***t*-bit semiclassical quantum Fourier transform**

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Because of the difficulty of building a high-dimensional quantum register, this paper presents an implementation of the high-dimensional quantum Fourier transform (QFT) based on a low-dimensional quantum register. First, we define the *t*-bit semiclassical quantum Fourier transform. In terms of probability amplitude, we prove that the transform can realize quantum Fourier transformation, illustrate that the requirement for the two-qubit gate reduces obviously, and further design a quantum circuit of the transform. Combining the classical fixed-window method and the implementation of Shor's quantum factorization algorithm, we then redesign a circuit for Shor's algorithm, whose required computation resource is approximately equal to that of Parker's. The requirement for elementary quantum gates for Parker's algorithm is $O(\lceil \log N \rceil^3)$, and the quantum register for our circuit requires *t*–1 more dimensions than Parker's. However, our circuit is *t*² times as fast as Parker's, where *t* is the width of the window.

semiclassical quantum Fourier transform, Shor's quantum algorithm, window method

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Quantum computing has a strong ability for parallel computing, which poses a great challenge to the security of the modern cipher. In 1994, Shor [1] presented a polynomial-time quantum algorithm for factorization. Apparently, the emergence of this algorithm greatly threatens the public-key cryptography, such as RSA cryptography, whose security is based on factorization and a discrete logarithm. In 1996, Grover [2] presented a quantum searching algorithm that reduces the computational complexity of current exhaustive search attacks from $O(2^n)$ to $O(2^{n/2})$. To overcome this problem, scholars [3–11] in China and abroad have since intensively investigated quantum computation and quantum cryptology.

Although the correctness of quantum computation and Shor's algorithm has been proved to hold true [12], it is still difficult to break 2048-bit RSA or 191-bit ECC in the case of the quantum computer for the reason that the kilobit quantum computer does not exist. Thus, how to reduce the computation resource required for Shor's algorithm is an

issue of common concern.

In 1996, Vedral et al. [13] published a circuit with a $7n+1$ -dimensional quantum register and $O(n^3)$ elementary gates for modular exponentiation (where *n* denotes the length of the integer to be factorized). It has been mentioned that the quantum register requirement can be reduced to $4n+3$ if unbounded Toffoli gates (*n*-controlled NOT gates) are available. Also in 1996, Beckman et al. [14] presented an extended analysis of modular exponentiation, with a circuit of a $4n+1$ -dimensional quantum register if unbounded Toffoli gates are available. In 1998, Zalka [15] described a method for factorization with a $3n+O(\log N)$ -dimensional quantum register using only elementary gates. However, these achievements are the optimization of Shor's algorithm, which is based only on reducing the dimensions of the quantum register.

Generally speaking, in the implementation of the quantum Fourier transform (QFT), the *n*-bit quantum state is once inputted into a *n*-dimensional quantum register and the circuit needs $n^2/2$ two-bit quantum gates and *n* one-bit quantum gates in total [16]. To date, for larger *n*, there is no

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implementation method for the n -dimensional QFT within existing technology. In 1996, Griffiths et al. [17] presented the single-bit semiclassical QFT. Since the transform's input is from bit flipping, the transform only needs a one-dimensional quantum register and one one-bit quantum gate, and the QFT can thus be implemented without a two-bit quantum gate. In 2000, on the basis of the single-bit semiclassical QFT, Parker et al. [18] presented a circuit of the $n+1$ -dimensional quantum register for factorization. Thus, the single-bit semiclassical QFT provides a solution for implementing a high-dimensional QFT, with which Shor's algorithm can be used to break public-key cryptology (e.g., RSA and ECC) under the condition of limited computation resources.

The QFT, compared with the single-bit semiclassical QFT, has higher implementation speed but needs more quantum computation resources. Considering that a two-bit quantum gate is more difficult to implement than a one-bit quantum gate, the two-bit quantum gate should be avoided as much as possible in a quantum computation circuit. To replace the QFT with a single-bit semiclassical QFT is a good choice. However, it is at the expense of implementation speed. Because of the difficulty of realizing a high-dimensional quantum register, how to balance the tradeoff between the quantum resource and speed in the implementation of the QFT (e.g., how to implement a high-dimensional QFT based on a low-dimensional quantum register) requires further study.

In this paper, we define the t -bit semiclassical QFT. In terms of the probability amplitude, we prove that the transform can realize quantum Fourier transformation, illustrate that the requirements for the two-bit and one-bit quantum gates are respectively reduced to $1/l^2$ and $2/l$ times those of the QFT (where n denotes the dimension of the QFT; if t divides n exactly, $l=[n/t]-1$, otherwise $l=[n/t]$), and clarify that the speed of the t -bit semiclassical QFT increases to t times that of the single-bit case. Finally, combining the classical fixed-window representation and the implementation method for Shor's algorithm, we design a circuit for Shor's quantum factorization algorithm based on the t -bit semiclassical QFT, whose computation resource is approximately equal to that of Parker's. However, our circuit is t^2 times as fast as Parker's, where t is the width of the window representation.

1 QFT and single-bit semiclassical QFT

Definition 1 (Quantum Fourier transform) [16] The QFT on an orthonormal basis $|0\rangle, |1\rangle, \dots, |N-1\rangle$ is defined to be a linear operator U_F with an action on the basis states described by

$$U_F : |j\rangle \mapsto \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega_N^{jk} |k\rangle.$$

Therefore, the action on an arbitrary state may be written as

$$U_F : \sum_{j=0}^{N-1} x_j |j\rangle \mapsto \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} x_j \omega_N^{jk} |k\rangle,$$

where the symbol " $\mapsto$ " stands for an invertible transform and $\omega_N = e^{2\pi i/N}$.

If $N = 2^n$, the QFT U_F acting on the state $|a\rangle$ can be written as

$$U_F |a\rangle = \prod_{j=0}^{n-1} \otimes |p(\phi_j)\rangle. \quad (1)$$

U_F is also called the n -dimensional QFT, where

$a = \sum_{j=0}^{n-1} a_j 2^j$ is the binary representation of a , a_j is 0 or 1,

$$\phi_j = \sum_{k=0}^{n-j} a_k 2^{j+k-n-1} \quad \text{for } j=0, 1, \dots, n-1 \quad \text{and} \quad |p(\phi)\rangle = (|0\rangle + e^{2\pi i \phi} |1\rangle) / \sqrt{2}.$$

It has been pointed out [16] that the circuit of the QFT requires a $\lceil \log N \rceil$ -dimensional quantum register. In 1996, Griffiths et al. presented a method that only required a one-dimensional quantum register and one-bit gate to implement the QFT, which was called single-bit semiclassical quantum Fourier transform [17]. Its circuit is illustrated in Figure 1.

Figure 1 is explained as follows.

(1) Each box, standing for an oracle, performs the operations

$$\begin{cases} |0\rangle \mapsto (|0\rangle + |1\rangle) / \sqrt{2} \\ |1\rangle \mapsto e^{2\pi i \phi} (|0\rangle - |1\rangle) / \sqrt{2} \end{cases}. \quad (2)$$

(2) A single line represents a quantum signal and a double line represents a classical signal.

(3) The outputs of each box are two classical signals, one of which is the result of the measurement and the other represents the phase $\phi' = \phi/2 + c/4$; ϕ' is the input of the next box, whose initial value is zero.

(4) The final measurement result is $c = \sum_{k=0}^{n-1} 2^k c_k$.

The single-bit semiclassical QFT carries each basis state

Figure 1 Circuit of the single-bit semiclassical QFT.

$|a\rangle$ into $|c\rangle$, whose amplitude is equal to that of the QFT [16]. Thus, the single-bit semiclassical QFT can realize quantum Fourier transformation and the input of $|a\rangle$ is provided from left to right by bit flipping. That is to say, the single-bit semiclassical QFT can carry out the parallel computing of 2^n inputs within n steps. Measurements made after transforming each bit show that the single-bit semiclassical QFT requires less quantum computation resources than the QFT.

2 t -bit semiclassical QFT

The QFT has a higher implementation speed but needs more quantum computation resources, whereas the single-bit semiclassical QFT needs less resources but has lower speed. To overcome the disadvantages of the two methods, we design a new QFT to balance the tradeoffs of the quantum resource and speed.

Definition 2 Suppose t is a positive integer and $l = \begin{cases} \lceil n/t \rceil, & t \nmid n \\ \lfloor n/t \rfloor - 1, & t \mid n \end{cases}$, $|a\rangle = |a_{n-1}, a_{n-2}, \dots, a_0\rangle$ is an arbitrary orthonormal state with orthonormal basis $|0\rangle, |1\rangle, \dots, |N-1\rangle$. If U_{F_t} transforms a block of bits of $|a\rangle$ entering on the left to one leaving on the right through

$$\begin{aligned} & |a_{n-1}, a_{n-2}, \dots, a_{lt}\rangle \mapsto \\ & \frac{1}{2^{(n-lt)/2}} \sum_{c'_0=0}^{2^{n-lt}-1} \omega_{2^{n-lt}}^{a'_0 c'_0} |c_{n-lt-1}, c_{n-lt-2}, \dots, c_0\rangle, \\ & |a_{lt-1}, a_{lt-2}, \dots, a_{(l-1)t}\rangle \mapsto \\ & \frac{1}{2^{t/2}} \sum_{c'_1=0}^{2^{t-1}-1} \omega_{2^t}^{a'_1 c'_1} e^{2\pi i(a_{lt-1} + \frac{a_{lt-2}}{2} + \dots + \frac{a_{(l-1)t}}{2^{t-1}})\varphi_1} |c_{n-(l-1)t-1}, c_{n-(l-1)t-2}, \dots, c_{n-lt}\rangle \\ & \vdots \\ & |a_{t-1}, a_{t-2}, \dots, a_0\rangle \mapsto \\ & \frac{1}{2^{t/2}} \sum_{c'_j=0}^{2^{t-1}-1} \omega_{2^t}^{a'_j c'_j} e^{2\pi i(a_{t-1} + \frac{a_{t-2}}{2} + \dots + \frac{a_0}{2^{t-1}})\varphi_l} |c_{n-1}, c_{n-2}, \dots, c_{n-t}\rangle, \end{aligned}$$

where $a'_0 = \sum_{r=lt}^{n-1} 2^{r-lt} a_r$, $a'_j = \sum_{r=(l-j)t}^{(l-j+1)t-1} 2^{r-(l-j)t} a_r$, $c'_0 = \sum_{r=0}^{n-lt-1} 2^r c_r$,

$c'_j = \sum_{r=n-(l-j+1)t}^{n-(l-j)t-1} 2^{r-n+(l-j+1)t} c_r$, $\varphi_1 = \frac{c_0}{2^{n-lt+1}} + \dots + \frac{c_{n-lt-1}}{2^2}$ and

$\varphi_k = \frac{\varphi_{k-1}}{2^t} + \frac{c_{n-(l-k+2)t}}{2^{t+1}} + \dots + \frac{c_{n-(l-k+1)t-1}}{2^2}$ ($j=1, \dots, l$, $k=2, \dots, l$), then U_{F_t} is called the t -bit semiclassical Fourier transform.

In essence, U_{F_t} is a block transform. That is to say, $|a\rangle$

is transformed from left to right in blocks, the first of which is $n-lt$ bits and the other is t bits.

Theorem 1 Suppose $|0\rangle, |1\rangle, \dots, |N-1\rangle$ is an orthonormal basis, U_F is the QFT and U_{F_t} is the t -bit semiclassical QFT. U_F carries $|a\rangle = |a_{n-1}, a_{n-2}, \dots, a_0\rangle$ into $|c\rangle = |c_{n-1}, c_{n-2}, \dots, c_0\rangle$, whose probability is equal to that for U_{F_t} .

Proof: To prove the theorem, we only need to prove that U_F carries $|a\rangle = |a_{n-1}, a_{n-2}, \dots, a_0\rangle$ into $|c\rangle = |c_{n-1}, c_{n-2}, \dots, c_0\rangle$, whose amplitude is equal to that for U_{F_t} .

From Definition 2 and $e^{2\pi i} = 1$, we obtain

$$\omega_{2^{n-lt}}^{a'_0 c'_0} = \omega_{2^{n-lt}}^{a'_0 c'_0} = e^{\pi i(\frac{ca_{n-1}}{2^0} + \frac{ca_{n-2}}{2^1} + \dots + \frac{ca_{lt}}{2^{n-lt-1}})}.$$

Therefore, the amplitude of $|c_{n-lt-1}, c_{n-lt-2}, \dots, c_0\rangle$ is

$$\frac{1}{2^{(n-lt)/2}} e^{\pi i(\frac{ca_{n-1}}{2^0} + \frac{ca_{n-2}}{2^1} + \dots + \frac{ca_{lt}}{2^{n-lt-1}})}.$$

Since

$$\begin{aligned} & \omega_{2^t}^{a'_1 c'_1} e^{2\pi i(a_{lt-1} + \frac{a_{lt-2}}{2} + \dots + \frac{a_{(l-1)t}}{2^{t-1}})\varphi_1} \\ &= e^{\frac{2\pi i a'_1 c'_1}{2^t}} e^{2\pi i(a_{lt-1} + \frac{a_{lt-2}}{2} + \dots + \frac{a_{(l-1)t}}{2^{t-1}})(\frac{c_0}{2^{n-lt+1}} + \dots + \frac{c_{n-lt-1}}{2^2})} \\ &= e^{\frac{\pi i a_{lt-1}}{2^{n-lt}}(c_0 + 2c_1 + \dots + 2^{n-(l-1)t-1}c_{n-(l-1)t-1})} \dots e^{\frac{\pi i a_{(l-1)t}}{2^{n-(l-1)t-1}}(c_0 + 2c_1 + \dots + 2^{n-(l-1)t-1}c_{n-(l-1)t-1})} \\ &= e^{\pi i(\frac{ca_{lt-1}}{2^{n-lt}} + \frac{ca_{lt-2}}{2^{n-lt+1}} + \dots + \frac{ca_{(l-1)t}}{2^{n-(l-1)t-1}})}, \end{aligned}$$

the amplitude of $|c_{n-(l-1)t-1}, c_{n-(l-1)t-2}, \dots, c_{n-lt}\rangle$ is

$$\frac{1}{2^{t/2}} e^{\pi i(\frac{ca_{lt-1}}{2^{n-lt}} + \frac{ca_{lt-2}}{2^{n-lt+1}} + \dots + \frac{ca_{(l-1)t}}{2^{n-(l-1)t-1}})}.$$

Similarly, the amplitude of $|c_{n-(l-2)t-1}, c_{n-(l-2)t-2}, \dots,$

$$c_{n-(l-1)t}\rangle \text{ is } \frac{1}{2^{t/2}} e^{\pi i(\frac{ca_{(l-1)t-1}}{2^{n-(l-1)t}} + \frac{ca_{(l-1)t-2}}{2^{n-(l-1)t+1}} + \dots + \frac{ca_{(l-2)t}}{2^{n-(l-2)t-1}})},$$

$\vdots$

the amplitude of $|c_{n-1}, c_{n-2}, \dots, c_{n-t}\rangle$ is

$$\frac{1}{2^{t/2}} e^{\pi i(\frac{ca_{n-1}}{2^{n-t}} + \frac{ca_{n-2}}{2^{n-t+1}} + \dots + \frac{ca_0}{2^{n-1}})}.$$

In conclusion, after U_{F_t} acts on $|a\rangle$, the amplitude of $|c\rangle = |c_{n-1}, c_{n-2}, \dots, c_0\rangle$ is

$$\begin{aligned} A &= \frac{1}{2^{n/2}} e^{\pi i(\frac{ca_{n-1}}{2^0} + \frac{ca_{n-2}}{2^1} + \dots + \frac{ca_{lt}}{2^{n-lt-1}})} \cdot e^{\pi i(\frac{ca_{lt-1}}{2^{n-lt}} + \frac{ca_{lt-2}}{2^{n-lt+1}} + \dots + \frac{ca_{(l-1)t}}{2^{n-(l-1)t-1}})} \dots \\ & e^{\pi i(\frac{ca_{(l-1)t-1}}{2^{n-(l-1)t}} + \frac{ca_{(l-1)t-2}}{2^{n-(l-1)t+1}} + \dots + \frac{ca_0}{2^{n-1}})}. \end{aligned}$$

Furthermore, $A = \frac{1}{2^{n/2}} \omega_{2^n}^{ac}$.

From Definition 1, we obtain

$$U_F |a\rangle = \frac{1}{2^{n/2}} \sum_{c=0}^{2^n-1} \omega_{2^n}^{ac} |c\rangle.$$

Then, after U_F acts on $|a\rangle$, the amplitude of $|c\rangle = |c_{n-1}, c_{n-2}, \dots, c_0\rangle$ is

$$\frac{1}{2^{n/2}} \omega_{2^n}^{ac}.$$

We thus obtain Theorem 1.

According to Theorem 1, the t -bit semiclassical QFT can realize quantum Fourier transformation. The circuit of the transform is illustrated in Figure 2.

Figure 2 is explained as follows.

(1) Each bold virtual box denotes VB1, while the fine virtual box containing E_1 denotes VB2 and that containing E_2 denotes VB3, where E_1 is the $n-lt$ -dimensional QFT and E_2 is the t -dimensional QFT. The circuits of E_1 and E_2 are detailed in [16].

(2) The outputs of VB2 have $n-lt+1$ values, namely $c_{n-lt-1}, c_{n-lt-2}, \dots, c_0$, $\varphi' = \frac{c_0}{2^{n-lt+1}} + \dots + \frac{c_{n-lt-1}}{2^2}$. If VB3 enters as $a'_{t-1}, a'_{t-2}, \dots, a'_0$, it leaves as $t+1$ values, namely $c'_{t-1}, c'_{t-2}, \dots, c'_0$, $\varphi' = \frac{\varphi}{2^t} + \frac{c'_0}{2^{t+1}} + \dots + \frac{c'_{t-1}}{2^2}$. $R_j = \begin{bmatrix} 1 & 0 \\ 0 & e^{2\pi i \varphi / 2^{j-1}} \end{bmatrix}$ for $j=1, 2, \dots, t$, where φ' is sent to the next VB1, φ is the input of VB1, and the very first VB1 uses $\varphi = 0$.

(3) $c = \sum_{j=0}^{n-1} 2^j c_j$ is the final measurement.

We now analyze the required resources of the circuit of the t -bit semiclassical QFT.

Figure 2 shows that the t -bit semiclassical QFT requires t -dimensional quantum registers. Since the n -dimensional

Figure 2 Circuit of the t -bit semiclassical QFT.

QFT needs $n^2/2$ two-bit gates and n one-bit gates [16], our circuit in Figure 2 needs $t^2/2$ two-bit gates and t one-bit gates to realize t -dimensional quantum Fourier transformation. It also needs t one-bit gates to realize R_j ($j=1, 2, \dots, t$). Therefore, the two-bit gates and one-bit gates for the t -bit semiclassical QFT are respectively $1/l^2$ and $2/l$ times as many as those of the n -dimensional QFT. It is obvious that the t -bit semiclassical QFT requires the preparation of superposition $\frac{1}{2^{(n-lt)/2}} \sum_{c=0}^{2^{n-lt}-1} |c\rangle$ once and that of $\frac{1}{2^{t/2}} \sum_{c=0}^{2^t-1} |c\rangle$ l' times while the single-bit semiclassical QFT requires preparation of superposition $(|0\rangle + |1\rangle)/\sqrt{2}$ L times. Therefore, the running speed of the t -bit semiclassical QFT is t times that of the single-bit semiclassical QFT.

3 Parker's circuit of Shor's quantum factorization algorithm

In 2000, Parker et al. presented a circuit of Shor's quantum factorization algorithm, which is illustrated in Figure 3 [18].

In Figure 3, H is a Hadamard gate [16]. $R_j = \begin{pmatrix} 1 & 0 \\ 0 & \phi_j \end{pmatrix}$,

where $\phi_j = e^{-2\pi i \sum_{k=2}^j m_{j-k}/2^k}$. U_a is the operation $U_a : |r\rangle|x\rangle \rightarrow |r\rangle|a^r x \bmod N\rangle$ controlled by one bit. $c = \sum_{i=0}^L 2^{L-i} m_i$ is the final measurement result, where $L = 2\lceil \log N \rceil + 1$ and $\lceil \log N \rceil$ is the minimum integer greater than $\log N$. The quantum state $|0\rangle + |1\rangle$ is $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, which can be obtained by the Hadamard transform acting on $|0\rangle$.

Parker's circuit needs a $\lceil \log N \rceil + 1$ -dimensional quantum register and $O(\lceil \log N \rceil^3)$ elementary gates [18]. Furthermore, it has been pointed out that the circuit requires nL addition operations (the addend length being $n = \lceil \log N \rceil$) when $\alpha \bmod N, \alpha^2 \bmod N, \dots, \alpha^{2^{L-1}} \bmod N$ are precomputed classically [19].

4 Circuit of the Shor algorithm based on the window method and analysis

In the circuit of Shor's algorithm, the modular exponential-

Figure 3 Parker's circuit of Shor's quantum factorization algorithm.

tion $\alpha^k \bmod N$ is the most time-consuming calculation [16]. In classical computation, there are two methods to accelerate implementation of modular exponentiation; i.e., the modified representation of integer k and increasing the precomputation amount. The window method [20] can accelerate the implementation of modular exponentiation by space for time, and its general concept is expressed as follows.

According to some rules, k can be represented as

$$k = \sum_{i=0}^{v-1} 2^{iu_i},$$

where $0 \leq k_i \leq 2^t - 1$ and $0 \leq i \leq v-1$. If $\alpha, \alpha^{2^t}, \alpha^{2^{2t}}, \dots, \alpha^{2^{(v-1)t}}$ are precomputed, the time taken by the addition operation can be reduced in modular exponentiation and u_i has already been inputted before each operation. If the classical fixed-window method is combined with the implementation of Shor's algorithm, all bits of u_i should be inputted once.

Employing the window method and t -bit semiclassical QFT, we design a quantum circuit of Shor's algorithm as presented in Figure 4.

The explanation of Figure 4 is as follows.

(1) $c = \sum_{i=0}^{L-1} 2^i m_i$ is the final measurement; if t divides L exactly, $l' = \lfloor L/t \rfloor$, otherwise $l' = \lfloor L/t \rfloor - 1$.

(2) The structures and above outputs of VB2 and VB3 are the same as in Figure 2; the inputs of VB2 and VB3 are $L-l't$ and t bits respectively; the initial value of ϕ is zero.

(3) If the left input of the black box $U_{\alpha}^{u_i 2^{it}}$ is A , then the output is $A \cdot \alpha^{u_i 2^{it}} \bmod N$, where $\alpha^{u_i 2^{it}} \bmod N$ is precomputed classically and $u_i = \sum_j 2^j u'_j$ ($u'_0, u'_1, \dots$ are the above inputs of the black box $U_{\alpha}^{u_i 2^{it}}$ from right to left).

(4) The quantum state $|0\rangle + |1\rangle$ is actually $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, which can be obtained from the Hadamard transform acting on $|0\rangle$.

(5) The lengths of $|1\rangle$ and $|0\rangle + |1\rangle$ are $\lceil \log N \rceil$ bits and 1 bit respectively.

From Figure 4, $\alpha^k \bmod N$ can be computed by the black box $U_{\alpha}^{u_{l'} 2^{l't}}, U_{\alpha}^{u_{l'-1} 2^{(l'-1)t}}, \dots, U_{\alpha}^{u_0 2^0}$, and the output result is $\alpha^{\sum_{i=0}^{l'} u_i 2^{it}} \bmod N$ (u_i is the above input of black box $U_{\alpha}^{u_i 2^{it}}$) with $\sum_{i=0}^{l'} u_i 2^{it}$ just being k 's window representation, where t is the width of the window. Therefore, the width of the window is the same as the dimension of the input of the t -bit semiclassical QFT.

The circuit needs a $\lceil \log N \rceil + t$ -dimensional quantum register. $O(1)$ elementary quantum gates are needed for R_i running once. The QFT needs $O(t^2)$ elementary quantum gates. Therefore, VB2 and VB3 running once both need $O(t^2)$ elementary quantum gates. Since modular exponentiation running once needs $O(\lceil \log N \rceil^3)$ elementary quantum gates, the new circuit needs $O(\lceil \log N \rceil^3)$ elementary quantum gates in all. Comparing with the circuit in [18], both need $O(\lceil \log N \rceil^3)$ elementary quantum gates, but the quantum register for the former requires $t-1$ dimensions more than the latter.

The runtime of the circuit of Shor's algorithm can be depicted in terms of modular exponentiation and quantum Fourier transformation [15]. Modular exponentiation running once needs n addition operations [19] and the new circuit needs $l'+1$ modular exponentiations. Therefore, the new circuit needs $(l'+1)n$ addition operations when $\alpha^{1+2^{it}} \bmod N, \alpha^{2+2^{it}} \bmod N, \dots, \alpha^{2^{l'+1}-1+2^{it}} \bmod N$ are precomputed classically ($j = 0, 1, \dots, l'$). The circuit of Shor's algorithm in [18] needs nL addition operations.

Figure 4 Quantum circuit of Shor's algorithm based on the window method and t -bit semiclassical QFT.

Additionally, modular exponentiation in the new circuit is t times as fast as that in [18]. As noted in section 2, the t -bit semiclassical QFT is t times as fast as the single-bit semiclassical QFT. Thus, compared with the circuit in [18], the new circuit is t^2 times as fast, and the new circuit's precomputation amount increases by a factor of $2^t/t$.

5 Conclusions

Because of the difficulty of building a high-dimensional quantum register, this paper presented an implementation of the high-dimensional QFT based on a low-dimensional quantum register. We defined the t -bit semiclassical QFT. The transform can balance the tradeoff of the quantum resources required and the speed of implementation of the QFT. Additionally, we design a quantum circuit for the transform. Combining the classical fixed-window representation with the implementation method for Shor's algorithm, we design a circuit for Shor's quantum factorization algorithm, which is t^2 times as fast as Parker's, where t is the width of the window.

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