Constructal entransy dissipation rate minimization for heat conduction based on a tapered element

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Based on constructal theory, the structure of a tapered element and high-conductivity link is optimized by taking the minimization of the entransy dissipation rate as the optimization objective. The results show that the mean temperature difference of the heat transfer cannot always decrease when the internal complexity of the control-volume increases. There exists an optimal constructal order leading to the minimum mean temperature difference for heat transfer. The thermal current density in high-conductivity links with variable shapes does not linearly depend on the length. Therefore, the optimized constructs based on the minimization of the entransy dissipation rate are different from those based on the minimization of the maximum temperature difference. Compared with the construct based on the minimization of the maximum temperature difference, the construct based on the minimization of the entransy dissipation rate can reduce the mean temperature difference, and improve the heat transfer performance significantly. Because entransy describes the heat transfer ability more suitably, various constructal problems in heat conduction may be addressed more effectively using this basis.

constructal theory, entransy dissipation rate, volume-to-point heat conduction, generalized thermodynamic optimization

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Many of the volume-to-point flows that occur in nature are shaped like tree networks. These flows include river basins and formative processes of cay. A volume-to-point heat conduction problem in engineering is the determination of the optimal distribution of a high-conductivity material through a finite volume, which results in the heat generated at every point being transferred most effectively to the boundary of the medium. Constructal theory was put forward by Bejan [1] and was applied to the optimization of the volume-to-point heat conduction problem. To obtain better heat transfer structure, many scholars [2–15] have made researched conduction elements with different shapes and have used various optimization objectives based on constructal theory.

Minimization of the maximum temperature difference is one of the most common optimization objectives. Bejan [1]

used the minimization of the maximum temperature difference as the optimization objective and assumed that the amount of high-conductivity material needed was finite. First, the rectangular element was optimized and the corresponding optimal elemental shape (aspect ratio) was obtained. Then, the first-order assembly that was designed with a number of optimized elemental volumes was optimized. There exists an optimal shape for the first-order assembly (or the number of the rectangular elements) that corresponds to the minimization of the maximum temperature difference of the first-order assembly. The analogy continues until the control volume is recovered by the assemblies. From [1], it can be seen that the thermal current in the high-conductivity link of a rectangular element increases continuously. Ledezma et al. [16] showed that when the constraint of a constant cross-section high-conductivity link is released and the rectangular element is re-optimized, the maximum temperature difference decreased by 6%.

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Moreover, when the constraint that the high-conductivity link be perpendicular to the low-conductivity link is released, the maximum temperature difference decreases by 5.8%. Furthermore, Almogbel and Bejan [17] proposed that when the constraint of uniform distribution of high-conductivity links through assembly was released the best high-conductivity link distribution of the first-order assembly was found to be nonuniform. Also, there exist some offsets and the maximum temperature difference decreased by 5.7% when the number of the first-order assemblies was six. Ghodoossi and Egriçan [18] analyzed the problem and obtained the exact results without the premise that the thermal current in the high-conductivity link of the rectangular element increases continuously, which was used in [1]. They showed that the approximate solution in [1] deviated from the obtained exact solution by 25% and that the deviation originated from the simplification that the thermal current increased continuously, but that equivalent coefficients of thermal conductivity were not equivalent in [1]. Wu et al. [19] analyzed the deviation and proved that the reason was not the simplification that the thermal current increased continuously, but that equivalent coefficients of thermal conductivity were not equivalent in [1]. Wu et al. [20] re-optimized the high-conductivity link distribution by releasing the constraint that the new-order assembly must be assembled using the optimized last-order assemblies and taking the aspect ratio of last-order assembly as a new optimized variable. The results showed that the maximum temperature difference of each assembly decreased by 30%. Karakas et al. [21] proposed that elemental width H_0 could be substituted with $(D_0 + H_0)$, and obtained optimized results which were different from those obtained in [1]. Based on a rectangular element with a variably shaped high-conductivity link, Zhou et al. [22] optimized the structure of the control-volume. The results showed that the thermal resistance cannot always decrease when the internal complexity of the control-volume increases. There exists an optimal constructal order that leads to the minimum thermal resistance. For the case in which the thermal current in the high-conductivity link increases discretely in a first- or higher-order assembly, Wei et al. [23] established a constructal optimization model with discrete variable cross-section high-conductivity links. The results showed that the minimum maximum thermal resistance of the assembly which was obtained by assembling the last-order assemblies could be obtained by changing the cross-section of high-conductivity link at the same assembly's order. Assuming that the hot spots were distributed along the boundary uniformly and taking the minimization of maximum temperature difference as optimization objective, Neagu and Bejan [24] optimized the tapered element and high-conductivity link and obtained the optimized structure that leads to a uniform temperature field in the inner element. In addition, they showed that the minimum thermal resistance decreases by 33% compared with that obtained in [1], and by 29% compared with that when only the variable cross-section high-conductivity link was opti-

mized. The thermal resistance at every point tends to be uniform in this case. Ghodoossi and Egriçan [25] proposed a triangular element and obtained the optimal structure that leads to the minimum thermal resistance.

The average heat transfer effect cannot be reflected by taking the minimization of the maximum temperature difference as constructal optimization objective. Guo et al. [26] proposed a new physical quantity "entransy", which represents the global heat transfer ability. They also proposed the entransy dissipation extremum principle: For a fixed boundary heat flux, the conduction process is optimized when the entransy dissipation is minimized. However, for a fixed boundary temperature, the conduction process is optimized when the entransy dissipation is maximized. An equivalent thermal resistance for multi-dimensional heat conduction problems was defined based on the entransy dissipation. Equivalent thermal resistance is an aspect of the average heat transfer effect. The physical meaning of entransy was further explained using the physical mechanisms of heat conduction [27] and electrothermal simulation experiments [28]. Many scholars [29–37] have shown interest in heat transfer optimization based on the minimization of entransy dissipation rate. Using the minimization of entransy dissipation rate as optimization objective, Wei et al. [38–40] combined the entransy dissipation extremum principle with constructal theory for the first time and found that the construct based on the minimization of the entransy dissipation rate decreased the mean temperature difference more efficiently than the construct based on the minimization of maximum temperature difference for the same conditions. They then optimized the volume-to-point heat conduction problem with a triangular element and obtained the structure that leads to the optimal heat transfer efficiency and the equivalent thermal resistance for every assembly's order [39]. Next, they used the entransy dissipation rate minimization as optimization objective and obtained the optimal construct using the constraint that new-order assembly must be assembled by the optimized last-order assembly [40]. Finally, Wei et al. [41] derived the mean temperature difference for an electromagnet and made a multidisciplinary constructal optimization for electromagnets, which combined the magnetic density problem with the heat transfer problem. Based on the entransy dissipation rate minimization, Xie et al. [42,43] optimized the heat generating bodies with rectangular cavity and T-shaped cavity. They showed that the mean temperature difference for heat transfer could be decreased more efficiently than that with maximum temperature difference minimization. This improved the global heat transfer effect in the system. The constraints that were removed from the cavity; the better the global system performance was. Chen et al. [44] optimized the disc cooling problem using the entransy dissipation rate minimization as the optimization objective.

Using a tapered element with a variable cross-section high-conductivity link, this paper will use the minimization

of entransy dissipation rate as the constructal optimization objective. From this, we will obtain the mean temperature difference for a given control-volume and the optimal shapes of the element and high-conductivity link. We will then present some interesting conclusions via a comparison with results obtained based on the maximum temperature difference minimization in [24].

1 Definition of entransy dissipation rate

Entransy, which is a new physical quantity that reflects the heat transfer ability of an object, was defined in [26] as

$$E_{vh} = \frac{1}{2} Q_{vh} T, \quad (1)$$

where Q_{vh} is the thermal capacity of an object with a constant volume, and T represents the thermal potential. The entransy dissipation function, which represents the entransy dissipation per unit time and per unit volume, was shown to be [26]

$$\dot{E}_{h\phi} = -\dot{q} \cdot \nabla T, \quad (2)$$

where $\dot{q}$ is thermal current density vector, and ∇T is the temperature gradient. During steady-state heat conduction, $\dot{E}_{h\phi}$ can be calculated as the difference between the entransy input and the entransy output of the object, i.e.

$$\dot{E}_{h\phi} = \dot{E}_{h,in} - \dot{E}_{h,out}. \quad (3)$$

The entransy dissipation rate of the whole volume in the volume-to-point conduction is

$$\dot{E}_{vh\phi} = \int_v \dot{E}_{h\phi} dv. \quad (4)$$

The equivalent thermal resistance for multi-dimensional heat conduction problems with specified heat flux boundary conditions is given as follows [26]

$$R_h = \dot{E}_{vh\phi} / \dot{Q}_h^2, \quad (5)$$

where $\dot{Q}_h$ is the thermal current. The mean temperature difference for multi-dimensional heat conduction can be expressed as

$$\Delta \bar{T} = R_h \dot{Q}_h. \quad (6)$$

2 Elemental optimization

As shown in Figure 1 [24], the heat is generated uniformly at a rate q''' (W/m³) and the heat source is uniformly distributed. The heat is first directed to a high-conductivity link

Figure 1 Tapered element [24].

(the thermal conductivity of the material is k_p , the thermal conductivity of the other material is k_0 , and $k_p \gg k_0$) of variable width D_0 (D_0 is free to vary along the x -direction and is denoted as $D_0(x)$) and is removed from the system through a path of temperature $T_{\min}$ located at $x=L_0$ and $y=0$ (M_0). The rest of the A_0 boundary is adiabatic. The heat current flows continuously. The hot spots ($T_{\max}$) are distributed along the boundary that does not make direct contact with M_0 . This boundary is defined by the unspecified curves, $y=\pm H_0(x)/2$. To allow for an analytical solution, we assume that $H_0(x)/L_0 \ll 1$, and that the heat conduction in the k_0 material is along the y -direction and the heat conduction in the k_p material is along the x -direction.

The equation [24] for heat conduction in the k_0 material is

$$\frac{\partial^2 T}{\partial y^2} + \frac{q'''}{k_0} = 0, \quad (7)$$

with the boundary conditions

$$\partial T / \partial y = 0, \quad y = \pm \frac{H_0}{2}, \quad (8)$$

$$T = T_0(x), \quad y = 0, \quad (9)$$

where $T_0(x)$ is the temperature of central k_p blade. When $y>0$, solving eq. (7) yields

$$T(x, y) = \frac{q'''}{2k_0} (-y^2 + H_0 y) + T_0(x). \quad (10)$$

Setting $T=T_{\max}$ (constant) at $y=\pm H_0/2$ yields

$$T_{\max} - T_0(x) = \frac{q''' H_0^2}{8k_0}. \quad (11)$$

The equation for heat conduction through the k_p blade is

$$\frac{d}{dx} \left(k_p D_0 \frac{dT_0}{dx} \right) + q''' H_0 = 0, \quad (12)$$

with boundary conditions:

$$\partial T_0 / \partial x = 0, \quad x = 0, \quad (13)$$

$$T_0 = T_{\min}, \quad x = L_0. \quad (14)$$

Eliminating $T_0(x)$ using eqs. (11) and (12) yields

$$\frac{d}{dx} \left(\frac{\tilde{k} D_0 H_0}{4} \frac{dH_0}{dx} \right) = H_0, \quad (15)$$

where $\tilde{k}$ is the thermal conductivity ratio, and $\tilde{k} = k_p / k_0 \gg 1$.

H_0 and D_0 are the power-law functions, $H_0 = c_0 x^m$, $D_0 = b_0 x^n$, where c_0 and b_0 are constant factors, and m and n are dimensionless real numbers. From [24], one can get

$$\frac{\tilde{k} b_0 c_0 m (2m + n - 1)}{4} = 1, \quad (16)$$

$$m + n = 2. \quad (17)$$

And then [24]

$$c_0 = \frac{2 L_0^{1-m}}{[\tilde{k} \phi_0 m (3-m)]^{1/2}}, \quad (18)$$

$$b_0 = \frac{2 [\phi_0 (3-m)]^{1/2} L_0^{m-1}}{(\tilde{k} m)^{1/2} (m+1)}. \quad (19)$$

The temperature distribution is obtained by solving eq. (12):

$$T_0(x) = \frac{c_0 q''' }{b_0 k_p (m+1)(m-n+2)} (L_0^{m-n+2} - x^{m-n+2}) + T_{\min}. \quad (20)$$

Eliminating $T_0(x)$ using eqs. (10) and (20) yields the temperature difference distribution when $y > 0$:

$$T(x, y) - T_{\min} = \frac{q'''}{2k_0} (-y^2 + H_0 y) + \frac{c_0 q'''}{b_0 k_p (m+1)(m-n+2)} (L_0^{m-n+2} - x^{m-n+2}). \quad (21)$$

The temperature difference distribution has x -axis symmetry. Therefore, H_0 can be substituted with $-H_0$ in eq. (21) when $y < 0$. The entransy dissipation rate of the element is

$$\begin{aligned} \dot{E}_{\text{vht},0} &= 2 \int_0^{L_0} \int_0^{\frac{H_0}{2}} q''' [T(x, y) - T_{\min}] dx dy \\ &= \frac{q''' A_0^2 (m+1)(4m+1)}{6k_0 \sqrt{\tilde{k} \phi_0} (3m+1) \sqrt{m(3-m)}}. \end{aligned} \quad (22)$$

Setting $d\dot{E}_{\text{vht},0}/dm = 0$ yields

$$m_{\text{opt}} = 1/2, \quad (23)$$

Therefore, the minimum entransy dissipation rate is

$$\dot{E}_{\text{vht},0,m} = \frac{3}{5\sqrt{5}} A_0^2 q''' k_0^{-1} (\tilde{k} \phi_0)^{-1/2}. \quad (24)$$

The corresponding minimum equivalent thermal resistance and mean temperature difference are

$$R_{h0,m} = \frac{3}{5\sqrt{5}} k_0^{-1} (\tilde{k} \phi_0)^{-1/2}, \quad (25)$$

$$\Delta \bar{T}_0 = \frac{3}{5\sqrt{5}} A_0 q''' k_0^{-1} (\tilde{k} \phi_0)^{-1/2}. \quad (26)$$

The optimal construct of the element is:

$$H_{0,\text{opt}} = \frac{4}{\sqrt{5}} \left(\frac{L_0}{\tilde{k} \phi_0} x \right)^{1/2}, \quad (27)$$

$$D_{0,\text{opt}} = \frac{4\sqrt{5}}{3} \sqrt{\frac{\phi_0}{\tilde{k} L_0}} x^{3/2}, \quad (28)$$

$$\left(\frac{H_{0,\text{base}}}{L_0} \right)_{\text{opt}} = \frac{4}{\sqrt{5}} (\tilde{k} \phi_0)^{-1/2} \ll 1, \quad (29)$$

$$\frac{D_{0,\text{base}}}{H_{0,\text{base}}} = \frac{5}{3} \phi_0. \quad (30)$$

The optimal element is shown in Figure 2. Eq. (29) shows that $\tilde{k} \phi_0 \gg 3.2$, which is in agreement with the assumed condition $H_0(x)/L_0 \ll 1$. The ratio between the profile area, A_0 , and the rectangle circumscribing its area, $L_0 \times H_{0,\text{base}}$, is $1/(m+1) = 2/3$. The ratio does not depend on $\tilde{k} \phi_0$.

Using on the minimization of the entransy dissipation rate, the optimal construct was found to be $H_0 \propto x^{1/2}$, $D_0 \propto x^{3/2}$. Moreover, the mean temperature difference and maximum temperature difference are

$$\Delta \bar{T}_0 = \frac{3}{5\sqrt{5}} A_0 q''' k_0^{-1} (\tilde{k} \phi_0)^{-1/2} = 0.2683 A_0 q''' k_0^{-1} (\tilde{k} \phi_0)^{-1/2}, \quad (31)$$

Figure 2 Optimal element.

$$\Delta T_{0,\max} = \frac{3}{4\sqrt{5}} A_0 q''' k_0^{-1} (\tilde{k}\phi_0)^{-1/2} = 0.3354 A_0 q''' k_0^{-1} (\tilde{k}\phi_0)^{-1/2}. \quad (32)$$

However, using the minimization of the maximum temperature difference in [24], the optimal construct was found to be $H_0 \propto x^{3/5}$, $D_0 \propto x^{7/5}$. In this case, the mean temperature difference and maximum temperature difference are

$$\Delta \bar{T}' = \frac{17}{63} A_0 q''' k_0^{-1} (\tilde{k}\phi_0)^{-1/2} = 0.2698 A_0 q''' k_0^{-1} (\tilde{k}\phi_0)^{-1/2}, \quad (33)$$

$$\Delta T'_{0,\max} = \frac{1}{3} A_0 q''' k_0^{-1} (\tilde{k}\phi_0)^{-1/2} = 0.3333 A_0 q''' k_0^{-1} (\tilde{k}\phi_0)^{-1/2}. \quad (34)$$

3 First-order assembly

As shown in Figure 3 [24], the profile area, A_1 , in the first-order assembly is covered by a large number of optimized elements with the slendernesses indicated by eq. (29). The tips of the elements describe the unspecified curves, $y = \pm H_1(x)/2$. The size of each element and its generated heat current ($q''' A_0$) depend on the axial position, $A_0(x)$. The heat current is directed to the high-conductivity link, D_1 , which is free to vary along the x -direction. It is denoted as $D_1(x)$. Wu et al. [45] showed that when using the minimization of the maximum temperature difference as optimization objective, the maximum temperatures before and after applying equivalent thermal conductivity were equal. Therefore, the entransy dissipation rates, $\dot{E}_{\text{vth}\phi 0}$ and $\dot{E}'_{\text{vth}\phi 0}$, should be equal. They are

$$\begin{aligned} \dot{E}_{\text{vth}\phi 0} &= \frac{3}{5\sqrt{5}} A_0^2 q'''^2 k_0^{-1} (\tilde{k}\phi_0)^{-1/2} \\ &= \frac{64}{75\sqrt{5}} q'''^2 k_0^{-1} (\tilde{k}\phi_0)^{-3/2} L_0^4, \end{aligned} \quad (35)$$

Figure 3 First-order assembly [24].

$$\begin{aligned} \dot{E}_{\text{vth}\phi 0} &= 2q''' \int_0^{L_0} \int_0^{H_0/2} \frac{q'''}{k_1} \left(-\frac{x^2}{2} + L_0 x \right) dx dy \\ &= \frac{36}{35\sqrt{5}} q'''^2 k_1^{-1} (\tilde{k}\phi_0)^{-1/2} L_0^4. \end{aligned} \quad (36)$$

Setting them to be equal yields the equivalent thermal conductivity:

$$k_1 = a_1 k_p \phi_0, \quad (37)$$

where $a_1 = 135/112$.

The equation for heat conduction through the k_p blade is

$$\frac{d}{dx} \left(k_p D_1 \frac{dT_1}{dx} \right) + \frac{2}{3} q''' H_1 = 0, \quad (38)$$

where the factor, $2/3$, accounts for the density with which the forest of elements fills the allocated area. H_1 and D_1 are the power-law functions:

$$H_1 = c_1 x^m, \quad D_1 = b_1 x^n, \quad (39)$$

where c_1 and b_1 are constant factors, and m and n are dimensionless real numbers. Applying eq. (15), one can obtain

$$\frac{k_p b_1 c_1 m (2m + n - 1)}{4k_1} x^{2m+n-2} = \frac{2}{3} x^m. \quad (40)$$

Therefore,

$$\frac{k_p b_1 c_1 m (2m + n - 1)}{4k_1} = \frac{2}{3}, \quad (41)$$

$$m + n = 2. \quad (42)$$

The profile area of the first-order assembly is

$$A_{p,1} = \int_0^{L_1} D_1(x) dx + A_1 \phi_0 = \frac{b_1 L_1^{n+1}}{n+1} + A_1 \phi_0. \quad (43)$$

The profile area of the k_p blade is

$$A_1 = \int_0^{L_1} \frac{2}{3} H_1(x) dx = \frac{2}{3} \frac{c_1 L_1^{m+1}}{m+1}. \quad (44)$$

This yields the volume fraction:

$$\phi_1 = \frac{A_{p,1}}{A_1} = \frac{3b_1(m+1)L_1^{n-m}}{2c_1(n+1)} + \phi_0. \quad (45)$$

Applying eq. (21), one obtains

$$\begin{aligned} T(x, y) - T_{\min} &= \frac{q'''}{2k_1} (-y^2 + H_1 y) \\ &+ \frac{2c_1 q'''}{3b_1 k_p (m+1)(m-n+2)} (L_1^{m-n+2} - x^{m-n+2}). \end{aligned} \quad (46)$$

The temperature difference distribution has x -axis symmetry. Therefore, H_1 can be substituted with $-H_1$ in eq. (46) when $y < 0$. The entransy dissipation rate for the first-order assembly is

$$\begin{aligned}\dot{E}_{vh\phi 1} &= 2 \int_0^{L_1} \int_0^{\frac{H_1}{2}} \frac{2}{3} q''' [T(x, y) - T_{\min}] dx dy \\ &= \frac{q'''^2 A_1^2 (m+1)(4m+1)}{4k_p (3m+1) \sqrt{m(3-m)}} [a_1 (\phi_1 - \phi_0) \phi_0]^{-1/2}.\end{aligned}\quad (47)$$

By optimizing eq. (47) with respect to m one obtains

$$m_{\text{opt}} = 1/2. \quad (48)$$

The corresponding minimum entransy dissipation rate is

$$\dot{E}_{vh\phi 1, m} = \frac{2\sqrt{21} A_1^2 q'''^2}{25k_p \sqrt{\phi_0 (\phi_1 - \phi_0)}}. \quad (49)$$

Eq. (49) shows that $\dot{E}_{vh\phi 1, m}$ decreases when either ϕ_0 or ϕ_1 increase. When the amount of k_p material is fixed (ϕ_1 is a constant), optimizing eq. (49) with respect to ϕ_0 yields

$$\phi_{0, \text{opt}} = \phi_1 / 2. \quad (50)$$

The corresponding entransy dissipation rate is

$$\dot{E}_{vh\phi 1, \text{mm}} = \frac{4\sqrt{21} A_1^2 q'''^2}{25k_p \phi_1}. \quad (51)$$

The equivalent thermal resistance and the mean temperature difference are

$$R_{h1, \text{mm}} = \frac{4\sqrt{21}}{25k_p \phi_1}, \quad (52)$$

$$\Delta \bar{T}_1 = \frac{4\sqrt{21} A_1 q'''}{25k_p \phi_1}. \quad (53)$$

The optimal construct of the first-order assembly is

$$H_{1, \text{opt}} = \frac{3\sqrt{21}}{7} (L_1 x)^{1/2}, \quad (54)$$

$$D_{1, \text{opt}} = \frac{5\sqrt{21}}{21\sqrt{L_1}} \phi_1 x^{3/2}, \quad (55)$$

$$\left(\frac{H_{1, \text{base}}}{L_1} \right)_{\text{opt}} = \frac{3\sqrt{21}}{7}, \quad (56)$$

$$\frac{D_{1, \text{base}}}{H_{1, \text{base}}} = \frac{5}{9} \phi_1. \quad (57)$$

The optimal first-order assembly is shown in Figure 4. Eq. (56) shows that $H_{1, \text{base}}/L_1 = 3\sqrt{21}/7 \doteq 1.96 > 1$. Because

Figure 4 Optimal first-order assembly.

there is a high-conductivity link in each element, the direction of heat conduction in the k_1 area is perpendicular to the high-conductivity link, D_1 . It is unnecessary to assume that $H_1(x)/L_1 \ll 1$.

Based on the entransy dissipation rate minimization, the optimal construct of the first-order assembly is $H_1 \propto x^{1/2}$, $D_1 \propto x^{3/2}$, and the mean temperature difference and maximum temperature difference are

$$\Delta \bar{T}_1 = \frac{4\sqrt{21} A_1 q'''}{25k_p \phi_1} = 0.7332 \frac{A_1 q'''}{k_p \phi_1}, \quad (58)$$

$$\Delta T_{1, \text{max}} = \frac{\sqrt{21} A_1 q'''}{5k_p \phi_1} = 0.9165 \frac{A_1 q'''}{k_p \phi_1}. \quad (59)$$

However, using the maximum temperature difference minimization in [24], the optimal construct of the first-order assembly is $H_1 \propto x^{3/5}$, $D_1 \propto x^{7/5}$, and the mean temperature difference and maximum temperature difference are

$$\Delta \bar{T}'_1 = \frac{68 A_1 q'''}{9\sqrt{105} k_p \phi_1} = 0.7373 \frac{A_1 q'''}{k_p \phi_1}, \quad (60)$$

$$\Delta T'_{1, \text{max}} = \frac{8 A_1 q'''}{9k_p \phi_1} = 0.8889 \frac{A_1 q'''}{k_p \phi_1}. \quad (61)$$

When the element and first-order assemblies have the same applied conditions, namely, $A_0 = A_1 = A$, $\phi_0 = \phi_1 = \phi$, one can obtain

$$\Delta \bar{T}_0 / \Delta \bar{T}_1 = \frac{3\sqrt{5}}{4\sqrt{21}} (\tilde{k}\phi)^{1/2}. \quad (62)$$

Two mean temperature difference curves between the element and first-order assemblies can be obtained. They are shown in Figure 5. When $\tilde{k}\phi > 112/15 \approx 7.47$, $\Delta \bar{T}_0 / \Delta \bar{T}_1 > 1$, namely, the mean temperature difference of the first-order design is smaller than that of the elemental design. When $\tilde{k}\phi < 7.47$, $\Delta \bar{T}_0 / \Delta \bar{T}_1 < 1$, namely, the mean temperature difference of the elemental design is smaller than that of the first-order design. The above section shows that $H_0(x)/L_0 \ll 1$, $\tilde{k}\phi > 3.2$. $\tilde{k}\phi = 7.47$ is a critical value. When $3.2 < \tilde{k}\phi < 7.47$, the optimal design is the elemental one; when $\tilde{k}\phi > 7.47$ the internal complexity should be increased, namely, the first-order design should be adopted.

4 Second-order assembly

As shown in Figure 6 [24], the profile area A_2 in the second-order assembly is covered by a large number of the optimized first-order assemblies. The tips of the first-order assemblies describe the unspecified curves $y = \pm H_2(x)/2$. The generated heat current is directed along the high-conductivity link, D_2 , which is free to vary along the x -direction. It is

Figure 5 Mean temperature difference for the element and first-order assembly.

Figure 6 Second-order assembly [24].

denoted as $D_2(x)$. Assuming that $H_2(x)/L_2 \ll 1$, the heat conduction in the k_2 material is along the y -direction and the heat conduction in the k_p blade is along the x -direction.

Using the entransy dissipation rate minimization as the optimization objective, the entransy dissipation rates before and after applying equivalent thermal conductivity, $\dot{E}_{vh\phi 1}$ and $\dot{E}'_{vh\phi 1}$, should be equal. They are

$$\dot{E}_{vh\phi 1} = \frac{4\sqrt{21}A_1^2 q^{m^2}}{25k_p \phi_1} = \frac{64\sqrt{21}q^{m^2} L_1^4}{525\phi_1 k_p}, \quad (63)$$

$$\begin{aligned} \dot{E}'_{vh\phi 1} &= 2q^{m^2} \int_0^{L_1} \int_0^{H_1/2} \frac{2}{3} \frac{q^{m^2}}{k_2} \left(-\frac{x^2}{2} + L_1 x \right) dx dy \\ &= \frac{18\sqrt{21}q^{m^2} L_1^4}{245k_2}. \end{aligned} \quad (64)$$

Setting them to be equal yields the equivalent thermal conductivity:

$$k_2 = a_2 k_p \phi_1, \quad (65)$$

where $a_2 = 135/224$. Analogous to the above two sections, one can obtain

$$\begin{aligned} \dot{E}_{vh\phi 2} &= 2 \int_0^{H_2/2} \int_0^{L_2} \left(\frac{2}{3} \right)^2 q^{m^2} [T(x, y) - T_{\min}] dx dy \\ &= \frac{3q^{m^2} A_2^2 (m+1)(4m+1)}{8k_p (3m+1) \sqrt{m(3-m)}} [a_2 (\phi_2 - \phi_1) \phi_1]^{-1/2}, \end{aligned} \quad (66)$$

where ϕ_2 is the fixed volume fraction of the k_p material allocated to the second-order assembly, $\phi_2 = A_{p,2}/A_2$; A_2 is the profile area of the second-order assembly, $A_2 = \int_0^{L_2} \left(\frac{2}{3} \right)^2 H_2(x) dx$; $A_{p,2}$ is the profile area of the k_p bladem, $A_{p,2} = \int_0^{L_2} D_2(x) dx + A_2 \phi_1$; and the factor $(2/3)^2$ accounts for the volume fraction that the heat generating material (A_2) comprises in the second-order assembly. Optimizing eq. (66) yields

$$m_{\text{opt}} = 1/2, \quad (67)$$

$$\phi_{1,\text{opt}} = \phi_2/2. \quad (68)$$

The minimum entransy dissipation rate of the second-order assembly is:

$$\dot{E}_{vh\phi 2, \text{mm}} = \frac{6\sqrt{42}A_2^2 q^{m^2}}{25k_p \phi_2}. \quad (69)$$

Its corresponding minimum equivalent thermal resistance and mean temperature difference are

$$R_{h2, \text{mm}} = \frac{6\sqrt{42}}{25k_p \phi_2}, \quad (70)$$

$$\Delta \bar{T}_2 = \frac{6\sqrt{42}A_2q'''}{25k_p\phi_2}. \quad (71)$$

The optimal construct of the second-order assembly is

$$H_{2,\text{opt}} = \frac{3\sqrt{42}}{14}(L_2x)^{1/2}, \quad (72)$$

$$D_{2,\text{opt}} = \frac{5\sqrt{42}}{63\sqrt{L_2}}\phi_2x^{3/2}, \quad (73)$$

$$\left(\frac{H_{2,\text{base}}}{L_2}\right)_{\text{opt}} = \frac{3\sqrt{42}}{14} \approx 1.39, \quad (74)$$

$$\frac{D_{2,\text{base}}}{H_{2,\text{base}}} = \frac{10}{27}\phi_2. \quad (75)$$

Eq. (74) shows that this construct does not agree with the assumed condition, $H_2(x)/L_2 \ll 1$. It is necessary to modify the second-order assembly. The modified second-order assembly is analyzed in the following text.

As shown in Figure 7 [24], the second-order assembly can be fitted using only two optimal first-order assemblies. Its profile area is $A_2=2A_1$. The heat currents ($q'_1=2q'''A_1$) collected by each first-order assembly flow through the k_p link along the x -direction. They are removed through the M spot:

$$q'_1 = k_p D_2 \frac{T_c - T_{\min}}{H_{1,\text{base}}/2}. \quad (76)$$

The entransy dissipation rate of the k_p link along the x -direction is

$$\dot{E}_{kp} = q'_1(T_c - T_{\min}) = \frac{2q'^2 A_1^2 H_{1,\text{base}}}{k_p D_2}. \quad (77)$$

The entransy dissipation rate of the second-order assembly

Figure 7 Modified second-order assembly.

constitutes of the entransy dissipation rate of the first-order assembly and $\dot{E}_{kp}$. Namely,

$$\dot{E}_{\text{vh}\phi_2} = \frac{2q''^2 A_1^2}{k_p} \left(\frac{H_{1,\text{base}}}{D_2} + \frac{2\sqrt{21}}{25\phi_1} \right). \quad (78)$$

The volume fraction of the k_p material allocated to the second-order assembly is

$$\phi_2 = \frac{A_{p,2}}{A_2} = \frac{\phi_1 A_2 + H_{1,\text{base}} D_2/2}{A_2} = \phi_1 + \frac{81D_2}{16\sqrt{21}H_{1,\text{base}}}. \quad (79)$$

Eliminating $H_{1,\text{base}}/D_2$ between eqs. (78) and (79) and optimizing eq. (78) with respect to ϕ_1 yields

$$\phi_1 = 0.3655\phi_2, \quad (80)$$

$$\dot{E}_{\text{vh}\phi_{2,m}} = \frac{1.3721q''^2 A_2^2}{k_p \phi_2}. \quad (81)$$

The corresponding minimum equivalent thermal resistance and mean temperature difference are

$$R_{h2,\text{mm}} = \frac{1.3721}{k_p \phi_2}, \quad (82)$$

$$\Delta \bar{T}_2 = \frac{1.3721A_2q'''}{k_p \phi_2}. \quad (83)$$

Eliminating ϕ_1 using eqs. (79) and (80) yields

$$\left(\frac{D_2}{H_{1,\text{base}}}\right)_{\text{opt}} = 0.574\phi_2. \quad (84)$$

Based on the minimization of the entransy dissipation rate, the maximum temperature difference of the second-order assembly is

$$\Delta T_{2,\text{max}} = \frac{1.8464A_2q'''}{k_p \phi_2}. \quad (85)$$

However, based on the minimization of the maximum temperature difference in [24], the mean temperature difference and the maximum temperature difference of the second-order assembly are

$$\Delta \bar{T}'_2 = \frac{1.4915A_2q'''}{k_p \phi_2}, \quad (86)$$

$$\Delta T'_{2,\text{max}} = 1.6877 \frac{A_2q'''}{k_p \phi_2}. \quad (87)$$

When the first-order assembly and second-order assembly have the same applied conditions, namely, $A_1=A_2=A$, $\phi_1=\phi_2=\phi$, one can obtain $\Delta \bar{T}'_1/\Delta \bar{T}'_2=0.40<1$. Namely, the mean temperature difference of the first-order design is

smaller than that of the second-order design.

5 Optimization results and analysis

If the volume fraction of high-conductivity material in each order assembly ($\phi_i(i=0,1,2)$) and the profile area of each order assembly (A_i) are kept constant, then $\phi_0=\phi_1=\phi_2=\phi$ and $A_0=A_1=A_2=A$. From above three sections, one can obtain that when $3.2<\tilde{k}\phi<7.47$, the mean temperature difference of the elemental design is smaller than that of the first-order design. Also, when $\tilde{k}\phi>7.47$, the conclusion is reversed. $\tilde{k}\phi>7.47$ is the critical point for a tree-like design. When $3.2<\tilde{k}\phi<7.47$, the optimal design for the given control-volume is the elemental one; when $\tilde{k}\phi>7.47$ the internal complexity increases, namely, the first-order design should be adopted.

Using the minimization of the entransy dissipation rate, the constructs are optimized. The optimization results from this study are different from those obtained in [24]. The major findings of this study are listed in Table 1. The major optimization results based on the minimization of the maximum temperature difference [24] are listed in Table 2. Comparing Tables 1 and 2, it can be seen that the optimized constructs based on the minimization of the entransy dissipation rate are different from those based on the minimization of the maximum temperature difference. This is because the thermal current density in the high-conductivity links of variable shape does not vary linearly with the length [39]. From the comparison between the two tables,

the optimized construct based on the maximum temperature difference minimization reduces the maximum temperature difference. However, the optimized construct based on the entransy dissipation rate minimization reduces the mean thermal resistance and improve the overall heat transfer effect.

6 Conclusions

By applying constructal optimization to a tapered element with a variable cross-section high-conductivity link using the minimization of entransy dissipation rate, the element is optimized. The optimal shapes of element and high-conductivity link were found to be $H_0\propto x^{1/2}$, $D_0\propto x^{3/2}$. These findings are different from those that were optimized ($H_0\propto x^{3/5}$, $D_0\propto x^{7/5}$) using on the minimization of the maximum temperature difference. The results show that the mean temperature difference of the heat transfer cannot always decrease when the internal complexity of the control-volume increases. There is an optimal constructal order that leads to the minimum mean temperature difference for heat transfer. When $3.2<\tilde{k}\phi<7.47$, the optimal design for the given control-volume is the elemental one. Namely, the optimal constructal order, which is zero, leads to the minimum mean temperature difference for heat transfer in the control-volume. When $\tilde{k}\phi>7.47$, the internal complexity is increased and the first-order design should be adopted. Namely, the optimal constructal order, which is one, leads to the minimum mean temperature difference for heat transfer in the control-volume. The equivalent thermal resistance defined based on the

Table 1 Constructal optimization results based on the minimization of the entransy dissipation rate (this study)

Order of the assembly, i	$\left(\frac{D_{i,\text{base}}}{H_{i,\text{base}}}\right)_{\text{opt}}$	$\left(\frac{H_{i,\text{base}}}{L_i}\right)_{\text{opt}}$	$\left(\frac{D_i}{H_{i-1,\text{base}}}\right)_{\text{opt}}$	$\frac{\Delta T_{i,\text{max}} \cdot k_p \phi_i}{q'' A_i}$	$\frac{\Delta \bar{T}_i \cdot k_p \phi_i}{q'' A_i}$
0	$\frac{5}{3}\phi_0$	$\frac{4}{\sqrt{5}}(\tilde{k}\phi_0)^{-1/2}$	—	$0.3354(\tilde{k}\phi_0)^{1/2}$	$0.2683(\tilde{k}\phi_0)^{1/2}$
1	$\frac{5}{9}\phi_1$	$\frac{3\sqrt{21}}{7}$	—	0.9165	0.7332
2	—	—	$0.574\phi_2$	1.8464	1.3721

Table 2 Constructal optimization results based on the minimization of the maximum temperature difference [24]

Order of the assembly, i	$\left(\frac{D_{i,\text{base}}}{H_{i,\text{base}}}\right)_{\text{opt}}^{\text{a)}$	$\left(\frac{H_{i,\text{base}}}{L_i}\right)_{\text{opt}}$	$\left(\frac{D_i}{H_{i-1,\text{base}}}\right)_{\text{opt}}$	$\frac{\Delta T_{i,\text{max}} \cdot k_p \phi_i}{q'' A_i}$	$\frac{\Delta \bar{T}_i \cdot k_p \phi_i^{\text{a)}}}{q'' A_i}$
0	$\frac{3}{2}\phi_0$	$\frac{5}{3}(\tilde{k}\phi_0)^{-1/2}$	—	$0.3333(\tilde{k}\phi_0)^{1/2}$	$0.2698(\tilde{k}\phi_0)^{1/2}$
1	$\frac{15}{32}\phi_1$	2	—	0.8889	0.7373
2	—	—	$0.513\phi_2$	1.6877	1.4915

a) These columns were derived from data in [24].

entransy dissipation rate reflects the heat conduction ability and heat transfer efficiency in the heat transfer process. The smaller the equivalent thermal resistance; the better is the heat transfer effect and the lower the mean temperature in the control-volume. The minimum thermal resistance reflects the maximum temperature difference, and represents the maximum temperature limit for the volume. The construct based on the minimization of the entransy dissipation rate is the optimal result for the mean heat transfer effect, which differs from the construct based on the minimization of the maximum temperature difference. Compared with the optimal construct based on the minimization of the maximum temperature difference, the optimal construct based on the minimization of the entransy dissipation rate decreases the mean temperature difference for heat transfer significantly. The improvement in the heat transfer effect resulting from the optimal construct based on the minimization of the entransy dissipation rate can be clearly seen in our results. Both the mean temperature difference and the maximum temperature difference should be combined when considering the efficiency and temperature limits simultaneously. This is important for addressing volume-to-point heat conduction problems. This paper has fully described the effect of the minimization of the entransy dissipation rate. Because the idea of entransy describes heat transfer ability more suitably [26] than the minimization of the maximum temperature difference, we suggest that all future studies of heat conduction constructal problems be based on entransy.

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