO}$ ($-i\lambda_{SO}$). For the down-spin atoms, the arrows are reversed. The dashed line represents the Wigner-Seitz unit cell, which contains three independent sites (A, B, C). (b) The trimer Kagomé lattice. Hopping amplitude corresponds to $t+\kappa$ ($t-\kappa$) for the thick (thin) bonds. (c) Contours of the effective magnetic field for up-spin atoms defined by eq. (20). Reprinted with permission from [35]. Copyright 2010 The American Physical Society.

Figure 9 The band structure of the lattice model in the stripe geometry. We take $\lambda_{SO}=0.1$, $\kappa=0$ for (a), $\lambda_{SO}=0$, $\kappa=0.1$ for (b) and $\lambda_{SO}=0.05$, $\kappa=0.1$ for (c). Reprinted with permission from [35]. Copyright 2010 The American Physical Society.

exists, it opens a band gap at Dirac point but no edge states connect the upper and lower bands (see Figure 9(b)), therefore the system is in normal insulator phase at $2/3$ -filling. On the other hand we also see that the trimer term cannot open a gap between the band 1 and 2 (Figure 9(b)). Therefore the system at $1/3$ -filling will be still in the topological insulator phase when two perturbations are present but the trimer term dominates (see Figure 9(c)).

The specific SO coupling means that the electrons with different spins feel totally opposite effective magnetic field when electrons hopping on the lattice. This effective field can be simulated in cold atom by use of laser-induced-gauge-field method [33, 34]. More details refer [35]. Therefore, the total effective vector potential and magnetic field can be written as

$$\mathbf{A}_{\text{eff}}^{\alpha} = \alpha \hbar k_1 \left[(\sin(2k_2y) - \cos(k_2y) \sin(\sqrt{3}k_2x)) \mathbf{e}_x + \sqrt{3} \sin(k_2y) \cos(\sqrt{3}k_2x) \mathbf{e}_y \right], \quad (19)$$

$$\mathbf{B}_{\text{eff}}^{\alpha} = -\alpha \frac{2\pi \hbar k_1}{\sqrt{3}} [2 \sin(k_2y) \sin(\pi x) + \cos(2k_2y)] \mathbf{e}_z. \quad (20)$$

Here, $\alpha=\pm 1$ representing the up- and down-spin. The total effect field is shown in Figure 8(c). It can be proved that this effect field equals to the SO coupling in eq. (18).

The Kagomé optical lattice with the trimer terms can be generated by use of the superlattice technique addressed in

[14, 15, 36]. In our proposal, each super-laser beam consists of four large detuned standing-wave lasers with the same polarization but different wave vector length in the x - y plane. The total potential is thus given by

$$V(\mathbf{r}) = V_0 \sum_{i=1}^3 [\cos(\mathbf{k}_i \cdot \mathbf{r} + 3\delta_i\varphi/2) + 2 \cos(\mathbf{k}_i \cdot \mathbf{r}/3 + \delta_i\varphi/2) + 4 \cos(\mathbf{k}_i \cdot \mathbf{r}/9 + \delta_i\varphi/6) + \zeta \cos(\mathbf{k}_i \cdot \mathbf{r}/9 + \delta_i(\varphi/6 + \pi/2))]^2, \quad (21)$$

with the wave vectors $\mathbf{k}_1 = (\frac{\sqrt{3}}{2}, \frac{1}{2})k$, $\mathbf{k}_2 = (-\frac{\sqrt{3}}{2}, \frac{1}{2})k$, $\mathbf{k}_3 = (0, 1)k$ and $\delta_1=\delta_2=-\delta_3=1$. Firstly, we consider the case with $\zeta=0$. One can get a triangular lattice when $\varphi=0$ or 2π and a Kagomé lattice when $0 < \varphi < 2\pi$. A uniform Kagomé lattice corresponds to $\varphi = \pi$, as shown in Figure 10(a). When we increase the strength of trimerized Hamiltonian, the Kagomé lattice will be distorted. To overcome this defect, we add another laser beam which corresponds to the $\zeta \neq 0$ in the eq. (21) and assume $\varphi = \pi$ all the time. The added laser will interfere with primary lasers and generate the trimerized Kagomé lattice. ζ is an adjustable parameter to control the strength of trimerized Hamiltonian. With this method, the Kagomé lattice will not have obvious offset from the uniform one even if ζ takes a relative large value. As shown in Figure 10(b), the parameter $\zeta=1.5$ is chosen as a typical example.

Figure 10 (Color online) (a) A uniform Kagomé lattice for $\varphi=\pi$ and $\zeta=0$. (b) A trimer Kagomé lattice for $\varphi=\pi$ and $\zeta=1.5$. Reprinted with permission from [35]. Copyright 2010 The American Physical Society.

We now introduce how to detect the QSH phase in ultracold-atomic optical lattice. In cold atom system, it has been shown that the conductivity σ_{xy} (Chern number) is related to the atomic density from the Streda formula $\sigma_{xy}=\partial\rho/\partial\mathcal{B}|_{\mu,T}$ when a uniform magnetic field $\mathcal{B}$ is applied in the system. Thus one can measure the Chern number through the detection of the density profile [37]. According to Streda formula, one can obtain the relation between the spin Chern number and the spin-atomic density as $C_\alpha=\rho_\alpha\phi_0/\mathcal{B}$ with $\alpha=\uparrow, \downarrow$ denote up and down spin. This formula provides us the approach to measure whether the system is in the QSH phase. Firstly, we measured the spin-atomic density and denoted it as $\rho_{\uparrow,\downarrow}^0$ at chemical potential $\mu=0$ in the absence of $\mathcal{B}$. Then the optical lattice is rotated to generate the effective uniform magnetic field $\mathcal{B}$, and the new density of the cold atoms $\rho_{\uparrow,\downarrow}^1$ is measured again. It can be proven that If $\rho_{\uparrow}^1 > \rho_{\uparrow}^0$ and $\rho_{\downarrow}^1 < \rho_{\downarrow}^0$, the system is in QSH insulator phase. However, if $\rho_{\uparrow,\downarrow}^1=\rho_{\uparrow,\downarrow}^0$, the system is in the normal insulator phase.

4 Conclusions

In summary, we have examined the quantum information and many body physics with cold atoms in various external potential, such as harmonic potential, Kagomé optical lattice, triangular optical lattice, and honeycomb lattice. We study the many body physics of BECs with tunable interactions and external potential, and present the integrable conditions for the one and two-dimensional Gross-Pitaevskii equations. Furthermore, we obtain a family of exact analytical solutions. The results show that the dynamics of the solitons can be effectively controlled by the Feshbach resonance. Moreover, the quantum phase transition and strongly correlated effect of cold atoms in triangular optical lattice and the interacting Dirac fermions on honeycomb lattice are investigated by using cluster dynamical mean-field theory and continuous time quantum Monte Carlo method. Finally, we simulate and detect the 2-dimensional quantum spin Hall insulator in a Kagomé optical lattice with a trimer and a nearest-neighbor spin-orbit coupling term.

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