Vortex definition and “vortex criteria”

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It is well known that the vorticity $\omega = \nabla \times \mathbf{u}$, where $\mathbf{u}$ is fluid velocity, has clear mathematic definition, but for viscous flow the definition of a vortex remains an open problem for long [1,2], which has now evolved to a big confusion after many “vortex identification criteria” (“vortex criteria” for short) were proposed to identify various turbulent coherent structures observed experimentally and numerically. The present communication attempts to clarify this confusing situation and provide a preliminary answer to the question: What is a vortex?¹⁾

1 Vorticity tubes and vortices

1.1 Classic definition of vortices by vorticity tubes

By the well defined vorticity, one can define vorticity lines, vorticity surfaces, and vorticity tubes. These concepts enabled Helmholtz [4] to establish three classic theorems for vorticity tubes (3Hs for short). In modern language [5], the first theorem (H1) is an unconditional and universal kinematic theorem, asserting that the strength of a vorticity tube or its circulation is invariant along the tube. Consequently, in three-dimensional space a vorticity tube of nonzero strength cannot end in the fluid. The second and third theorems (H2 and H3) involve the temporal evolution and hence dynamics. They are conditional, and assert that if and only if the flow is circulation-preserving (i.e., fluid acceleration is curl-free), a vorticity tube will consist of the same material particles moving with the fluid (H2) and its strength is time-invariant (H3).²⁾

On the other hand, the concept of vortices may be traced back to ancient time [2] and has been used to describe a big variety of rotational flows due to its intuitive nature, but so far without a consensus on its rigorous definition. During the establishment of the 3Hs, Helmholtz [4] was the first to call a very thin vorticity tube as a vortex filament. This was followed by Lamb [6], who added that a vortex filament is simply a vortex. Then, Saffman [7] defined a vortex as a “finite volume of vorticity immersed in irrotational fluid”. From the context of ref. [7] one finds that the “finite volume of vorticity” is specifically referred to a thin vorticity tube. Thus, in summary, we have:

Definition 1. A vortex is a vorticity tube surrounded by irrotational fluid.

Accordingly, the 3Hs have been simply called three vortex theorems. Recall that Helmholtz [4] was working on inviscid and incompressible fluid model without non-conservative external body force, in which the flows are all circulation-preserving. Thus, a vortex had all invariant properties of the vorticity tube, and the above simple Definition 1 was perfect.

It should be stressed that a vorticity tube is very different from vorticity lines therein. Unlike vortex filament, a vorticity line does not satisfy 3Hs at all. The vorticity can vary along the line and even becomes zero [8], nor it has to form closed loop in unbounded fluid, which happens rarely [5]. Thus, a vorticity tube may have a subdomain of vanishing vorticity. Actually, in the study of the motion of circular

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1) In so doing we follow the warning of Professor J. C. Wu [3]: a vorticity tube has to be clearly distinguished from a vortex before the consensus is reached on the latter’s definition.

2) The condition for H2 can be slightly relaxed, see ref. [5].

vortex rings, one of the models was a hollow ring [9]; and a vortex may contain vorticity of opposite signs as in the case of Taylor vortex [10]. Indeed, in reality vorticity tubes (vortices by Definition 1) may have very irregular geometry and complicated vorticity distribution.

1.2 Difficulty of vortex definition for viscous flows

Evidently, once we enter real viscous flows, Definition 1 can no longer be exact, but remains a “useful approximation for real fluids of small viscosity” [7]. It is this “useful approximation” that makes viscous vortices have been extensively used in practice for about a century. Applied fluid mechanicians, aerodynamicists, and engineers do not hesitate to work on some physically intuitive concepts no matter if their rigorous mathematic definition is available. As long as a viscous vortex is sufficiently isolated from the others, Definition 1 remains a useful approximation.

Definition 1 (in the sense of “useful approximation”) became seriously unsatisfactory and vortex definition became a controversial issue, only when it was necessary to investigate the viscous interactions among multiple vortices and between viscous vortex and solid wall, the formation of vortex inside the fluid or at a wall, and the topological change of tangled vortices due to viscosity, etc. In particular, this happens after turbulent coherent structures were discovered, which themselves are lack of clear definition too.

Evidently, the difficulty in vortex definition has just one single root: the violation of the circulation-preserving condition for H2 and H3 due to viscosity, baroclinic effect, or non-conservative body force. All 3Hs still hold true under their conditions, but a vorticity tube satisfying H1 is no longer a material tube. Thus, because a thorough clarification of this issue for the coherent structures in transitional and turbulent flows is beyond the reach of routine DPIV experimental measurements and direct numerical simulation (DNS), how to capture the evolution of a vorticity tube with non-material boundary had long been unknown. It is in this circumstance that various “vortex criteria” came into being. Note however that even in homogeneous and isotropic turbulence and at Kolmogorov’s dissipation scale, there still exists highly discrete vortices [11], which are the cause of turbulent intermittency.

1.3 Two properties of viscous vortices

Before proceed, it is necessary here to review two basic properties. Here, for convenience, a vortex in viscous flow will still be understood as approximately defined by Definition 1, and we call it “H1-based vortex” for short.

(1) Vortex as flow structure at large Reynolds numbers. Consider a flow over a body at the Reynolds number $Re=UL/\nu$, where U and L are the characteristic velocity and body length, respectively, and ν is the molecule kinematic viscosity for laminar flow or averaged eddy viscosity for turbulent flow. In a Stokes flow over a bluff body with $Re \ll 1$, the flow remains attached, with the vorticity field spreading smoothly to very far distance [12]. There is no

vortex. Attached separation bubble behind the body appears at $Re=O(10)$, which is still not a vortex since outside the bubble there is distributed vorticity field. All observations and experiments indicate that vortices are a special form of fluid motion only at $Re \gg 1$. Only for this case the support of vorticity field in a vortex is physically compact in the following sense: the vorticity decays exponentially and rapidly, so that there is a negligible small vorticity tail (“exponential vorticity tail” for short) at the side boundary of a viscous vortex. This boundary is no more fuzzier than the outer edge of a boundary layer, but no one asserts that the boundary layer is poorly defined. Mathematically, this fast decaying feature is ensured by the fact that the vorticity is governed by a parabolic convective-diffusion equation.

In particular, in the asymptotic limit $Re \rightarrow \infty$, all vortex layers, including boundary layers and free vortex layers, degenerate to infinitely thin vortex sheets. This important concept of Prandtl [13] distinguishes the viscous flow with $\nu \neq 0$ but $\nu \rightarrow 0$ from strictly inviscid flow with $\nu = 0$, and thereby removed the objection of Kelvin on the circulation generation from an airfoil (see ref. [14]). Batchelor [12] called the former “effectively inviscid flow”, which has actually served as the sole flow model in classic steady and unsteady “inviscid” aerodynamics and led to brilliant success. In this case Definition 1 recovers its perfect status.

(2) Axial vortex and layer-like vortex. Betz [15] presents convincing argument that at large Re a concentrated axial vortex can only be formed by the rolling up of free vortex layers. A vortex layer also has physically compact support, having an exponential vorticity tail at its side faces. In science and technology, for certain practical purposes one often just focuses on axial vortices and ignores their feeding vortex layers. This is understandable, since before tightly rolling-up, a vortex layer carries much smaller amount of vorticity per unit length than that in the core of axial vortex, so that it only has minor influence on the global flow. However, the vortex layer and its formation process is an inevitable ingredient vortex dynamics, and cannot be excluded.

The rolling-up of vortex layer is not only extensively observed in nature and technology but also in some theoretical models for axial vortices. In the study of aircraft trailing vortices, Moore and Saffman [16] include the rolling sheet as an outer part of the vortex core, which depends on the spanwise distribution of aerodynamic loading (circulation). Recent experiments have shown that the Moore-Saffman trailing-vortex model behaves better than Batchelor’s model [17], although both theories are based on the same set of simplified governing equations [18]. On the other hand, as can be observed by disturbing a cup of water, when a viscous axial vortex hits a non-rotating solid boundary, the vorticity lines therein must turn to spreading along the wall like a horn [8], implying the axial vortex is transforming to an attached vortex layer. Therefore, axial vortex and vortex layer may be

two closely related constituents of a single vortex, in either temporal evolution or spatial distribution, or both.

This being the case, vortex layer should be included in the definition of vortex: it is a layer-like vortex. This is still in consistency with defining vortex by vorticity tube; one merely needs to consider very thin loop in constructing the tube. Excluding vortex layer would make the vortex an incomplete physical existence.

2 “Vortex criteria” and vortical structures

As mentioned before, when various turbulent coherent structures observed experimentally and numerically (especially by DNS) exhibit certain compact vortex-like structures, but how to capture vorticity tubes in complex viscous flows was still unknown, various pattern-recognition theories for these structures were constructed, as reviewed comprehensively by Epps [19].³⁾ This kind of efforts has helped one’s physical understanding, though controversy still exists among different criteria and patterns deduced thereby. However, since the proposed criteria used the name of “vortex criteria” or even “vortex definition”, big confusion has happened on the alternative vortex definitions other than the century-long familiar concept. Below we look at whether those “vortex-criteria” are relevant to vortex definition.

2.1 Truesdell’s vorticity numbers

The primary motivation of those criteria was to identify some pieces of vortices from the observed complex flow structures, based on the degree of vorticity concentration. This task was pioneered by Truesdell [5], who introduced a kinematic vorticity number m_K as the ratio of the norms of the vorticity $\boldsymbol{\omega}$ and strain-rate tensor $\boldsymbol{D}$ (in our notation) to measure the fluid rotationality, and a dynamic vorticity number m_D as the ratio of the norms of the Lamb vector and the rest of fluid acceleration. The former,

$$m_K \equiv \frac{|\boldsymbol{\omega}|}{\sqrt{2\boldsymbol{D}:\boldsymbol{D}}} > 1,$$

has motivated quite a few modern vortex criteria as its variants. As said in sect. 1.1, vortices may have very different strengths, and in a vorticity tube there can be very non-uniform vorticity distribution; so these numbers have nothing to do with vorticity tubes and bear no relevance to the H1-based vortex definition.

It is worth noticing that, as a mathematician, Truesdell never used a concept without rigorous definition. His classic book [5] did not give any definition for vortex, so he used this word with great caution. The word “vortex” is mentioned only at two minor contexts [5]. But still he carefully calls 3Hs Helmholtz vorticity theorems. Obviously Truesdell accepted the H1-based vortex concept, but just put the topic

aside. Regrettably, while most modern criteria grew from Truesdell’s m_K , some authors did not follow Truesdell’s caution as they used the confusing nomenclature.

2.2 Can “vortex” be decoupled from vorticity tubes?

Almost all modern “vorticity criteria” are only used to identify axial vortices. Vortex layers or layer-like vortices are excluded. Consequently, the important processes of vortex formation and evolution are out of one’s vision. This is one of the major drawbacks of the criteria.

The most severe drawback of using these criteria to define a vortex, however, is their ignorance of the most basic property of the vortex, the H1 or the solenoidality of a divergence-free vector field. It is this tube-like property that distinguishes vortices from any other flow structures in the entire history of science and technology. Since those criteria are focused on the relative importance of rotation and strain rate as Truesdell’s m_K (some added built-in low-pressure condition in vortex core), they also have nothing to do with vorticity tubes. As one identifies a “vortex” when certain criterion-dependent number exceeds a threshold value, it could be broken into segments like earthworms as its stretching rate (measured by $\boldsymbol{\omega} \cdot \nabla \boldsymbol{u} = \boldsymbol{\omega} \cdot \boldsymbol{D}$) becomes very strong, which violates H1, despite that the strong vorticity-tube stretching is associated with vorticity enhancement. Indeed, most “vortex patterns” recognized by the existing criteria are worm-like rather than tube-like. Therefore, we strongly appeal using the word “vortical structures” instead of “vortices” to describe the recognized patterns of coherent structures by these criteria.

Although not common, there also appeared some voice to replace the concept of H1-based vortices by the structures identified by some other “vortex criteria”, which we definitely disagree. When Helmholtz and Kelvin investigated the self-induced motion of circular vortex rings, when Prandtl used the simplified wing vortex system (bound vortex with trailing vortex sheet) to establish the lifting-line theory, when meteorologists study typhoons, hurricanes and tornadoes, when vortex motion was viewed as the sinews and muscles by Küchemann [20], the voice of fluid by Müller and Obermeier [21], and the sinews of turbulence by Moffatt et al. [22], what in their mind can only be the H1-based vortices and by no means anything else. Can these well-established theories be reconstructed by any other vortex definition and thereby the history be rewritten?⁴⁾

2.3 Can there be a complete theory for coherent-structure dynamics?

So far all “vorticity criteria” stay in the level of pattern recognition. To rationally recognize an observed pattern, certain pieces of theory of fluid kinematic and/or dynamics are necessary, but none of them has grown to a complete dynamic

3) This review makes things more controversial, since it puts quite a few criteria under the same title “vortex identification” despite they were proposed for identifying Eulerian or Lagrangian coherent structures, rather than vortices.

4) Words like “sinews”, “muscles” and “voices” can never be applied to incompressible fluid tube although it also has the “H1-type” property.

theory for the pattern evolution and interaction, and can compete with the mature vorticity-dynamics theory. Because of the complete ignorance of the H1, what dynamic laws the recognized pattern obeys remains blank. This theoretical lacking is a very severe shortage of various criteria. Consequently, after recognizing certain structures, in practice one still has to appeal to vorticity dynamics to interpret their evolution and interaction; but since the structures are only some pieces of vortices, such interpretation can in no way be complete.

3 A basic vortex definition and discussion

The preceding discussions have come to a conclusion that various “vortex criteria” cannot serve as vortex definitions, nor can the structures identified by them replace real vortices. An H1-based vortex definition allowing for non-circulation-preserving flows is the only correct approach. Actually, such a definition is not an extremely difficult issue; it becomes apparently out of reach merely due to the big confusion caused by various “vortex criteria”. As a starting point for further discussion, we propose to specify the H1-based vortex definition to a tentative statement:

Definition 2. A vortex is a vorticity tube in large Reynolds-number flow, bounded by irrotational fluid with an exponential vorticity tail or by non-rotating solid boundary.

By this definition, a vortex must satisfy H1. Interface between two fluids, say the water-air interface, is not the boundary of a vortex that may penetrate it. Similarly, a vortex may partly run through a rotational solid body due to the continuity of normal vorticity [10,18]. Below we discuss the unique power and promising prospect of this definition.

3.1 Evolution of vorticity surface in viscous flow

We have seen that for non-circulation-preserving flow a major barrier of H1-based vortex definition is that the side boundary of the vortex is no longer a material surface as it evolves in time. Fortunately, a remarkable breakthrough of this barrier has been achieved by Yang and coworkers [23-25] with encouraging progresses. By a special scheme, these authors use DNS to capture a material surface and follow its time evolution. Viscous diffusion is accounted for, which continuously renews the surface as time to ensure it remains a vorticity surface. In a transitional channel flow, the authors observed that, initially parallel flat vorticity surfaces will roll into tube-like structure (here again the vortex layer is an inevitable ingredient), which evolves to a hairpin vortex [24].

In the late transition stage, the vortex reconnection occurs as the head of hairpin raises up to meet its counterpart from the opposite wall. This topological change by tracing the vorticity surfaces is more clearly seen than that inferred from any other “vortex criteria” [25], because the observed vorticity tube is never broken into pieces.

This progress is a sign that the prospect of using H1-based vortex definition to analyze complex vortex interactions is really promising. Further progresses along this direction is

of course highly desired.

3.2 Vortex dynamics and vorticity dynamics

The above success in identifying vorticity surfaces by CFD exemplifies a general fact: Compared with those “vortex criteria”, the greatest unmatched superiority of the H1-based vortex definition is that one can be fully backed up by the rational theory of vorticity dynamics, from which a complete theory of vortex dynamics can be developed to address the entire lifetime history of vortices, from the generation of vorticity, formation of vortices, their motions, interactions, evolution, instability, till becoming turbulence and being dissipated (refs. [10,18]).

The interaction of a vortex and a stationary solid is another example of this superiority of Definition 2. In contrast to the inherent difficulty of tracing the root of a “vortex” by those criteria to the wall, there is no principal difficulty to do so by Definition 2. Not only kinematically the vorticity lines in the vortex have to turn and spread tangent to the wall, but also the underlying dynamics processes can be fully investigated by boundary vorticity dynamics [26-29].

Again consider a hairpin vortex. Its roots at the wall is always cut off by those criteria, but rootless earthworms cannot be a complete structure. Actually, the roots of a hairpin are the two-dimensional on-wall pair of source and sink of the three-dimensional vorticity field associated with spirals of the wall-friction lines, which is the sink and source of normal vorticity near the wall, respectively, from which the hairpin grows in a near-wall strain-rate field along the maximal stretching direction. Zhao et al. [30] made the first numerical search of the hairpins root, and the numerical experiments of Chen [29] revealed a clear and complete chain of processes in this mechanism.

3.3 Further remarks

(1) The objectivity of a vortex. Vortices are undoubtedly the objective existence form of fluid motion at large Reynolds numbers. But its specific appearance may vary as different motion status of observers. It suffices that the vorticity is invariant with respected to any translational frame of references, and unnecessary to require a vortex be defined invariant even with rotational transformation of frame of references. An observer rotating with a tornado can feel its existence by strong inertial-force field even though he (she) cannot “see” the vortex. Unlike defining a material matter requires the satisfaction of so-called frame-indifference principle, we are now talking about the form of fluid motion.

(2) Vortex decomposition and connection. In reality a single vortex obtained by Definition 2 can be decomposed to several connected vortices without violating H1, each having its special origin and form. For example, the aircraft vortex system is a “big” vortex loop consisting of wing boundary layers (bound vortex), trailing vortex layer and wing-tip vortices formed by the rolling-up of the former, and a (somewhat idealized) starting vortex. The vortex loop generated by

an insect flapping wing can be decomposed to leading-edge vortex, wingtip vortex, trailing vortex, and wing-root vortex. Also, the boundary layer that covers the whole wing surface like a sock (open at the trailing edge) may be decomposed to upper and lower boundary layers. A U-shape vortex in a lake is actually continued into the air to form a closed loop, but it is often suffices to study the former only. Whenever so desired, these decomposed vortices can be connected to a whole.

(3) Vortex-pattern recognition. Owing to the fuzziness of the side boundary of a vortex, one may well choose appropriate vorticity surface away from the exponential vorticity tail, usually enclosing thinner vorticity tube (e.g. refs. [24,25]) to visualize the vortex pattern without violating the H1. This is similar to but better than the common practice of adjusting the threshold values in using “vortex criteria”. Since a vorticity tube can physically bifurcate into two or more tubes, to capture this behavior the arbitrariness in selecting vorticity surface for visualization may have a lower bound.

The popular nomenclature of “vortex criteria” should be replaced by criteria for identifying various vortical structures. The word “vortex” can only be applied to a flow structure at large Reynolds numbers satisfying Helmholtz’s first vorticity-tube theorem, which not only fully reflects the physics behind one’s intuitive concept in the whole history of science and technology but also, as shown here, is indeed powerful and promising. In particular, if the H1-based vortex dynamics could be introduced into the study of transitional and turbulent vortical structures, systematically rather than fragmentally, it would usher a more bright future.

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