

On the role of symmetry in solving maximum lifetime problem in two-dimensional sensor networks

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Abstract We analyze continuous and discrete symmetries of the maximum lifetime problem in two dimensional sensor networks. We show how a symmetry of the network and invariance of the problem under a given transformation group G can be utilized to simplify its solution. We prove that for a G -invariant maximum lifetime problem there exists a G -invariant solution. Constraints which follow from the G -invariance allows us to reduce the problem and its solution to the subset of the sensor network. The subset we call an optimal fundamental region of network with respect to the action of the symmetry group G . We analyze in detail solutions of the maximum network lifetime problem invariant under a group of isometry transformations of a two dimensional Euclidean plane.

Keywords Wireless sensor networks · Energy efficiency · Symmetry group

1 Introduction

Let us denote by S_N^K a sensor network built of N sensors and K data collectors. We split the set S_N^K into two subsets, the set of data collectors C_K and the set of sensors S_N , such that $S_N^K = C_K \cup S_N$. We identify elements p_i of the network S_N^K with points $p_i = (p_i^1, p_i^2)$ of a two dimensional plane R^2 , where $p_i \in C_K$ for $i \in [1, K]$ and $p_i \in S_N$ for $i \in [K + 1, K + N]$. Each sensor $p_i \in S_N$ periodically generates the amount Q_i of data and sends it, possibly via other sensors,

to the data collectors. The data transmission cost energy matrix $E_{i,j}$ defines the energy required to send one unit of data between two elements p_i, p_j of the network S_N^K . The energy consumed by the p_i sensor to send all of its data in one cycle of the network lifetime is given by the formula

$$E_i(q, \bar{p}) = \sum_{j=1, j \neq i}^{K+N} q_{i,j}(\bar{p}) E_{i,j}(\bar{p}), \quad (1)$$

where $q_{i,j}(\bar{p})$ is the amount of data sent by the p_i sensor to the p_j element of the network S_N^K and $\bar{p} = (p_1, \dots, p_K, p_{K+1}, \dots, p_{K+N})$ defines location of the data collectors and sensors in R^2 . By definition the data collectors p_i are elements of the network S_N^K which do not send or retransmit any data, i.e.,

$$\forall p_i \in C_K \quad \forall p_j \in S_N^K \quad q_{i,j}(\bar{p}) = 0.$$

From the above assumption it follows that for the data collectors the energy given by (1) is equal to zero, $E_i(q, \bar{p}) = 0, i \in [1, K]$. Because the sensors have limited energy resources, to extend the network lifetime we need to find such weighted graph for the data transmitted in the network that the energy consumed by the most overloaded sensor would be minimal. Namely, if we assume that all sensors have the same initial energy E_0 and for a given data transmission graph the most overloaded sensor consumes in a one cycle E_i^{\max} of energy, then $[E_0/E_i^{\max}]$ is the number of cycles until this sensor runs out of energy. We define the network lifetime as a number of cycles the data can be transmitted in the network until the first sensor runs out of energy [1, 2]. In this paper we will not discuss a particular solution of the maximum network lifetime problem. We analyze the structure of the solution when the sensor network is invariant under discrete symmetry group. We assume, that the initial parameters which define the

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problem are the data transmission cost energy matrix $E_{i,j}(\bar{p})$ and the amount of data generated by each node $Q_i(\bar{p})$. We also assume, that the sensors can send their data to any element of the network S_N^K , the data collectors can receive any amount of data without costs, the initial energy of each sensor and the maximum amount of data which the sensor can send to other nodes of the network S_N^K is sufficiently large that at least one solution of the problem exists, i.e., there is no upper bound for $q_{i,j}(\bar{p})$. Under the above assumptions the maximum lifetime problem for a sensor network S_N^K can be written in the form

$$\begin{cases} \min_q \max_i \{E_i(q, \bar{p})\}_{i \in [K+1, K+N]}, \\ E_i(q, \bar{p}) = (qE^T(\bar{p}))_{i,i}, \\ (q - q^T)v_0 - Q(\bar{p}) = 0, \\ E_{i,j}(\bar{p}) \geq 0, Q_i(\bar{p}) \geq 0, \quad q_{i,j}(\bar{p}) \geq 0, \end{cases} \quad (2)$$

where the vector v_0 has the form $(\underbrace{1, \dots, 1}_{K+N}, \dots)$

$(-q_1^{(c)}, \dots, -q_K^{(c)}, Q_{K+1}(\bar{p}), \dots, Q_{K+N}(\bar{p}))$ and E^T is a transposition of the data transmission cost energy matrix E .

The undetermined numbers $q_k^{(c)}$ define the amount of data received by the data collectors and satisfy the following formulas $q_k^{(c)} = \sum_i q_{i,k}(\bar{p})$, $\sum_{k=1}^K q_k^{(c)} = \sum_{n=1}^N Q_{K+n}(\bar{p})$. The first formula in (2) means, that we minimize the objective function

$$f(q, \bar{p}) = \max_i \{E_i(q, \bar{p})\}_{i \in [K+1, K+N]} \quad (3)$$

of the maximum network lifetime problem with respect to the $q_{i,j}$ variables. The second formula defines the energy consumed by each sensor to send all of its data in a one cycle of the network lifetime. This equation is a matrix form of (1). The third formula in (2) is a data transmission flow conservation constraint, which states that the amount of data $Q_i(\bar{p})$ generated by the p_i sensor and the amount of data received from other sensors $\sum_j q_{j,i}$ must be equal to the amount $\sum_j q_{i,j}$ of data the p_i sensor can send. In the coordinates, for $i \in [K + 1, K + N]$ it can be written in the form $\sum_{j=K+1}^{K+N} q_{i,j} - \sum_{j=K+1}^{K+N} q_{j,i} = Q_i$ and for $k \in [1, K]$ it defines the numbers $q_k^{(c)} =: \sum_{j=K+1}^{K+N} q_{j,k}(\bar{p})$. The notation in (2) indicates, that in general the functions $E_{i,j}(\bar{p})$, $Q_i(\bar{p})$ and the solutions $q_{i,j}(\bar{p})$ may depend not only on the coordinates of the transmitter p_i and receiver p_j but also on the coordinates of other elements of the network S_N^K and thus the functions may have a non-local character.

In [3] there were identified five power-aware metrics for data transmission in mobile ad-hoc networks, which can be used to define a network lifetime. The definition (2) of network lifetime problem is equivalent to the minimization of the “maximum node cost”, the fifth metric defined in [3].

In this paper we discuss two types of symmetries of the maximum network lifetime problem and the impact of these symmetries on the solution of (2). We show how a symmetry of the problem (2) and a symmetry of the set S_N^K can be used to simplify the solution of (2) and reduce it to some subset of S_N^K . The first type of symmetry is related to an invariance of the problem (2) under a continuous group of transformation G^{Space} of the two dimensional plane R^2 onto itself in which the sensor network S_N^K is embedded. Under a transformation $g \in G^{\text{Space}}$ the elements p of the network S_N^K are moved to another location $g(p) \in S_N^K$ of the plane R^2 . We assume, that the numbers of sensors N and data collectors K under these transformation remain unchanged. If the Eq. (2) are invariant under transformations group G^{Space} , then there arises a question whether their solutions $q_{i,j}(\bar{p})$ for S_N^K and $q'_{i,j}(\bar{p})$ for S_N^K coincide. We show in Sect. 3, that it is indeed the case. We call this type of symmetry the space symmetry, because it exhibits the global properties of the functions $E_{i,j}(\bar{p})$, $Q_i(\bar{p})$, $q_{i,j}(\bar{p})$ and the whole problem in R^2 . For example, if the matrix elements $E_{i,j}(\bar{p})$ are functions of the Euclidean distance between points p_i and p_j of S_N^K network, for simplicity we assume that $Q_i(\bar{p})$ are constant functions, then the problem (2) is invariant under group of isometries of the Euclidean plane R^2 [4]. Invariance of $E_{i,j}(\bar{p})$ under isometry transformation means that the cost of data transmission between two elements of S_N^K does not depend neither on the direction of the data transmission nor location of the network S_N^K in R^2 . This property is called an isotropy property of the problem (2) in R^2 . The second type of symmetry is related to an invariance of (2) under transformations group of the finite set S_N^K . In this paper we consider a bijective transformations of the set S_N^K onto itself. By definition, such transformations g are permutations, i.e. $g(p_i) = p_{g(i)}$, and form a subgroup of symmetric group Π of the set S_N^K [5]. Because we do not want to mix the sensors and data collectors we assume that the group G acts separately on the sets C_K and S_N , which means that

$$G \subseteq \Pi(C_K) \oplus \Pi(S_N). \quad (4)$$

Thus, in this paper we consider transformation groups G of the network S_N^K which are subgroups of the symmetric group $\Pi(C_K) \oplus \Pi(S_N)$. In general, symmetries of the functions $E_{i,j}(\bar{p})$ and $Q_i(\bar{p})$ and the whole problem (2) do not have to be related with the shape of the set S_N^K . We can establish the relation by requiring that, for a given symmetry group G of the set S_N^K , we will consider only a G -invariant functions $E_{i,j}(\bar{p})$, $Q_i(\bar{p})$ and search for a G -invariant solution of (2). If this is the case, the problem (2) we call a problem with an internal symmetry group, because it is related to the shape of the set S_N^K .

2 Related work

Symmetries quite naturally appeared in a several well known optimization and combinatorial problems such as partitioning or coloring problems. Examples of solving linear programming problems with symmetries can be found in [6]. A review of results and techniques for solving a symmetric constraint programming problems can be found in [7]. Existence of symmetries in a given optimization problem facilitates searching for a solution of it. Usually such problem splits into $|G|$ identical parts, where $|G|$ is an order of a symmetry group, and it is enough to solve the problem only for an one part. Solving the given problem on a reduced set of parameters is called a symmetry breaking procedure, because the reduced problem loses its symmetry [6, 7]. In this paper we utilize the technique of a symmetry breaking to simplify solution of the G -invariant maximum lifetime problem in sensor networks. A symmetry breaking of G -invariant problem (2) is performed by reduction of its solution to the subset of S_N^K , the optimal fundamental region F_0^* for the action of the group G . Because a selection of a fundamental region F for given symmetry group G is not unique, and the problem cannot be reduced for every fundamental region, we show in this paper how to construct the optimal one and we investigate its properties. It is not always evident that for a given optimization problem with a symmetry group G there exists a G -invariant solution of it. The main result of this paper is a theorem, which states that for a considered G -invariant maximum lifetime problem there exists a symmetric (G -invariant) solution. In this paper we investigate in detail properties of the isometry invariant solutions of the problem in two dimensional sensor networks. It seems, that the presented paper is a first attempt of analyzing symmetries and utilize methods of solving the optimization problems in sensor networks by means of their symmetry groups.

3 A space symmetry of the maximum network lifetime problem

We consider a group of a one-to-one transformations of the two dimensional plane R^2 onto itself. Elements g of the group, denoted as G^{Space} , transform points p of R^2 to some other points $g(p)$ in R^2 . Because $S_N^K \subset R^2$, the nodes of the network located at the points p_i are also transformed under g to the points $g(p_i)$ of R^2 . The transformed network $g(S_N^K)$ by an element g of G^{Space} we denote by $S_N'^K$. We assume, that the functions $E_{i,j}(\bar{p})$ and $Q_i(\bar{p})$ in (2) are invariant under transformations of the group G^{Space} , which means that $\forall g \in G^{Space}$ we have

$$\begin{cases} E_{i,j}(g(\bar{p})) = E_{i,j}(\bar{p}), \\ Q_i(g(\bar{p})) = Q_i(\bar{p}), \end{cases} \tag{5}$$

where $g(\bar{p}) = (g(p_1), \dots, g(p_{K+N}))$, $p_i \in R^2$, $i \in [1, K + N]$ and \bar{p} is defined in (1).

Lemma 1 *Let G^{Space} be a transformation group of two dimensional plane R^2 , $E_{i,j}(\bar{p})$ and $Q_i(\bar{p})$ be G^{Space} -invariant functions in R^2 , then any solution $q(\bar{p})$ of the maximum lifetime problem (2) is G^{Space} -invariant.*

Proof To prove the invariance of $q_{i,j}(\bar{p})$ under transformation group G^{Space} we show that $\forall g \in G^{Space}$, the equations $q_{i,j}(g(\bar{p})) = q_{i,j}(\bar{p})$ are satisfied. Since the functions $Q_i(\bar{p})$ are G^{Space} -invariant, the feasible set given by the second equation in (2) is G^{Space} -invariant

$$(q' - q^T)v_0 = Q(g(\bar{p})) = Q(\bar{p}).$$

The invariance of the feasible set under G^{Space} means that the scopes of the parameters q and q' in (2) for both networks S_N^K and $S_N'^K$ are the same. From the condition (5) and the invariance of a feasible set it follows that the function $f(q, \bar{p})$, given by (3), is G^{Space} -invariant with respect to the \bar{p} variable

$$\forall g \in G^{Space} \quad f(q, \bar{p}) = f(q, g(\bar{p})).$$

From the above it follows that $f(q, \bar{p})$ is a constant function for any $g \in G^{Space}$ and any solution $q_{i,j}(\bar{p})$ of (2), as a minimal value of $f(q, \bar{p})$, satisfies

$$\forall g \in G^{Space} \quad q_{i,j}(g(\bar{p})) = q_{i,j}(\bar{p}),$$

which is G^{Space} -invariance condition for $q(\bar{p})$. \square

As an example of the application of the above lemma, let us consider the data transmission cost energy matrix $E_{i,j}(\bar{p})$ to be a function of the Euclidean distance

$$d(p_i, p_j) = \sqrt{(p_i^1 - p_j^1)^2 + (p_i^2 - p_j^2)^2}, \tag{6}$$

between two elements p_i, p_j of the network S_N^K , i.e.,

$$E_{i,j}(\bar{p}) =: E_{i,j}(d(p_i, p_j)). \tag{7}$$

The group which leaves invariant the metric (6) is a direct sum of two continuous abelian groups $G^{Space} = O_2 \oplus T$, the orthogonal group O_2 and the translation group T in R^2 [4]. If $Q(\bar{p})$ is invariant under $O_2 \oplus T$ and $E_{i,j}(\bar{p})$ is of the form (7), then we know from the Lemma 1 that the solution of (2) is also G^{Space} -invariant, which means that it is a function of the distance d and the G^{Space} -invariant functions $Q(\bar{p})$

$$q_{i,j}(\bar{p}) = q_{i,j}(d, Q(\bar{p})).$$

Further analysis of the Eq. (2) allows us to deduce that the solution of (2) must be a linear function in $Q_i(\bar{p})$. In the

above example we see, that simple analysis of symmetries of the functions $E(\bar{p})$ and $Q(\bar{p})$ and the structure of the Eq. (2) allows us to predict general form of its solution which greatly facilitates searching for it. This is especially important when the approximation algorithms are applied to solve the problem.

4 An internal symmetry of the sensor network

As an internal symmetry group G of a sensor network S_N^K we consider a one-to-one transformations of S_N^K onto itself. Such groups are subgroups of a symmetric group $\Pi(S_N^K)$. Because we cannot mix the sensors and the data collectors, the group G must be a direct sum of two subgroups $\Pi(S_N)$ and $\Pi(C_K)$, which transform sensors into sensors and data collectors into data collectors, see (4). For an action of $g \in G$ on elements p_i of the network S_N^K

$$g(p_i) = p_{g(i)},$$

we have the following transformation rules for E and Q

$$\begin{cases} g(E_{i,j}) = E_{g(i),g(j)}, \\ g(Q_i) = Q_{g(i)}. \end{cases}$$

In a matrix representation of G , the elements $g \in G$ act on S_N^K as a linear transformations $\forall_{i \in [1, K+N]} p_i = \sum_{j=1}^{K+N} g_{i,j} p_j$, which induces the following transformations on the matrix E and the vector Q

$$E' = gEg^{-1}, \quad Q' = gQ, \tag{8}$$

In the coordinates the formulas (8) can be written in the form $E'_{i,j} = \sum_{k,l} g_{i,k} E_{k,l} g_{l,j}^{-1}$ and $Q'_i = \sum_j g_{i,j} Q_j$. The next proposition shows that the problem (2) is covariant under action of the group $\Pi(C_K) \oplus \Pi(S_N)$. It means, that for any transformation (8) of $E_{i,j}$ and Q_i , by an element $g \in \Pi(C_K) \oplus \Pi(S_N)$, there exist two solutions q and q' of (2) which are related by the transformation g .

Proposition 1 *Let V and V' be the set of all solutions of the maximum lifetime problem (2) for (S_N^K, E, Q) and (S_N^K, E', Q') sensor networks, where $S_N^K = g(S_N^K)$ and $g \in G = \Pi(C_K) \oplus \Pi(S_N)$, then the diagram*

$$\begin{array}{ccc} (E, Q) & \longrightarrow & V \\ \downarrow g & & \uparrow g^{-1} \\ (E', Q') & \longrightarrow & V' \end{array} \tag{9}$$

is commutative.

In (9) the horizontal arrows mean assignment for given E and Q a solution to the problem (2).

Proof Since any permutation can be written as a product of transpositions, it is enough to prove (9) for the transpositions

$$\forall_{i \neq r, r'} g(p_i) = p_i, \quad g(p_r) = p_{r'}, \quad p_r, p_{r'} \in S_N \quad \text{or} \quad p_r, p_{r'} \in C_K. \tag{10}$$

The feasible set, defined by the second equation in (2), for $i \in [K + 1, K + N]$ we write in the form

$$h_i(q, \bar{p}) = 0, \quad i \in [K + 1, K + N],$$

where $h_i(q, \bar{p}) = \sum_{j=1}^{K+N} (q_{i,j} - q_{j,i}) - Q_i$. The transpositions (10) swap the functions h_i for $i \in [K + 1, K + N]$, i.e.,

$$g(h_r) = h_{r'}, \quad g(h_i) = h_i, \quad i \neq r, r',$$

which means that the feasible set is G -invariant. For $i \in [1, K]$, the feasible set $h_i(q, \bar{p}) = 0$ is trivially G -invariant because of the identity $Q_i = q_i^{(c)}$. The transpositions (10) exchange the energies consumed by the r -th and r' -th sensors $g(E_r) = E_{r'}$ and other sensor energy levels remain unchanged $g(E_i) = E_i$, for $i \neq r, r'$, where E_i is given by (1). From this it follows that the objective function of the maximum lifetime problem given by (3) is invariant under (10). The invariance of $f(q, E, Q)$ under $\Pi(C_K) \oplus \Pi(S_N)$ means that $\forall g \in G$ and for $E' = g(E), Q' = g(Q), q' = g(q)$ the functions are equal, $f(q, E, Q) = f(q', E', Q')$, and have the same minimal value with respect to the q variable

$$\min_q f(q, E, Q) = \min_{q'} f(q', E', Q').$$

From the above equation it follows that, if $q_{i',j'} \in V'$ is a solution of (2) for E', Q' then the matrix

$$q_{i,j} = g^{-1}(q_{i',j'}) \in V$$

is a solution of (2) for E, Q .

In the next section, based on the result of the Proposition 1 we show that for a G -invariant functions $E_{i,j}$ and Q_i , where G is a subgroup of $\Pi(C_K) \oplus \Pi(S_N)$, there exists a G -invariant solution of (2).

4.1 The G invariant solution of the maximum network lifetime problem

The G -invariance of $E_{i,j}$ and Q_i means that for any transformation (8) we have

$$\begin{cases} E_{g(i),g(j)} = E_{i,j}, \\ Q_{g(i)} = Q_i. \end{cases}$$

In the following theorem we construct a G -invariant solutions of (2).

Theorem 1 *Let G be a symmetry group of the set S_N^K , which transforms sensors into sensors and data collectors*

into data collectors, E and Q be a G -invariant functions in (2), then there exists a G -invariant solution q of (2), i.e.,

$$\forall g \in G \quad gqg^{-1} = q. \tag{11}$$

Proof We assume that for a given G -invariant matrix E and a vector Q there exists, not necessary a G -invariant, solution of (2). We denote it by q_0 . From the Proposition 1 we know that

$$\forall g_m \in G \quad q_m = g_m q_0 g_m^{-1}$$

are solutions of (2) for the same E and Q . Since, for any solution g_m of (2) the minimal value of the objective function (3) must be the same, i.e. $\forall m, m' f(q_m) = f(q_{m'})$, then any linear combination of g_m , i.e., $q(\lambda_0, \dots, \lambda_{M-1}) = \frac{1}{\lambda_0 + \dots + \lambda_{M-1}} \sum_{m=0}^{M-1} \lambda_m q_m$, where $\lambda_m \geq 0, \sum_m \lambda_m > 0, M = |G|$, is also a solution of (2).

In particular, the matrix

$$q = \frac{1}{|G|} \sum_{g_m \in G} g_m q_0 g_m^{-1} \tag{12}$$

is the solution of (2) and it is G -invariant, which means that

$$\begin{aligned} \forall g_n \in G \quad g_n q g_n^{-1} &= \frac{1}{|G|} \sum_{g_m \in G} (g_n g_m) q_0 (g_n g_m)^{-1} \\ &= \frac{1}{|G|} \sum_{g_{m'} \in G} g_{m'} q_0 g_{m'}^{-1} = q, \end{aligned}$$

where we used the fact that the left action of G on itself is transitive. \square

5 Reduction of the maximum network lifetime problem to the subset of S_N^K

By definition a G -invariant solution of the maximum network lifetime problem (2) satisfies the constraint (11). This constraint reduces the number of variables $q_{i,j}$ and by this simplifies the solution of (2). Additionally, we would like to relate the shape of the network S_N^K and its symmetry with the invariance of solutions of (2). This would allow us to determine the solution of (2) based on the shape of the network S_N^K . One can easily relate the shape of the network S_N^K with a symmetry of the data transmission cost energy matrix $E_{i,j}(\bar{p})$ by requiring that the matrix elements $E_{i,j}(\bar{p})$ are functions of the distance $d(p_i, p_j)$ between points of S_N^K . Transformations which preserve the distance (6) form a group of isometries of an Euclidean plane R^2 . Every isometry of a real Euclidean space is a composition of a translation and an orthogonal transformation [4]. For a finite set S_N^K in R^2 every isometry is an orthogonal transformation $g \in O_2$, since there is no translation which transforms a finite set onto itself. In this section we

consider as a symmetry group G of the sensor network S_N^K subgroups of the orthogonal group O_2 and assume that the functions $E_{i,j}(\bar{p})$ and $Q_i(\bar{p})$ are O_2 -invariant.

The data transmission cost energy matrices $E(\bar{p})$, which are functions of the distance between elements of the sensor network are widely used in data transmission models in sensor networks. One may ask, what is the general form of the O_2 -invariant data transmission cost energy matrix E which is a distance between transmitter and receiver. The matrix $E = \sum_n \lambda_n d^{a_n}$, which is a linear combination of powers of the matrix d , see (6), is trivially O_2 -invariant. If $\exists_n : a_n \neq 1$, then the costs of data transmission between two nodes depend on distances between other nodes. Such data transmission cost energy matrices have non-local character and have no applications. Because the matrix $E_{i,j} = d(p_i, p_j)^a$, for any real number a is also trivially O_2 -invariant, we write the more general form of the O_2 -invariant data transmission cost energy matrix $E_{i,j}$ which is a function of the distance between transmitter and receiver

$$E_{i,j}(\bar{\lambda}, \bar{a}) = \sum_n \lambda_n d(p_i, p_j)^{a_n}, \tag{13}$$

where $\bar{\lambda} = (\lambda_1, \dots), \bar{a} = (a_1, \dots)$ and λ_n, a_n are non-negative real numbers. Most of the data transmission models in sensor networks utilize the data transmission cost energy functions of the form (13). As an example, let us consider the cost function $E_{i,j} = d(p_i, p_j)^2 + \lambda_1 d(p_i, p_j)^4$. The first component describes the cost of data transmission in the vacuum and the second component appears when the network is in an industrialized environment [8]. By inserting in (13), the distance function in one dimension $d_{i,j} = |i - j|, \lambda_n \sim \gamma^n \frac{1}{n!}, a_n = \alpha + n, \alpha \geq 2, \gamma \geq 0$, we obtain the data transmission cost matrix $E_{i,j} \sim |i - j|^\alpha e^{\gamma|i-j|}$ utilized in [2].

Because a finite subgroup of the orthogonal group O_2 is either a dihedral group D_M or a rotation group $R^{(M)}$ [4, 5], we consider these groups as symmetries of the sensor network S_N^K and the functions $E_{i,j}$ and Q_i . We show, that in both cases solution of the problem (2) can be reduced to subset of S_N^K , which we call an optimal fundamental region F_0^* in S_N^K . For the dihedral symmetry group D_M the optimal fundamental region can be easily determined. In case of the rotation group $R^{(M)}$ the optimal fundamental region must be determined for a particular distribution of elements of S_N^K over the plane R^2 . Existence of a reduction of the maximum network lifetime problem (2) to the optimal fundamental region will be proven under two assumptions. We assume, that two sensors cannot exchange the data

$$q_{i,j} \neq 0 \Rightarrow q_{j,i} = 0, \quad p_i, p_j \in S_N, \tag{14}$$

which means that if the sensor p_i sends some data to the sensor p_j , then the sensor p_j cannot send any data to p_i .

Second assumption is that we consider only a G -invariant $E_{i,j}$ matrices having the property

$$d(p_i, p_j) \leq d(p_{i'}, p_{j'}) \Rightarrow E_{i,j} \leq E_{i',j'}, \tag{15}$$

which means that for such matrices the cost of data transmission grows whenever the distance between elements of the network S_N^K grows.

5.1 The sensor network with a dihedral symmetry group

The dihedral group D_M is a symmetry group of a regular polygon with M sides. It is a semidirect product $R^{(M)} \rtimes S^{(M)}$ of the rotation group $R^{(M)}$ and the reflection group $S^{(M)}$. It consists of $2M$ elements, M reflections $\{S_m\}_{m=0}^{M-1}$ and M rotations $\{R_m\}_{m=0}^{M-1}$. Because any rotation can be represented as a product of even reflections then the dihedral group can be generated by M reflections, $D_M = \langle S_m \rangle_{m=0}^{M-1}$. For a given finite set S_N^K with the symmetry group D_M , a subset F of S_N^K is called a fundamental region for D_M , if $S_N^K = \bigcup_{g \in D_M} g(F)$ [6]. This requirement means that the set S_N^K is a union of $|D_M| = 2M$ subsets $F_m = g_m(F)$, i.e., $S_N^K = \bigcup_{m \in [0, 2M-1]} F_m$. To block the transmission between sensors from different sets $g(F)$, we require that on the reflection lines X_m , $m \in [0, M - 1]$ there are no sensors placed. This requirement can be written in the following form

$$\forall_{p \in S_N} \text{St}(p, D_M) = \{g \in D_M : g(p) = p\} = \{I\},$$

which means that the stabilizer of any point p of the set S_N is trivial.

A fundamental region F of the S_N^K set can be selected in many ways. Among fundamental regions in S_N^K there is only one optimal region for which the D_M -invariant problem (2) can be reduced. We show that to optimize the data transmission and to extend the network lifetime the sensors from the optimal fundamental region do not send data outside the region. This means that the solution to the maximum lifetime problem can be factorized into $2M$ identical functions, each function represents the solutions to the problem in $g_m(F) \subset S_N^K$, $m \in [0, 2M - 1]$. We denote by S_0 and S_1 the elements of the dihedral group D_M which are a reflection along the X -axis and the X_1 line respectively. By V_0 We denote the region between X -axis, $X_0 \geq 0$, and the reflection line $X_1 \geq 0$. The two half-lines $X_0 \geq 0$ and $X_1 \geq 0$ belong to V_0 . There exists only one fundamental region F_0 which is a subset of V_0 . From the set F_0 , by the following sequence of transformations $F_m = S_m(F_{m-1})$ for $m \in [1, M - 1]$ and $F_{M+m} = S_m^{-1}(F_{M+m-1})$ for $m \in [0, M - 1]$, where S_m are reflections of D_M , we can generate $|G| = 2M$ sets F_m and represent the network S_N^K as

a sum of them, $S_N^K = \bigcup_{m=0}^{2M-1} F_m$. We show that the fundamental region F_0 is an optimal one, in the sense that there exists a D_M -invariant solution $q_{i,j}$ of (2) for which $q_{i,j} \neq 0 \Rightarrow p_i, p_j \in F_m$ and the matrix $q_{i,j}$ can be factorized into $2M$ identical sub-matrices, a one sub-matrix for each region F_m . In other words, we show that the regions F_m are closed for data transmission and inside each of them the data transmission paths are identical, so it is enough to find the solution of (2) in one of them, for example in F_0 . Note that, the reduction of the D_M -invariant problem (2) to the minimal fundamental domain F_0 of S_N^K is also possible when the data collectors are located on the reflection lines X_m . Let us denote by ∂V_0 the border of the sector V_0 . The set ∂V_0 is a sum of two half-lines $X_0 \geq 0$ and $X_1 \geq 0$. By C_0 and C_1 we denote the set of data collectors which are located on the half-line $X_0 \geq 0$ and $X_1 \geq 0$ respectively. The fundamental domain F_0 we write as the sum of three subsets $F_0 = F'_0 \cup C_1 \cup C_0$, where F'_0 is a set of elements of the sensor network S_N^K which lie inside V_0 . An example of optimal fundamental region $F_0 = F'_0 \cup C_0 \cup C_1$ for D_4 group is shown in the Fig. 1. In the set F'_0 there are seven sensors and one data collector, three data collectors are located at the border of V_0 and lie in the sets C_0 and C_1 .

The following proposition shows that the D_M -invariant problem (2) over D_M -invariant network S_N^K , with the data transmission cost energy matrix $E_{i,j}$ having the property (15), can be reduced to the region F_0 .

Proposition 2 *Let D_M be a symmetry group of the set S_N^K with trivial stabilizer for each sensor $p_i \in S_N$, $F_0 \subset V_0$ a fundamental region in S_N^K for the group D_M with the sets C_0, C_1 of data collectors which lie on the half-line $X_0 \geq 0$ and $X_1 \geq 0$ respectively, then the solution of a D_M -invariant problem (2) with E satisfying (15) can be reduced to F_0 .*

Proof From the Theorem 1 we know that for a D_M -invariant problem (2) there exists a D_M -invariant solution. We show, that the D_M -invariant solution $q_{i,j}$ of (2) can be factorized to $2M$ copies, and each copy is identical to the solution of (2) in the region F_0 . For a given reflection $S_m \in D_M$ we write the set S_N^K as sum of three subsets

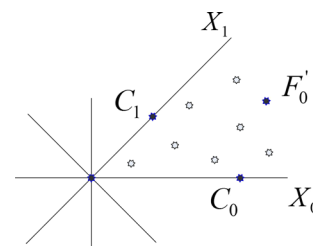


Fig. 1 Minimal fundamental region $F_0 = F'_0 \cup C_0 \cup C_1$ for the dihedral group D_4

$S_N^K = (S_N^K)^{(1)} \cup C^m \cup (S_N^K)^{(2)}$, where C^m is a set of data collectors which lie on the reflection line X_m and $(S_N^K)^{(a)}$, $a = 1, 2$ are sets of sensor network S_N^K elements which lie on both sides of X_m . On the reflection line X_m there is a set of data collectors isometric to the set $C_0 \cup C_1$ for M odd, $C_0 \cup C_0$ for M even and m even, and $C_1 \cup C_1$ for M even and m odd. The elements of $(S_N^K)^{(a)}$ we denote by $p_{a,i}$, $a = 1, 2$, where $S_m(p_{1,i}) = p_{2,i}$. The invariance of the solution q of (2) under transformation $S_m q (S_m)^{-1} = q$ can be written in the form

$$\begin{cases} q_{1,i,2,j} = q_{2,i,1,j}, \\ q_{1,i,1,j} = q_{2,i,2,j}, \\ q_{1,i,2,i} = q_{2,i,1,i} = 0, \\ q_{1,j,2,j} = q_{2,j,1,j} = 0, \end{cases} \quad (16)$$

for all $p_{a,i} \in (S_N^K)^{(a)}$, $a = 1, 2$. The second equation in (16) follows from the requirement (14). From the geometric properties of the reflection symmetry we have $d(p_{1,i}, p_{1,j}) \leq d(p_{1,i}, p_{2,j})$, and because of the assumption (15), we get the set of inequalities $E_{1,i,1,j} \leq E_{1,i,2,j}$ and $E_{2,i,2,j} \leq E_{2,i,1,j}$. From these inequalities it follows that for a D_M -invariant solution q of (2) for which $q_{1,i,2,j} = q_{2,i,1,j} \neq 0$, we can find a solution q' for which $q'_{1,i,2,j} = q'_{1,j,2,i} = 0$ and $q'_{1,i,1,j} = q'_{2,i,2,j} = q_{1,i,2,j}$. This means that we can construct a D_M -invariant solution q' of (2) for which there is no data transmission across the reflection line X_m and inside the sets $(S_N^K)^{(1)}$, $(S_N^K)^{(2)}$ the data transmission is given by the same solution. From the definition of the maximum network lifetime problem (2) and the property (15) of the data transmission cost energy matrix E it follows, that to optimize the data transmission the sensors always send their data to the nearest data collector

$$\forall p_n \in S_N, p_k \in C_K, \quad q_{n,k} \neq 0 \Rightarrow \forall p_{k'} \in C_K \\ d(p_n, p_k) \leq d(p_n, p_{k'}).$$

This means that the data collectors cannot receive any data from sensors which lie behind the reflection line X_m . These properties are valid for any reflection S_m , $m \in [0, M - 1]$ of D_M and from this it follows that the data is not sent across any reflection line X_m , $m \in [0, M - 1]$. Because of the symmetry, inside each of the $2M$ regions the solutions of (2) are identical and can be represented by a solution in F_0 . \square

In Fig. 2, the dashed arrows indicate the optimal data transmission path between sensors which lie on both sides of the reflection line X_m .

In Proposition 2 the assumption that the set of all data collectors which lie on the reflections lines is D_M -invariant can be omitted. If we add to the set $C^{(X)} = \bigcup_{m=0}^{M-1} (S_m(C_0) \cup S_m(C_1))$ in S_N^K an arbitrary set $\tilde{C}^{(X)}$ of data collectors, then

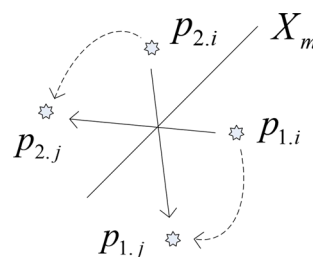


Fig. 2 The dashed arrows optimal data transmission path between two sensors

the solution of the problem (2) in $S_N^K \cup \tilde{C}^{(X)}$ can be factorized also into $2M$ parts, but the solutions in each part are different due to the difference of the data collector sets on various reflection lines.

5.2 The sensor network with a rotation symmetry group

A rotation group in R^2 is a cyclic group generated by M elements $R^{(M)} = \langle R_m \rangle_{m=0}^{M-1}$, where R_m denotes a rotation by the angle $\alpha_m = \frac{2\pi}{M} m$. We assume that the rotations are around the point $p_0 = (0, 0) \in R^2$. The point p_0 is unique for which the stabilizer is non-trivial and it is equal to the whole group $R^{(M)}$, $\text{St}(p_0, R^{(M)}) = R^{(M)}$. From this reason, we assume that at the point p_0 there is no element of the network S_N^K . If it is necessary to consider a sensor network with an element located at p_0 , then we will build a $R^{(M)}$ -invariant sensor network $S_N^K \cup C^{(0)}$, where $C^{(0)}$ is a set which consists of a one element located at p_0 , a data collector. Let us denote by V_0 the area between X-axis, $X \geq 0$, and the half-line $p^2 = \tan[\alpha_1] p^1$, $p^1 \geq 0$, where $\alpha_1 = \frac{2\pi}{M}$ and $(p^1, p^2) \in R^2$. For the set S_N^K there exists only one fundamental region F_0 in S_N^K which is a subset of V_0 . By rotation of F_0 by elements of $R^{(M)}$ $R_m(F_0) = F_m$,

we can obtain M regions, $F_m \subset V_m$, $m \in [0, M - 1]$, where $V_m = R_m(V_0)$, such that $S_N^K = \bigcup_{m=0}^{M-1} F_m$ and $\bigcap_{m=0}^{M-1} F_m = \{\emptyset\}$. We describe the properties of a $R^{(M)}$ -invariant solution of (2) in terms of orbits of the symmetry group $R^{(M)}$. The orbit of the point $p \in S_N^K$ under action of the group $R^{(M)}$ is a subset of S_N^K

$$\text{Orb}(p, R^{(M)}) = \{p \in S_N^K : p = g(p), g \in R^{(M)}\}.$$

Since we assumed that $p_0 \notin S_N^K$, then any fundamental region F for $R^{(M)}$ can be defined as a set of orbits $F = S_N^K / \sim_{R^{(M)}}$, where for $p_1 \neq p_2 \in S_N^K$, $p_1 \sim_{R^{(M)}} p_2 \Leftrightarrow \exists g \in R^{(M)} : g(p_1) = p_2$. The points of the set F_m we denote by $p_{m,i}$, where the number $m \in [0, M - 1]$ indexes the elements of the i -th orbit. We will count the points $p_{m,i}$ on the

i -th orbit anticlockwise starting from the X -axis, $X \geq 0$. The numbers $m \in [0, M - 1]$ and i uniquely identify the points of the set S_N^K . For the set $S_N^K \cup C^{(0)}$, $i \in [0, \frac{N+K}{M}]$, and for S_N^K , $i \in [1, \frac{N+K}{M}]$. The effect of rotation of the point $p_{n,i}$ by the angle $\alpha_m = \frac{2\pi}{M}m$, $m \in [0, M - 1]$ can be written by the formula

$$R_m(p_{n,i}) = p_{(n+m),i},$$

where $n + m$ denotes $(n + m) \bmod M$. For $p_{0,0} \in C^{(0)}$, $R_m(p_{0,0}) = p_{0,0}$. The invariance of the solution of (2) under R_k rotation can be rewritten in the form

$$R_k q R_k^{-1} = q \Rightarrow q_{(m+k),i,(n+k),j} = q_{m,i,n,j}. \tag{17}$$

The following proposition states that for $R^{(M)}$ -invariant maximum lifetime problem (2) for the sensor network S_N^K there exists a $R^{(M)}$ -invariant solution for which there is no data transmission between sensors which lie on the same orbit.

Proposition 3 *Let q be a solution of $R^{(M)}$ -invariant problem (2) for the S_N^K network, then the sensors from the same orbit do not transmit data to each other, i.e.,*

$$\forall_{i,m,n} p_{m,i}, p_{n,i} \in S_N \quad q_{m,i,n,i} = 0.$$

Proof From the Theorem 1 we know that for $R^{(M)}$ -invariant maximum lifetime problem (2) there exists a $R^{(M)}$ -invariant solution q . For such solution, if the sensor $p_{m,i}$ sends $q_{m,i,n,i}$ of data to the $p_{n,i}$ sensor then, from (17) we know that for $k = m - n$, the $p_{(2m-n),i}$ sensor sends the same amount of data $q_{(2m-n),i,m,i} = q_{m,i,n,i}$ to the $p_{m,i}$ sensor. Because any amount of data which is sent by the $p_{m,i}$ sensor to the sensor on the same orbit “returns” to it, then we can find a $R^{(M)}$ -invariant solution q' of (2) for which $\forall_{i,m,n} q'_{m,i,n,i} = 0$. \square

In the next proposition we prove that if the requirement (15) for the data transmission cost energy matrix $E_{i,j}$ is satisfied, then there exists a $R^{(M)}$ -invariant solution q of (2) for which a sensor from one orbit sends its data to the nearest sensor from another orbit.

Proposition 4 *Let q be a solution of $R^{(M)}$ -invariant problem (2) for $S_N^K \cup C^{(0)}$ sensor network with $E_{i,j}$ satisfying (15), then for any sensor $p_{m,i}$ and any element $p_{n,j}$ of $S_N^K \cup C^{(0)}$ from different orbits, $i \neq j$, the only non zero element of the matrix $q_{m,i,n,j}$ has the property*

$$\forall_{i \neq j, m, n} q_{m,i,n,j} \neq 0 \Rightarrow \forall_{p_{n',j} \in S_N} d(p_{m,i}, p_{n,j}) \leq d(p_{m,i}, p_{n',j}).$$

Proof Let us assume that for $R^{(M)}$ -invariant solution q of (2) the $p_{m,i}$ sensor sends to the sensor or to the data collector $p_{n,j}$ the amount $q_{m,i,n,j}$ of data, $i \neq j$. From (17) we know that the same amount of data is sent from the $p_{(m+k),i}$ sensor to the

$p_{(n+k),j}$ sensor or to the data collector, $k \in [0, M - 1]$. As a result, each element of a sensor network from the j -th orbit receives the same amount of data $q_{m,i,n,j}$ from a one sensor from the i -th orbit. Due to the assumption that the data transmission cost energy matrix $E_{i,j}$ satisfies (15), the minimum energy of sending the amount of data $q_{m,i,n,j}$ by the $p_{m,i}$ sensor from the i -th orbit to the element $p_{n,j} \in S_N^K \cup C^{(0)}$ from the j -th orbit is achieved when the distance between $p_{m,i}$ and $p_{n,j}$ is minimal, i.e., it has the property $\forall_{p_{n',j} \in S_N^K \cup C^{(0)}} d(p_{m,i}, p_{n,j}) \leq d(p_{m,i}, p_{n',j})$. \square

In Fig. 3 the dashed arrows indicate the optimal data transmission path between sensors $p_{m,i}$ from the i -th orbit and sensors or data collectors $p_{m,j}$ from the j -th orbit, $m \in [0, M - 1]$.

Because the rotation group $R^{(M)}$ is abelian the constrains $R_m q R_m^{-1} = q$, $m \in [0, M - 1]$ which satisfy the $R^{(M)}$ -invariant solutions of (2) are not very restrictive. In general, from the rotation invariance we cannot determine the optimal fundamental region F_0^* in S_N^K , i.e., a region to which the problem (2) can be reduced. The next three propositions describe the size and location of the optimal fundamental region F_0^* in S_N^K . Let us denote by V_0^\pm two sub-regions of V_0 such that

$$V_0 = V_0^- \cup V_0^+,$$

where the points of V_0^- lie on or between the X -axis, $X \geq 0$ and the half-line $p^2 = \tan[\frac{\pi}{2}]p^1$, $p^1 \geq 0$. The region V_0^+ lie above of V_0^- . The Fig. 4 shows the location of the regions V_0^\mp in V_0 .

The regions V_m^\pm can be obtained by rotating V_0^\pm by elements of the group $R^{(M)}$, $V_m^\pm = R_m(V_0^\pm)$. The next proposition shows, that the sensors which lie in V_m can send their data to the data collectors or other sensors only when they lie in V_m or in the neighboring regions $V_{m \mp 1}^\pm$.

Proposition 5 *Let q be a solution of $R^{(M)}$ -invariant problem (2) for $S_N^K \cup C^{(0)}$ network with $E_{i,j}$ satisfying (15), then the sensors $p_{m,i}$ from the region V_m can send their data to the elements $p_{m',i}$ of the sensor network $S_N^K \cup C^{(0)}$ only when they lie in $V_m \cup V_{m \mp 1}^\pm$, i.e.,*

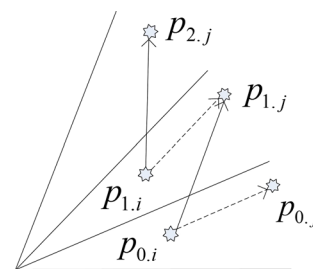


Fig. 3 The dashed arrows optimal data transmission path between elements of S_N^K which lie on different orbits

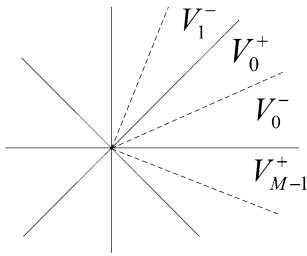


Fig. 4 Location of the regions V_0^\mp in V_0

$$\forall_{i \neq j} p_{m,i} \in V_m, q_{m,i,m',j} \neq 0, \Rightarrow p_{m',j} \in V_m \cup V_{m+1}^+ \\ \text{or } p_{m',j} \in V_m \cup V_{m+1}^-.$$

Proof For a sensor $p_{m,i} \in V_m^+$ from the i -th orbit which sends $q_{m,i,n,j}$ of data to the sensor $p_{n,j} \in V_n$ from the j -th orbit, $i \neq j$, we can find a sensor $p_{m',j}$ from the j -th orbit which $p_{m',j} \in V_m \cup V_{m+1}^-$, and $d(p_{m,i}, p_{m',j}) = \min_{n'} d(p_{m,i}, p_{n',j})$. Because of (15) we can find $R^{(M)}$ -invariant solutions q of (2), for which the sensors from the set V_m^+ send their data to the sensors or data collectors from $V_m \cup V_{m+1}^-$. Similarly, sensors $p_{m,i}$ from the set V_m^- can send their data to the elements of the sensor network from the subset $V_m \cup V_{m+1}^+ \subset S_N^K \cup C^{(0)}$ and the proposition is proven.

From the Proposition 4 we know that there exists a $R^{(M)}$ -invariant solutions q of (2) for which a sensor from a given orbit sends its data only to a one sensor or a data collector from another orbit. The following proposition describes conditions under which a fundamental region F_0 for $R^{(M)}$ is the optimal one.

Proposition 6 *Let F_0 be a fundamental region in S_N^K and the set $F_0 \cup C^{(0)}$ fulfills the requirement*

$$\forall p_{0,i} \in F_0 \cup C^{(0)}, \forall_j d(p_{0,i}, p_{0,j}) = \min_m d(p_{0,i}, p_{m,j}),$$

$m \in [0, M - 1]$, then the solution of the $R^{(M)}$ -invariant problem (2) for $S_N^K \cup C^{(0)}$ with $E_{i,j}$ satisfying (15) can be restricted to the set $F_0 \cup C^{(0)}$.

Proof From the Proposition 4 we know that there exists $R^{(M)}$ -invariant solutions q of (2) for which sensors send their data to the nearest sensor or data collector from other orbits. We select the sensor $p_{0,1} \in S_N^K \cup C^{(0)}$ from the first orbit of $R^{(M)}$ and build a set $F_0 \cup C^{(0)}$ by picking up a one element $p_{0,j}$ from each orbit, such that

$$\forall_i d(p_{0,1}, p_{0,i}) = \min_m d(p_{0,1}, p_{m,i}).$$

If all elements $p_{0,i}$ of a constructed set $F_0 \cup C^{(0)}$ have the property, that from the inequality $\forall_j d(p_{0,i}, p_{m',j}) =$

$\min_m d(p_{0,i}, p_{m,j})$ it follows that $p_{m',j} \in F_0 \cup C^{(0)}$, then from the Proposition 4 we know that the set is closed for data transmission. This means that for $p_{0,i} \in F_0, p_{m,j} \notin F_0 \Rightarrow q_{0,i,m,j} = 0$. Because F_0 is a fundamental region, then $S_N^K = \cup_{m=0}^{M-1} R_m(F_0)$, and the solution of (2) splits into M copies, one for each region $R_m(F_0) \cup C^{(0)}$. \square

The next proposition describes the location of the optimal fundamental region F_0^* for $R^{(M)}$ -invariant solutions of (2) in the set S_N^K .

Proposition 7 *Let q be a solution of $R^{(M)}$ -invariant problem (2) for S_N^K network with $E_{i,j}$ satisfying (15), then the optimal fundamental region F_0^* is a subset of $V_1^+ \cup V_0 \cup V_{M-1}^-$.*

Proof Follows from the Proposition 5. \square

It is easy to see that the Proposition 5 is also valid if we consider a sensor network with a data collector located at the point p_0 , i.e., for the $S_N^K \cup C^{(0)}$ network.

6 Conclusions

We have analyzed a continuous and discrete symmetries of the maximum lifetime problem in two dimensional sensor networks S_N^K built of K data collectors and N sensors. We showed that, invariance of the problem under a continuous group of transformations G implies that the solution is also G -invariant and can be expressed in terms of the symmetry group invariants. As we showed, this fact greatly facilitates searching for a strict or approximate solution of the problem.

In this paper, we also investigated properties of the solutions of the maximum lifetime problem for sensor networks S_N^K invariant under transformation groups G which are subgroups of the symmetric group $\Pi(C_K) \oplus \Pi(S_N)$, where C_K and S_N are subsets of S_N^K which consist of the data collectors and sensors respectively. We showed that for such groups a G -invariant maximum lifetime problem has a G -invariant solution. In this paper we analyzed in detail invariance of the sensor network and solutions of the problem under group of isometry transformations O_2 in R^2 . Constrains which follow from the O_2 -invariance of a solution allowed us to reduce it to a subset, an optimal fundamental region of the network. The fact that, the G -invariant maximum lifetime problem and its solution can be factorized and reduced to the fundamental region of the symmetry group G can be utilized to design sensor networks with symmetries and with known solution in the optimal fundamental region of the network.

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in mobile wireless networks

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