A NOTE ON “CONSTANT LATENT ODDS-RATIOS MODELS AND THE MANTEL–HAENSZEL NULL HYPOTHESIS” HESSEN, 2005

GUNTER MARIS

CITO, NATIONAL INSTITUTE FOR EDUCATIONAL MEASUREMENT ARNHEM

In a recent paper, Hessen (Psychometrika 70(3):497–516, 2005) introduces the class of constant latent odds-ratios models as an extension of the Rasch model for which the sum score is still the sufficient statistic for ability. In this paper the relation between both the general and the general parametric constant latent odds-ratios model and the Rasch model is considered.

Key words: four-parameter logistic model, constant latent odds-ratios model, Rasch model.

Hessen (2005) introduces the class of *constant latent odds-ratios models* (CLOR, for short) as a generalization of the Rasch model (Rasch, 1960), and shows how differential item functioning by means of a Mantel–Haenszel procedure can be studied. In this paper we focus on the class of CLOR models and clarify their relation to the Rasch model. In the Rasch model, the probability with which a person with ability t gives a correct response to item i is the following:

$$P(X_i = 1|t) = \pi_i(t) = \frac{b_i t}{1 + b_i t} = \frac{t}{c_i + t}, \quad (1)$$

where b_i (c_i) is the easiness (difficulty) of the i th item. This function is usually called the *item response function* (IRF, for short). In the definition of the class of CLOR models, the odds-ratio,

$$V_i(t) = \frac{P(X_i = 1|t)}{P(X_i = 0|t)} \quad (2)$$

and the ratio of odds-ratios, for items i and j ,

$$\omega_{ij}(t) = \frac{V_i(t)}{V_j(t)} \quad (3)$$

play a crucial role. For the Rasch model it holds that $\omega_{ij}(t)$ does not depend on the ability t .

The class of CLOR models is characterized by three properties:

1. Unidimensionality.
2. Local independence (LI, for short): $\Pr(\mathbf{X} = \mathbf{x}|t) = \prod_i \Pr(X_i = x_i|t)$.
3. Constant ratio of odds-ratios $\omega_{ij}(t) = \omega_{ij}$.

CLOR models may have both monotone and non-monotone IRFs. Hessen (2005) claims that the class of CLOR models is an extension of the Rasch model for which the property that the sum score is sufficient for the latent trait is retained. This claim is illustrated with some parametric and nonparametric IRT models. In this paper we show that whenever unidimensionality and LI are assumed, the assumption of constant ratio of odds-ratios is equivalent to the assumption that the IRF can be expressed as in (1). For non-monotone CLOR models we show that the latent

trait is not identifiable from $\mathbf{X}$, and that the same distribution can be specified with the IRF in (1). Furthermore, we show that the general parametric CLOR model is equivalent to the Rasch model together with the assumption that the range of the latent trait is constrained.

1. CLOR Models in Relation to the Rasch Model

From the assumptions of CLOR models we readily obtain that

$$\frac{\pi_i(t)}{1 - \pi_i(t)} = \omega_{ij} \frac{\pi_j(t)}{1 - \pi_j(t)} \quad (4)$$

and hence

$$\pi_i(t) = \frac{\omega_{ij}\pi_j(t)}{1 - \pi_j(t) + \omega_{ij}\pi_j(t)}. \quad (5)$$

If, without loss of generality, we choose a parametrization for the latent trait t such that it equals the odds-ratio for an arbitrary item (denoted as item 1 in what follows):

$$t = V_1(t) = \frac{\pi_1(t)}{1 - \pi_1(t)} \quad (6)$$

we obtain that the IRF for item 1 can be written in the following form:

$$\pi_1(t) = \frac{t}{1 + t}, \quad (7)$$

whereas the IRFs for the other items can be obtained from (5),

$$\pi_i(t) = \frac{\omega_{i1} \frac{t}{1+t}}{1 - \frac{t}{1+t} + \omega_{i1} \frac{t}{1+t}} = \frac{\omega_{i1}t}{1 + \omega_{i1}t}. \quad (8)$$

We see that for any CLOR model the IRF can be expressed, as in (1), with item easiness equal to the ratio of odds-ratios ω_{i1} , and ability as defined by the odds-ratio $V_1(t)$. Observe that both the item easiness parameters as well as the ability parameters can be uniquely determined from $\mathbf{X}$. In other words, these parameters are identifiable. Observe, furthermore, that specific CLOR models may impose additional constraints beyond those characterizing the class of CLOR models, as illustrated below. In a CLOR model we not only have that the sum score is the sufficient statistic for the latent trait, but also that the number of persons giving a correct answer to item i is the sufficient statistic for the item easiness parameter ω_{ij} (j fixed).

For future reference we state our result as a theorem.

Theorem 1. *Whenever unidimensionality and local independence are assumed, the assumption of a constant ratio of odds-ratios is equivalent to the assumption that, for every item, the IRF can be expressed as in (1) with item easiness equal to ω_{ij} (j fixed).*

Observe that unless the range of the latent trait is explicitly constrained, as in the general parametric CLOR model considered in the next section, the IRF for every item satisfies the following limits: $\lim_{t \rightarrow \infty} \pi_i(t) = 1$ and $\lim_{t \rightarrow 0} \pi_i(t) = 0$.

Hessen (2005) also considers CLOR models where the IRF is *not* a monotone function of the latent trait. From Theorem 1 it follows that for *any* CLOR model the IRF can be written as in (1), and hence the same applies to every non-monotone CLOR model. An example may

clarify the strange phenomenon that the same distribution can at the same time have monotone and non-monotone IRFs. Let the IRFs for a set of items be defined as follows:

$$\pi_i(\theta) = \frac{b_i \exp(\theta^2)}{1 + b_i \exp(\theta^2)}, \quad (9)$$

where b_1 equals one and θ can assume both positive and negative values. A routine derivation shows that this model is indeed a CLOR model with non-monotone IRFs. However, the same distribution can be written in terms of a latent trait $t = \exp(\theta^2)$ as follows:

$$\pi_i(t) = \frac{b_i t}{1 + b_i t}, \quad (10)$$

which is recognized as (1), with the additional assumption that the latent trait t is larger than or equal to one. This means that the latent trait θ is not identifiable from $\mathbf{X}$, whereas the latent trait t is identifiable from $\mathbf{X}$. In other words, all that is identifiable from $\mathbf{X}$ about θ is its magnitude $\|\theta\|$, whereas its sign is not identifiable from $\mathbf{X}$. The model in (9) gives an example of a specific CLOR model that imposes an additional assumption beyond those characterizing the class of CLOR models.

As a special case of a non-monotone IRF we consider an IRF which remains constant over some part of the domain of the latent trait t^* . First, if the IRF of one item is constant over the range from t^* equal to a to t^* equal to b , the IRFs of all other items are constant over this same range. This follows readily from the constant latent odds-ratio property. First, if for $t_1^* \neq t_2^*$ it holds that the IRF for item 1 has the same value, then also the odds-ratio has the same value. Second, because the ratio of latent odds-ratios is constant we have that

$$\frac{V_1(t_1^*)}{V_j(t_2^*)} = \omega_{1j} = \frac{V_1(t_2^*)}{V_j(t_2^*)} \quad (11)$$

and hence

$$V_j(t_2^*) = \frac{V_1(t_1^*)}{\omega_{1j}} = \frac{V_1(t_2^*)}{\omega_{1j}} = V_j(t_2^*). \quad (12)$$

It is readily seen that also in this case the latent trait t^* cannot be identifiable from $\mathbf{X}$.

2. The General Parametric CLOR Model is Equivalent to the Rasch Model with a Constraint on the Range of the Latent Trait

Hessen (2005) introduces the general parametric CLOR model, which is a special case of the four-parameter logistic model of Barton and Lord (1981), and has the following IRF:

$$\Pr(X_{pi} = 1 | t_p^*, c_i^*, \gamma, \delta) = \frac{[1 - \gamma c_i^*][\delta + t_p^*]}{[1 - \delta\gamma][c_i^* + t_p^*]}, \quad (13)$$

where $0 \leq \delta < c_i^*$ and $0 \leq \gamma < 1/c_i^*$ for all i . The general parametric CLOR model defines non-intersecting item response functions for which the lower (upper) asymptotes may differ from zero (one) depending on the value of δ and γ .

A routine derivation shows that the IRF in (13) can also be expressed in the form of (1):

$$\Pr(X_{pi} = 1 | t_p^*, c_i^*, \gamma, \delta) = \frac{t_{\gamma, \delta}(t_p^*)}{c_{\gamma, \delta}(c_i^*) + t_{\gamma, \delta}(t_p^*)}, \quad (14)$$

where $t_{\gamma,\delta}$ and $c_{\gamma,\delta}$ are the following monotone functions:

$$t_{\gamma,\delta}(t_p^*) = \frac{\delta + t_p^*}{1 + \gamma t_p^*}, \quad (15)$$

and

$$c_{\gamma,\delta}(c_i^*) = \frac{c_i^* - \delta}{1 - \gamma c_i^*}. \quad (16)$$

Notice that in order to obtain a positive result in (15) and (16) the parameters δ and γ have to be constrained. Specifically, the constraints imposed by Hessen (2005) ($0 \leq \delta < c_i^*$ and $0 \leq \gamma < 1/c_i^*$) meet these requirements. Moreover, the transformed latent trait satisfies the following constraints:

$$\delta \leq t_{\gamma,\delta}(t^*) \leq \frac{1}{\gamma}, \quad (17)$$

which means that the general parametric CLOR model in (13) is equivalent to the Rasch model with a constraint on the range of the latent trait. By restricting the range of the latent trait, we obtain that at the lowest possible level of the latent trait (i.e., $t_{\gamma,\delta}(t^*) = \delta$) the probability to give a correct response can be larger than zero and, similarly, that at the highest possible level of the canonical latent trait (i.e., $t_{\gamma,\delta}(t^*) = 1/\gamma$) the probability to give a correct response can be smaller than one.

3. Discussion

We have shown for the general CLOR model that the IRF can be expressed as in (1). Whether this result can be interpreted as saying that the general CLOR model is equivalent to the Rasch model, or not, depends on how one defines the Rasch model. Specifically, it depends on whether the range of the latent trait is part of the definition or not. If the Rasch model is defined as (1) together with local independence, the result states that the general CLOR model is equivalent to the Rasch model. If the Rasch model is defined as (1) together with local independence and the assumption that t ranges over the non-negative real number, the result states that the general CLOR model is not equivalent to the Rasch model. We have also shown that, for non-monotone CLOR models, the latent trait parameter is not identifiable from $\mathbf{X}$. Different CLOR models correspond to different ways to parametrize the latent trait. If the latent trait t in (1) is transformed monotonically we obtain a monotone CLOR model, whereas if the latent trait is transformed in a non-monotone way we obtain a non-monotone CLOR model. Observe that in order to retain the sufficiency property of the sum score the same transformation is to be used for the latent trait with all items. Specific instances of a CLOR model, such as the general parametric CLOR model, may add additional assumptions to those of Theorem 1. In the case of the general parametric CLOR model the assumption is added that the range of the latent trait is restricted. In this way, it becomes possible that the IRFs have lower and upper asymptotes different from zero and one, respectively.

References

- Barton, M., & Lord, F. (1981). An upper asymptote for the three-parameter logistic item-response model (Technical Report No. 81-20). New York: Educational Testing Service.

- Hessen, D.J. (2005). Constant latent odds-ratios models and the Mantel–Haenszel null hypothesis. *Psychometrika* 70(3), 497–516.
- Rasch, G. (1960). *Probabilistic models for some intelligence and attainment tests*. Copenhagen: The Danish Institute of Educational Research. (Expanded edition, 1980. Chicago: The University of Chicago Press.)

Manuscript received 27 JUL 2006

Final version received 1 JUN 2007

Published Online Date: 25 SEP 2007