Stability Analysis of Uniform Equilibrium Foam States for EOR Processes

E. Ashoori · D. Marchesin · W. R. Rossen

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Abstract The use of foam for mobility control is a promising mean to improve sweep efficiency in EOR. Experimental studies discovered that foam exhibits three different states (weak foam, intermediate foam, and strong foam). The intermediate-foam state is found to be unstable in the lab whereas the weak- and strong-foam states are stable. The model of Kam (Colloids Surf A Physicochem Eng Asp 318(1–3): 62–77, 2008) is the only mechanistic foam model that can fit a variety of steady-state experimental data including multiple steady states. This model is modified from a previous mechanistic foam model to resolve the intrinsic instability of the strong-foam state. Simple finite-difference simulations have found that an arbitrary perturbation grows for the unstable intermediate foam but diminishes for the strong- and weak-foam states. The issue of the stability of foam states, especially the strong-foam state, is a serious concern in application of foam in EOR. Instabilities may rule out one or more states and consequently have considerable effect on reservoir sweep efficiency and injection pressure. Here, for the first time the stability of the various equilibrium foam states is investigated by an analytical stability-analysis method together with numerical simulations. We demonstrate the instability of most intermediate states, consistent with the laboratory observations. However, our analysis reveals an instability of the strong-foam state. We show that the diffusion, whether introduced artificially by the finite-difference scheme or representing physical dispersion, damps this instability. We obtain good agreement with finite-element simulations with and without additional diffusion. We also prove that all states are unconditionally stable for a local-equilibrium-foam model.

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List of symbols

C_c	Foam parameter in Kam model ($\text{m}^{-3} \text{s}^{-1}$)
C_f	Foam parameter in Kam model
C_g	Foam parameter in Kam model ($\text{m}^{-3} \text{s}^{-1}$)
f_w	Water fractional-flow function (excluding capillarity-driven flow)
k_{rg}^0	Gas relative-permeability in the absence of foam
k_{rw}	Water relative-permeability
n	Foam parameter in Kam model (Appendix A)
n_D	Dimensionless foam texture ($= n_f/n_{\max}$)
n_f	Foam texture (number of lamellae per unit volume) (m^{-3})
$n_{\max}$	Maximum foam texture (m^{-3})
∇p	Pressure gradient in Kam model (psi ft^{-1})
P_c	Gas-water capillary pressure (Pa)
∇p_0	Foam parameter in Kam model (psi ft^{-1})
r_1, r_2	Eigenvectors
r_g	Foam generation function ($\text{m}^{-3} \text{s}^{-1}$)
r_c	Foam-coalescence function ($\text{m}^{-3} \text{s}^{-1}$)
S_g	Gas saturation
S_{gr}	Residual gas saturation in a water-gas system
S_w	Water saturation
S_{wc}	Connate water saturation in a water-gas system
S_w^*	Limiting water saturation
t	Time (s)
u	Total superficial velocity (m s^{-1})
u_g	Gas superficial velocity (m s^{-1})
u_w	Water superficial velocity (m s^{-1})
x	Standing coordinate for displacement in 1-D (m)

Superscripts and subscripts

⁰ Equilibrium point

Greek symbols

ϕ	Porosity
K	Oscillation wave number, i.e., inverse wavelength (m^{-1})
λ_g	Mobility of gas ($\text{m}^2 (\text{Pa s})^{-1}$)
λ_w	Mobility of water ($\text{m}^2 (\text{Pa s})^{-1}$)
μ_g^f	Gas viscosity in the presence of foam (Pa s)
μ_g^0	Gas viscosity in the absence of foam (Pa s)
μ_w	Viscosity of water (Pa s)
v_1, v_2	Eigenvalues (solutions of Eq. 12)
σ	Gas-water interfacial tension (N m^{-1})

1 Introduction

Gas flooding for enhanced oil recovery (EOR) can be nearly 100% efficient in oil recovery where gas sweeps, but gas sweep efficiency is reduced by reservoir heterogeneity, gravity segregation, and viscous instability (Lake 1989). Foam can help fight all three causes of poor gas sweep (Kovscek and Radke 1994; Rossen 1996; Rossen et al. 2010).

Gas and surfactant solution can form multiple local-equilibrium (LE) states for the same injection conditions. It is commonly observed that at low superficial velocity there is “weak” or “coarse” foam, with a modest reduction in gas mobility (Ransohoff and Radke 1988; Rossen and Gauglitz 1990; Friedmann et al. 1991, 1994; Tanzil et al. 2002; Gauglitz et al. 2002). Creation of strong foam is triggered by an increase in superficial velocity or pressure gradient; thereafter injection rate can be returned to the lower values and the strong foam remains at velocities for which weak foam was previously observed. In experiments with pressure gradient constrained rather than injection rates (Gauglitz et al. 2002; Kam and Rossen 2003), one observes three LE states for the same total superficial velocity u and water fractional flow f_w : a weak-foam state (with little reduction in gas mobility), a strong-foam state (with large reduction in gas mobility), and an intermediate state that appears to be unstable in laboratory core floods: specifically, the state of foam fluctuates continually when one tries to impose the superficial velocities and ∇P of this state. Besides theoretical interest, the intermediate state may be important in field applications where the well-to-well pressure difference is not sufficient to maintain a large bank of strong foam.

There are two main approaches to modeling complex mechanisms governing foam texture: population-balance modeling (Falls et al. 1988; Friedmann et al. 1991; Kovscek et al. 1995; Kam et al. 2007; Kam 2008; Chen et al. 2010) and local-equilibrium (LE) modeling (Fisher et al. 1990; de Vries and Wit 1990; Persoff et al. 1991; Ettinger and Radke 1992; Vassenden and Holt 2000; Computer Modeling Group 2009; Chen et al. 2010; Ashoori et al. 2010). For simplicity, LE foam models assume that foam reaches the LE bubble texture corresponding to its water saturation very fast. Therefore, no explicit balance equation for foam bubbles is required in these models. One should keep in mind that even in population-balance models, there are large regions in the solution where the flow is at local equilibrium, as far as foam is concerned (see Chen et al. 2010).

There are a few modeling efforts to predict multiple steady-state states seen in the laboratory. Friedmann et al. (1991) fit the jump to strong foam with a population-balance model where lamella creation requires exceeding a minimum ∇P : the model is discontinuous at this value of ∇P . More recently, the only foam models to represent the multiple LE states for foam are variants of the population-balance model of Kam and Rossen (2003) (Kam et al. 2007; Kam 2008; Afsharpoor et al. 2010), where creation of lamellae depends on pressure gradient ∇P . (In this article, ∇P represents the magnitude of pressure gradient, not a vector quantity.) The model is based loosely on the model for foam generation of Rossen and Gauglitz (1990). In that model, the key step in the jump from weak foam to strong foam at a given superficial velocity is the mobilization of liquid lenses out of pore throats, initiating a process of lamella division; this mobilization depends on ∇P . Rossen and Gauglitz fit the trend of minimum gas superficial velocity for foam generation with f_w using their model.

The population-balance model of Kam (2008) and related older models are the only ones that fit data for all three LE states as a function of ∇P and also the two flow regimes at low and high f_w within the strong-foam state (cf. Alvarez et al. 2001). Furthermore, most other population-balance models are based on lamella creation by so-called Roof snap-off (e.g., Holm 1968; Kovscek et al. 1995). This mechanism for foam generation at steady state is called to question (Rossen 2003, 2008).

The Kam model assumes an upper limit in foam texture corresponding to a minimum bubble size, assumed to be about the volume of one pore. This limit is required by experimental observations of a “low-quality foam regime” within the strong-foam state (Osterloh and Jante 1992; Alvarez et al. 2001). This limit can be explained by two mechanisms. First, foam generation by either snap-off or lamella division is not expected to create bubbles much smaller than a pore. Second, if such bubbles were created, they would rapidly disappear by diffusion until bubble size reaches roughly the volume of a pore, at which point diffusion ceases (Rossen 1996). Because of the change in mechanisms for bubble creation and destruction for bubbles of roughly pore size, the model of Kam (2008) and related models simply postulate an upper limit to foam texture; if it exceeds the limit, it is reset to that limit. For an alternate approach, see Chen et al. (2010).

Kam et al. (2007) examined the stability of the strong-, weak-foam, and intermediate states numerically by checking the growth or decay of an arbitrary perturbation in foam texture (inversely related to bubble size) and water saturation using numerical simulation on one LE state from each group. The low-quality strong- and weak-foam states appeared stable, and the intermediate state unstable. More recently, Kam (2008) found the high-quality strong-foam state to be unstable in the original model using the same method (specifically, an arbitrary perturbation and upwind-finite-difference numerical simulation), and modified the model. He then found the strong- and weak-foam states of his modified model to be stable, and the intermediate state unstable, using the same method. It is this foam model we examine here. In both variants of this model, simulation of the strong-foam state is difficult because the water saturation S_w of this state is extremely close to the water saturation S_w^* at which foam collapses completely. Besides the extreme sensitivity of foam properties to S_w in this vicinity, the foam-coalescence function in the model is undefined for $S_w < S_w^*$.

It is important to determine the stability of foam in all its LE states in order to predict which state would be observed in field application. These investigations must include both laboratory studies and examination of the models used to extrapolate laboratory results to the field. Here we examine the stability of uniform LE states in both LE foam models and in the model of Kam (2008) as the most current population-balance model predicting multiple LE states for a given injection condition. We study the latter in several ways: analysis of the linearized behavior of the foam flow in a region with states representing non-LE states in the immediate vicinity of a given LE state and numerical simulation of the growth or decay of a perturbation to that uniform state. Since numerical dispersion is an unavoidable artifact of numerical simulation, and physical dispersion is present in all porous media, we also examine the effect of dispersion on the stability of the LE states.

Stability here presumes the existence of an LE state, where foam generation and destruction rates are in balance; we examine whether a perturbation to such a state would grow or decay with time. This issue is distinct from our earlier studies of whether a given LE state could displace another during foam injection (Ashoori et al. 2011a) or could be created from injected gas and liquid at the entrance of a porous medium (Ashoori et al. 2011b). Here we assume initially a large uniform region at the LE state, and then perturb it. Also, our study is distinct from the issue of “foam stability” in the presence of oil (Law et al. 1992; Schramm 1994; Rossen 1996; Mannhardt and Svorstøl 1999). Strictly, the issue of “foam stability to oil” asks whether an LE strong-foam state *exists* in the presence of a given saturation of a given oil, not whether such a state would be stable to perturbations. Here we seek to understand the stability of foam states in the absence of oil, as part of the larger effort to understand foam under all conditions of field application.

1.1 Governing Equations for Foam Flow in Porous Medium

The governing equation for an immiscible two-phase (gas–water), two component incompressible displacement in rectilinear flow through a porous media is given by the Rapoport–Leas equation (Barenblatt et al. 1991; Bedrikovetsky et al. 1995)

$$\varphi \frac{\partial S_w}{\partial t} + \frac{\partial u_w}{\partial x} = 0 \quad (1)$$

or equivalently,

$$\varphi \frac{\partial S_w}{\partial t} + u \frac{\partial f_w}{\partial x} + \frac{\partial}{\partial x} \left(\frac{\lambda_w \lambda_g}{\lambda_w + \lambda_g} \frac{dP_c}{dS_w} \frac{\partial S_w}{\partial x} \right) = 0 \quad (2)$$

where S_w and f_w are water saturation and fractional flow, φ is porosity, u total superficial velocity, x position, t time, and λ_w and λ_g the mobilities of water and gas. Besides the dependency of f_w on S_w in normal gas–water flow, in foam processes λ_g and f_w also depend on foam texture or bubble size n_f , defined as the number of lamellae per unit volume of gas phase. The last term in the Eq. 2 is the capillarity-driven water velocity. There could also be dependency on surfactant concentration, but we assume for simplicity that surfactant is present in full-strength concentration in the aqueous phase everywhere, e.g., because of injection of a large preflush of surfactant before the co-injection of gas and surfactant solution. Therefore no separate material balance on surfactant is needed.

In foam processes, gas properties depend on bubble texture as well as water saturation (Falls et al. 1988, 1989; Kovsky and Radke 1994; Rossen 1996). Therefore modeling requires, in addition to the water mass-balance equation (Eq. 2), a so-called population-balance equation:

$$\varphi \frac{\partial}{\partial t} (S_g n_D) + \frac{\partial}{\partial x} (u_g n_D) = \frac{\varphi}{n_{\max}} S_g (r_g - r_c) \quad (3)$$

where $n_D \equiv n_f/n_{\max}$ is the dimensionless bubble texture and $n_{\max}$ is upper limit to foam texture (see Appendix A for more details). In the rest of this article we work with dimensionless foam texture n_D rather than n_f . In this equation, we assume for simplicity that n_D takes the same value for trapped and flowing portions of the foam. The term on the right-hand side describes the net foam generation rate; r_g and r_c denote the rates of bubble generation and destruction, respectively. The gas superficial velocity u_g is given by

$$u_g = (1 - f_w) u - f_w \lambda_g \frac{\partial P_c}{\partial S_w} \frac{\partial S_w}{\partial x}. \quad (4)$$

Local-equilibrium (LE) for foam means that the uniform foam state has reached the equilibrium bubble density corresponding to the fractional flow and saturation at that position. In other words, $r_g = r_c$ for an LE state.

1.2 Stability

A local-equilibrium state is called stable if the state moves back toward the same equilibrium when it is slightly perturbed. In the immediate vicinity of the LE state, representing a region with this uniform state, Eqs. 2 and 3 can be linearized; the stability of an equilibrium of a linear system is relatively simple to determine. A system of differential equations is linear if the dependent variables and their derivatives appear to the power one (products are not allowed) and nonlinear otherwise.

We are interested in the linear second-order PDE in the general form

$$\frac{\partial U}{\partial t} + A \frac{\partial U}{\partial x} = B \frac{\partial^2 U}{\partial x^2} + CU \quad (5)$$

where $U = \begin{pmatrix} u_1(x, t) \\ u_2(x, t) \end{pmatrix}$ and A , B , and C are 2×2 matrixes with constant coefficients. We are interested in studying the stability of the equilibrium $U = 0$. The general solution of this system can be written as the superposition in the real parameter κ of oscillatory solutions of the form $U = \exp(i\kappa x + \nu t) \begin{bmatrix} a(\kappa) \\ b(\kappa) \end{bmatrix}$, where a , b , and ν depend on κ . This is the classical normal mode analysis (Boyce and DiPrima 1986). It has appeared in petroleum literature at least since the analysis of fingering instability in oil banks was performed (see, e.g., Scheidegger 1958). Substituting this expression for U in Eq. 5, typically two complex eigenvalues ν_i with corresponding eigenvectors $\begin{bmatrix} a_i(\kappa) \\ b_i(\kappa) \end{bmatrix}$ ($i = 1, 2$) can be found. Notice that the absolute value of the oscillatory exponential factor in U is $\exp(t \operatorname{Real}(\nu_i))$ which grows or decays with time according to the sign of the real part of ν_i . Therefore, the eigenvalues determine three types of behavior:

- Both eigenvalues have negative real parts. In this case the equilibrium point is stable, i.e., oscillations approach the equilibrium point as t grows unboundedly.
- At least one of the eigenvalues has positive real part. This equilibrium point is not stable.
- One of the eigenvalues is purely imaginary and the other one has nonpositive real part. This equilibrium point is weakly stable, i.e., the state moves boundedly near the equilibrium point.

The behavior of a nonlinear system very close to an equilibrium point is determined by the behavior of a much simpler linearized system of the form (5). This applies to cases (a) and (b) but not in case (c).

1.2.1 Non-LE Stability Analysis of Uniform States in Population-Balance Foam Flow

Equations 2 and 3 are the two governing equations of a foam process. At first we need to rewrite the system equations to be explicitly in terms of the dependent variables S_w and n_D . We expand the derivative of $f_w = f_w(S_w, n_D)$ with respect to x in Eq. 2, obtaining

$$\frac{\partial S_w}{\partial t} + A_{11} \frac{\partial S_w}{\partial x} + A_{12} \frac{\partial n_D}{\partial x} = \frac{\partial}{\partial x} \left(B_{11} \frac{\partial S_w}{\partial x} \right) \quad (6)$$

where $A_{11} = \frac{u}{\varphi} \frac{\partial f_w}{\partial S_w}$, $A_{12} = \frac{u}{\varphi} \frac{\partial f_w}{\partial n_D}$, $B_{11} = -\frac{\lambda_g f_w}{\varphi} \frac{dP_c}{dS_w}$.

Expanding Eq. 3 similarly leads to the following equation:

$$n_D \left(\varphi \frac{\partial S_g}{\partial t} + \frac{\partial u_g}{\partial x} \right) + \varphi S_g \frac{\partial n_D}{\partial t} + u_g \frac{\partial n_D}{\partial x} = \frac{\varphi}{n_{\max}} S_g (r_g - r_c). \quad (7)$$

According to Eq. 1, the term in the left parenthesis vanishes in Eq. 7 for a constant total-velocity-process such as this one. Inserting u_g from Eq. 4 in Eq. 3 and expanding the equation, Eq. 7 can be written as

$$\frac{\partial n_D}{\partial t} + \frac{(1 - f_w) u}{\varphi S_g} \frac{\partial n_D}{\partial x} - \frac{f_w \lambda_g}{\varphi S_g} \frac{\partial P_c}{\partial S_w} \frac{\partial S_w}{\partial x} \frac{\partial n_D}{\partial x} = \frac{1}{n_{\max}} (r_g - r_c). \quad (8)$$

The system (6, 8) is nonlinear. To investigate the stability of an LE state we study the behavior of the linearized form close vicinity of a LE point (S_w^0, n_D^0) denoted by superscript 0 . We can use Taylor expansions of, e.g., $A_{ij}(S_w, n_D)$ about the point (S_w^0, n_D^0) provided that it has continuous partial derivatives up to order two:

$$A_{ij}(S_w, n_D) \cong A_{ij}(S_w^0, n_D^0) + \left. \frac{\partial A_{ij}(S_w, n_D)}{\partial S_w} \right|_0 \Delta S_w + \left. \frac{\partial A_{ij}(S_w, n_D)}{\partial n_D} \right|_0 \Delta n_D. \quad (9)$$

We substitute the Taylor expansions for the other terms around (S_w^0, n_D^0) similarly. Next, we replace $S_w(x, t)$ by $S_w^0 + \Delta S_w(x, t)$ and $n_D(x, t)$ by $n_D^0 + \Delta n_D(x, t)$. Note that $(r_g - r_c)$ evaluated at (S_w^0, n_D^0) vanishes because this point is at equilibrium. Ignoring all the terms consisting of ΔS_w or Δn_D raised to power greater than one, the nonlinear system (6, 8) is linearized in the form of Eq. 5 in terms of dependent variables $U = \begin{bmatrix} \Delta S_w \\ \Delta n_D \end{bmatrix}$ as follows

$$\frac{\partial}{\partial t} \begin{bmatrix} \Delta S_w \\ \Delta n_D \end{bmatrix} + \begin{bmatrix} A_{11}^0 & A_{12}^0 \\ 0 & A_{22}^0 \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} \Delta S_w \\ \Delta n_D \end{bmatrix} = \frac{\partial}{\partial x} \left(\begin{bmatrix} B_{11}^0 & 0 \\ 0 & 0 \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} \Delta S_w \\ \Delta n_D \end{bmatrix} \right) + \begin{bmatrix} 0 & 0 \\ C_{21}^0 & C_{22}^0 \end{bmatrix} \begin{bmatrix} \Delta S_w \\ \Delta n_D \end{bmatrix} \quad (10)$$

where $A_{22}^0 = \left. \frac{(1-f_w)u}{\phi S_g} \right|_0$, $C_{21}^0 = \left. \frac{1}{n_{\max}} \frac{\partial(r_g - r_c)}{\partial S_w} \right|_0$ and $C_{22}^0 = \left. \frac{1}{n_{\max}} \frac{\partial(r_g - r_c)}{\partial n_D} \right|_0$.

Inserting the oscillatory perturbation $e^{i\kappa x + \nu t} \begin{bmatrix} r \\ 1 \end{bmatrix}$ into system (10) we have

$$\left\{ \begin{bmatrix} -i\kappa A_{11}^0 - \kappa^2 B_{11}^0 & -i\kappa A_{12}^0 \\ -\kappa^2 + C_{21}^0 & -i\kappa A_{22}^0 - \kappa^2 + C_{22}^0 \end{bmatrix} - \nu \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} r \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (11)$$

This system has a nontrivial solution if the determinant of the matrix of coefficients is zero:

$$\nu^2 + (i\kappa A_{22}^0 + \kappa^2 - C_{22}^0 + i\kappa A_{11}^0 + \kappa^2 B_{11}^0) \nu + (i\kappa A_{11}^0 + \kappa^2 B_{11}^0) (i\kappa A_{22}^0 + \kappa^2 - C_{22}^0) - i\kappa (\kappa^2 - C_{21}^0) A_{12}^0 = 0. \quad (12)$$

This is called the *dispersion relation* in wave propagation theory (Whitham 1974). Solving this quadratic Eq. 12 we obtain

$$\nu_{1,2} = \frac{+C_{22}^0 - \kappa^2 B_{11}^0}{2} - i\kappa \frac{A_{22}^0 + A_{11}^0}{2} \pm \frac{\sqrt{(C_{22}^0 + \kappa^2 B_{11}^0 + i\kappa(A_{11}^0 - A_{22}^0))^2 - 4i\kappa A_{12}^0 C_{21}^0}}{2}. \quad (13)$$

The values of the $r_{1,2}$ in the eigenvector are calculated by replacing $\nu_{1,2}$ in one of the equations in (11) leading to

$$r_{1,2} = \frac{-i\kappa A_{12}^0}{\nu_{1,2} + \kappa^2 B_{11}^0 + i\kappa A_{11}^0}. \quad (14)$$

The linear stability or instability of an LE state can be determined by studying the sign of the real part of the eigenvalues as already explained. The eigenvectors reveal the path of growing or diminishing of an oscillation wave around the LE point.

2 Results

2.1 LE Linear-Stability Analysis of Uniform States in LE Foam Flow

For the simple case of a foam in LE foam models, foam is always at local equilibrium, i.e., $r_g = r_c$ everywhere. Therefore, the perturbations also follow LE assumptions. Owing to LE assumption, n_D can be written as a function of S_w , defined implicitly by $r_g(S_w, n_D) = r_c(S_w, n_D)$. The governing equation is the same as Eq. 6 except that $f_w = f_w(S_w, n_D(S_w))$ as follows:

$$\frac{\partial S_w}{\partial t} + A_{11} \frac{\partial S_w}{\partial x} + A_{12} \frac{\partial n_D(S_w)}{\partial x} = \frac{\partial}{\partial x} \left(B_{11} \frac{\partial S_w}{\partial x} \right), \quad (15)$$

where $A_{11} = \frac{u}{\varphi} \frac{\partial f_w}{\partial S_w} \Big|_{(S_w, n_D(S_w))}$, $A_{12} = \frac{u}{\varphi} \frac{\partial f_w}{\partial n_D} \Big|_{(S_w, n_D(S_w))}$, $B_{11} = - \frac{\lambda_g f_w}{\varphi} \frac{dP_c}{dS_w} \Big|_{(S_w, n_D(S_w))}$.

Note that $\frac{dP_c}{dS_w}$ just depends on S_w . Linearizing the equation above we have

$$\frac{\partial \Delta S_w}{\partial t} + A_{11}^0 \frac{\partial \Delta S_w}{\partial x} + A_{12}^0 \frac{\partial \Delta n_D(\Delta S_w)}{\partial x} = \frac{\partial}{\partial x} \left(B_{11}^0 \frac{\partial \Delta S_w}{\partial x} \right). \quad (16)$$

For a foam where $r_g = r_c$ everywhere, one can show that $\Delta n_D(\Delta S_w) = -C_{21}^0 \Delta S_w / C_{22}^0$, based on linearizing the nonlinear relation $r_g - r_c = 0$. Replacing Δn_D in Eq. 16, we obtain

$$\frac{\partial \Delta S_w}{\partial t} + \left(A_{11}^0 - \frac{A_{12}^0 C_{21}^0}{C_{22}^0} \right) \frac{\partial \Delta S_w}{\partial x} = \frac{\partial}{\partial x} \left(B_{11}^0 \frac{\partial \Delta S_w}{\partial x} \right). \quad (17)$$

Finally, inserting the oscillatory solution $e^{i\kappa x + \nu t}$ in we find an eigenvalue

$$\nu = -\kappa^2 B_{11}^0 + i\kappa \left(-A_{11}^0 + \frac{A_{12}^0 C_{21}^0}{C_{22}^0} \right). \quad (18)$$

The stability of the state depends on the real part of the only eigenvalue ν , which is $-\kappa^2 B_{11}^0$. This term is always negative according to the definition of B_{11}^0 in (15). Thus all uniform states in a foam model that assume (no deviation from) local equilibrium are unconditionally stable. Note that this conclusion is independent of the foam model as long as it assumes local equilibrium even in perturbations.

The same conclusion applies to the model of Kam (2008) for the strong-foam state where n_D is fixed at 1. In that case as well, there is only one balance equation, for water, and $n_D = n_D(S_w) = 1$. Therefore, the third-term vanishes in equation and ν becomes $-\kappa^2 B_{11}^0 - i\kappa A_{11}^0$ which has a negative real part as in (18).

2.2 Linearized Stability of LE States in Kam (2008) Model

Here we examine the stability of LE states from the linearized governing system (10) applying the model of Kam (2008) for foam in the rest of this article. For a given injected fractional flow and total superficial velocity, one can find the corresponding water saturation and foam texture from the foam model as explained in Appendix A. Next, to construct system (10), A_{11}^0 , A_{12}^0 , A_{22}^0 , B_{11}^0 , C_{21}^0 and C_{22}^0 must be calculated for known (S_w^0, n_D^0, f_w^0) . The linear stability of the state is determined by plotting the real parts of $\nu_{1,2}$ versus κ according to Eq. 13.

For $u = 5.29 \times 10^5$ m/s and water fractional flow of 0.05, Kam's model predicts three possible LE states (see Kam (2008), or see Fig. 4): weak foam on the right-most curve, strong

foam on the left-most curve and an intermediate-strength foam in between. Figures 1, 2, and 3 show $v_{1,2}$ versus κ for these states. According to Fig. 3, the weak-foam state has two negative eigenvalues and thus is stable. In Fig. 2 we see that the intermediate-strength foam

Fig. 1 Solutions of Eq. 12 for the strong-foam state $(S_w, n_D, f_w)^0 = (0.058585, 0.652177, 0.05)$ for $u = 5.29 \times 10^{-5}$ m/s ($C_c = 1$). The *superscript* 0 represents properties at LE. Note that the critical water saturation below which foam does not exist is 0.0585

Fig. 2 Solutions of Eq. 12 for the intermediate-strength-foam state $(S_w, n_D, f_w)^0 = (0.126306, 0.027812, 0.05)$ for $u = 5.29 \times 10^{-5}$ m/s ($C_c = 1$)

Fig. 3 Solutions of Eq. 12 for the weak-foam state $(S_w, n_D, f_w)^0 = (0.371342, 0.000950, 0.05)$ for $u = 5.29 \times 10^{-5}$ m/s ($C_c = 1$)

Fig. 4 Linear stability according to the solutions of Eq. 12 for the fractional-flow curves ($C_c = 1$) for increasing values of u : **a** 2.798×10^{-5} m/s, **b** 4×10^{-5} m/s, **c** 4.5×10^{-5} m/s, and **d** 5.29×10^{-5} m/s. The black and gray colors represent stability and instability, respectively. Small figures are enlargements of the LE fractional-flow curves around critical water saturation S_w^* . Notice that the horizontal enlargement is much more than the vertical one

has one positive eigenvalue and is thus unstable. In Fig. 1 the strong-foam state has also one positive eigenvalue, which makes it unstable. However, the positive unstable eigenvalue is very small. This means that one cannot probably see the instability of strong-foam state unless introducing an unstable perturbation to the strong-foam state directly along its unstable eigenvector, or by waiting some time for the instability to reveal itself.

Figure 4 summarizes the linear-stability results for all the LE states for four different total velocities and foam-kinetic rates corresponding to $C_c = 1$. The black and gray colors in this figure and the rest of this article represent stability and instability, respectively. Figure 4 makes several points. First, all the strong-foam states with bubble density less than one ($n_D^{\text{LE}} \leq 1$) are unstable. This model postulates $n_D = 1$ if $n_D^{\text{LE}} > 1$, as discussed above. The strong-foam LE states have dimensionless foam texture equal to 1 for most values of water fractional-flow on the left-hand branch in Fig. 4 except for the small gray parts which are shown to be unstable. One can see the abrupt change in slope of the fractional-flow curves at the point where $n_D = 1$ in the magnified f_w versus S_w curves in Fig. 4. If we did not impose the condition $n_D^{\text{LE}} \leq 1$, most of the strong-foam branches of the fractional-flow curves would be unstable. As noted in Sect. 2.1, the part where n_D is fixed at 1 is inherently stable.

Second, the middle-state-LE foam is mostly unstable for all the values of u in Fig. 4. This is consistent with experiments of Gauglitz et al. (2002). Third, the LE-weak-foam states, the right-most branch, is likely to be unstable at higher velocities.

Moreover, the linear-stability analysis shows that there is always one stable eigenvalue for all the LE points in Fig. 4.

For some fractional-flow values in Fig. 4 there are three LE states but none of them is stable: for instance, at small water fractional-flow in Fig. 4d.

Linear-stability analysis for slower foam-kinetic rates (corresponding to smaller C_c and C_g) show more serious instability of strong foam and weaker instability (less-negative v) but more widespread instability of weak foam. Comparing Fig. 5 and 1 show that slowing down foam kinetics by a factor of 100 makes the unstable eigenvalue about 100 times greater in magnitude and the stable one about 100 times smaller.

An approximate way of investigating the stability of the linearized system (10) is to solve this system numerically. Figure 6 shows the stability of the same strong-foam state as in Fig. 1 determined by COMSOL® (COMSOL Multiphysics 2008). We introduce an oscillatory differentiable initial perturbation to a finite (100-m) medium with boundary conditions specifying no perturbation at the ends. Our initial perturbations are selected along the real part of the unstable eigenvector $r_1 = 0.130379 - 0.0000461i$ for $\kappa = 3$. We choose a small perturbation $0.00003 \times \cos(3x)$. To make the perturbation decay away from its center we multiply it by $\exp(-0.03 \times (x - x_0)^2)$, a Gaussian function centered at x_0 , obtaining the initial perturbation used in our simulations:

Fig. 5 Solutions of Eq. 12 for the strong-foam state $(S_w, n_D, f_w)^0 = (0.058585, 0.652177, 0.05)$ for $u = 5.29 \times 10^{-5}$ m/s ($C_c = 0.01$). Figure 1 shows the linear stability of identical state at 100 times faster foam kinetics

Fig. 6 Linear stability of strong-foam state $(S_w, n_D, f_w)^0 = (0.058585, 0.652177, 0.05)$ for $u = 5.29 \times 10^{-5}$ m/s ($C_c = 1$) by COMSOL. The initial perturbation is given in (19) with $x_0 = 50$. Thin black curve corresponds to $t = 0$ s and the gray curve corresponds to $t = 30$ s. Total medium length is 100 m

$$\begin{pmatrix} \Delta S_w \\ \Delta n_D \end{pmatrix} = 0.00003 \times \exp(-0.03 \times (x - x_0)^2) \times \cos(3x) \begin{pmatrix} 0.130379 \\ 1 \end{pmatrix}. \quad (19)$$

Note that perturbation in S_w needs to be very small to preserve physical meaning; otherwise it goes below the critical water saturation (here 0.0585), below which foam cannot survive. According to Fig. 6, the perturbation in S_w gradually grows, while the perturbation in n_D increases 2 orders of magnitude after just 30 s. Thus, the simulation of the perturbation of the linear system agrees with the analytical result.

2.2.1 Linearized Stability in the Presence of Diffusion

Local velocity gradients (as between pore wall and pore center), locally heterogeneous streamlines and mechanical mixing in pore bodies give rise to physical dispersion in flow in porous media. This dispersion depends on the length scale under study. The “echo” dispersion coefficient (i.e., coefficient that reflects true mixing, not merely spreading effect of advection) can vary from roughly 10^{-4} m²/s on the reservoir scale to less than 10^{-7} m²/s on the lab scale for the superficial velocity of order of 0.1 to 1×10^{-4} m/s (about 1–10 m/d) (John et al. 2010). This difference can be explained qualitatively by comparing the scale of heterogeneity captured by a given measurement in real reservoir to a core in the lab (Lake 1989). In addition, some numerical schemes introduce artificial, or numerical, dispersion, depending on the grid size.

For the remainder of this article, to avoid confusion with the use of *dispersion* in the mathematical literature [e.g., in reference to Eq. 12 (Whitham 1974)], we refer to all processes giving rise to second-order-derivative terms in balance equations as *diffusion*. Thus physical dispersion, numerical diffusion, and capillary-driven flow are referred to hereafter as *diffusive processes* or *diffusion*.

We are interested to see how this diffusion affects the stability of the strong-foam state. See Appendix B for the foam governing equations in the presence of diffusion and derivation of eigenvalues and eigenvectors regarding linear stability.

Figure 7 shows the stability of the strong-foam state (the same unstable state as in Fig. 1) in the presence of diffusion. A small value of diffusion coefficient ($D = 2 \times 10^{-5}$ m²/s) makes the strong-foam stable; i.e., it changes the tiny positive unstable eigenvalue to a small negative one. This relatively small value of diffusion can make most of strong-foam states

Fig. 7 Solutions of Eq. 12 for the state $(S_w, n_D, f_w)^0 = (0.058585, 0.652177, 0.05)$ for $u = 5.29 \times 10^{-5}$ m/s ($C_c = 1$) with artificial diffusion ($D = 2 \times 10^{-5}$ m²/s). Figure 1 shows the linear instability of the identical state in the absence of diffusion

Fig. 8 Linear stability according to the solutions of Eq. 12 for the fractional-flow curves ($C_c = 1$) with artificial diffusion ($D = 2 \times 10^{-5} \text{ m}^2/\text{s}$) for different values of u : **a** $2.798 \times 10^{-5} \text{ m/s}$, **b** $4 \times 10^{-5} \text{ m/s}$, **c** $4.5 \times 10^{-5} \text{ m/s}$, and **d** $5.29 \times 10^{-5} \text{ m/s}$. The *black* and *gray* colors represent stability and instability, respectively. Notice that the *black* stable strong-foam branches in the enlarged plots in this figure (in the presence of diffusion) are unstable in Fig. 4 without diffusion

stable as seen in Fig. 8 (compare the black parts on the left-most branch in Fig. 8 with Fig. 4). This holds also even if one does not impose the condition $n_D^{\text{LE}} \leq 1$ for larger values of water fractional-flow. Our analysis shows that the slower the foam kinetics (i.e., the smaller the values of C_c and C_g), the larger diffusion coefficient needed to make the strong-foam stable.

Figure 9 illustrates how an initial perturbation evolves with time for the strong-foam state modeled by COMSOL as in Fig. 6. Note that here we are solving the linearized system. One sees the very gradual decrease of perturbation for ΔS_w but increase of perturbation for Δn_D till roughly 1 s and gradually diminishing behavior afterwards. Diffusion controls the instability in water saturation; for bubble density, the perturbation shrinks eventually. This discrepancy in the response of n_D to perturbations at early time, between the analytical result of the Eq. 12 seen in Fig. 7 and COMSOL results in Fig. 9, could be due to the following reasons. First, the solution of the numerical scheme used by COMSOL slightly differs from the PDE in discretization error. Second, the damping term $\exp(-0.03 \times (x - x_0)^2)$ that we implement on the initial perturbation in COMSOL might cause small changes in the stability results in COMSOL. However, the long-term result is the same for both methods: the perturbation decays at long times in the presence of diffusion.

2.2.2 Linear Stability in the Upwind Scheme

Kam (2008) reports the strong-foam state (in the same foam model as used in this article) is stable. He mentions in his article that a dozen different perturbations lead to the same stable strong foam. The main question that arises is why his stability tests obtain end with different results from ours.

Fig. 9 Linear stability of state $(S_w, n_D, f_w)^0 = (0.058585, 0.652177, 0.05)$ for $u = 5.29 \times 10^{-5}$ m/s ($C_c = 1$) with artificial diffusion ($D = 2 \times 10^{-5}$ m²/s) modeled using COMSOL. The initial perturbation is given in (19) with $x_0 = 20$. Thin black curve corresponds to $t = 1$ s and the thick black curve corresponds to $t = 600$ s. Total medium length is 100 m

Fig. 10 Solutions of Eq. 12 near the state $(S_w, n_D, f_w)^0 = (0.058585, 0.652177, 0.05)$ for $u = 5.29 \times 10^{-5}$ m/s ($C_c = 1$) for upwind-finite-difference scheme. $\Delta x = 0.012$ m, $\Delta t = 0.01$ s (solid line) and $\Delta t = 0.001$ s (dashed line). Notice the difference compared to Fig. 1

Kam's simulations use an upwind-finite-difference scheme. In Appendix C, we derive how this specific scheme could affect the eigenvalues of the system linearized near an equilibrium point. Figure 10 shows $v_{1,2}$ versus κ for the same strong-foam state as in Fig. 1 but for the finite-difference scheme. We set $\Delta x = 0.012$ m as in Kam (2008). Kam does not report the value of Δt he used, but due to very fast wave speed for the strong-foam states in this model, we believe that Δt should be smaller than 0.01 s (solid lines). We also repeat the procedure for a ten times smaller Δt (dashed lines). The strong-foam state is stable with this numerical scheme. Our interpretation is that the finite-difference scheme introduces some artificial numerical diffusion that makes the state stable: compare the small stable eigenvalue (v_1) in Fig. 10 with the one in Fig. 7. Figures 11 and 12 show the stability analysis for the upwind-finite-difference scheme for the intermediate foam and weak-foam states (comparable to Figs. 2 and 3), respectively. Figure 13 is the summary of stability analysis done with the finite-difference scheme for all the LE states with different total velocities.

Fig. 11 Solutions of Eq. 12 for the state $(S_w, n_D, f_w)^0 = (0.126306, 0.027812, 0.05)$ for $u = 5.29 \times 10^{-5}$ m/s ($C_c = 1$) for upwind-finite-difference scheme. $\Delta x = 0.012$ m, $\Delta t = 0.01$ s (solid line) and $\Delta t = 0.001$ s (dashed line)

Fig. 12 Solutions of Eq. 12 for the state $(S_w, n_D, f_w)^0 = (0.371342, 0.000950, 0.05)$ for $u = 5.29 \times 10^{-5}$ m/s ($C_c = 1$) for upwind-finite-difference scheme. $\Delta x = 0.012$ m, $\Delta t = 0.01$ s (solid line) and $\Delta t = 0.001$ s (dashed line)

Besides the numerical scheme that might affect the apparent stability of states, the instability of the strong foam is relatively weak (the unstable eigenvalue is small) so that one needs to either run a simulation for some time or introduce the exact unstable perturbation to see immediately the instability of the state. This also means that numerical stability analysis done with some *arbitrary* perturbations run for relatively short times cannot guarantee the accuracy of the result.

2.3 Nonlinear Stability of LE States in Kam's Model

Next we analyze stability by simulating the original nonlinear Eqs. 2 and 3 using COMSOL. The dependent variables are S_w and n_D . We assume a 100-m medium initially at a given LE state (S_w^0, n_D^0, f_w^0) . The inflow boundary conditions are water fractional-flow and foam texture as at the LE state. We introduce a perturbation for S_w and n_D along the unstable eigenvector (from linear analysis) and simulate the behavior of the system.

Figure 14 shows the stability of the same strong-foam state as in Fig. 6, except that it follows the original nonlinear equations. A close look at Fig. 14 shows a very gradual

Fig. 13 Linear stability according to the solutions of Eq. 12 for the fractional-flow curves ($C_c = 1$) with upwind-finite-difference scheme for different values of u : **a** 2.798×10^{-5} m/s, **b** 4×10^{-5} m/s, **c** 4.5×10^{-5} m/s, and **d** 5.29×10^{-5} m/s. The black and gray colors stability and instability, respectively. In these figures $\Delta x = 0.012$ m, $\Delta t = 0.01$ s. Contrast the stable strong-foam branches in the enlarged plots in this figure with the unstable ones in Fig. 4 (without diffusion) and stable ones in Fig. 8 (with diffusion)

Fig. 14 Nonlinear stability of state $(S_w, n_D, f_w)^0 = (0.058585, 0.652177, 0.05)$ for $u = 5.29 \times 10^{-5}$ m/s ($C_c = 1$) determined using COMSOL. The initial perturbation is given in (19) with $x_0 = 20$. Thin black curve corresponds to $t = 1$ s and the gray curve corresponds to $t = 600$ s. Total medium length is 100 m

growth in the perturbation in S_w . The perturbation in n_D grows intensely at small times, but it grows more gradually, similarly to S_w , later. The behavior of foam texture within the first second could be due to the numerical scheme used by COMSOL or/and the damping term $\exp(-0.03 \times (x - x_0)^2)$ in the initial perturbation, as we discussed earlier.

Fig. 15 Nonlinear stability of state $(S_w, n_D, f_w)^0 = (0.058585, 0.652177, 0.05)$ for $u = 5.29 \times 10^{-5}$ m/s ($C_c = 1$) with artificial diffusion ($D = 2 \times 10^5$ m²/s) determined using COMSOL. The initial perturbation is given in (19) with $x_0 = 20$. Thin black curve corresponds to $t = 1$ s and the thick black curve corresponds to $t = 600$ s. Total medium length is 100 m

Fig. 16 Nonlinear instability of state $(S_w, n_D, f_w)^0 = (0.058585, 0.652177, 0.05)$ for $u = 5.29 \times 10^{-5}$ m/s ($C_c = 0.01$) determined using COMSOL. The initial perturbation is given in (19) with $x_0 = 50$. Thin black curve corresponds to $t = 0$ s and the gray curve corresponds to $t = 10$ s. Total medium length is 100 m

Artificial diffusion has the same effect for the strong-foam state defined with original equations as for the linearized system (Fig. 9). According to Fig. 15, artificial diffusion controls the instability of the strong-foam state.

However, for slower foam kinetics as seen in Fig. 16, obvious instabilities appear even after 10 s for the strong-foam state. This means that the instability of strong foam is a more severe issue if foam kinetics is slower.

3 Discussion

Kam (2008) notes that his parameter fit is one out of several possible fits on the experimental data. In this study, we show that the stability of most foam states and the instability of middle states can well be predicted by this model. However, some instabilities that we find in this article could be limited to the model and parameters used and not necessarily describe foam behavior in reality. This and related models (cf. Kam and Rossen 2003; Kam et al. 2007; Kam 2008; Afsharpoor et al. 2010) are the only ones available now that describe multiple foam

states and the phenomenon of foam generation with increasing superficial velocity. Moreover, Kam developed his model to remedy an instability in an earlier version. It is important to fully explore the implications of foam models for scale-up beyond the laboratory to the field scale.

We find that instability increases as foam kinetics is slowed (C_c and C_g reduced). Another study (Ashoori et al. 2011b) finds that kinetic parameters in the low range of those tested are needed to represent an entrance region of order cm long in laboratory corefloods. Fitting such an entrance region could lead to problems in the stability of the strong-foam state for that choice of parameters.

4 Conclusions

We derive the equations needed to investigate the stability of the LE foam states. We also describe the effect of the numerical scheme on the stability of the simulated LE states. We apply our non-LE analysis to the model of Kam (2008), but it could be applied to any foam model. The results on states in LE foam models, however, are applicable to any model as long as it assumes LE condition everywhere. We reach the following conclusions:

- Uniform states in any model that assumes local equilibrium, i.e., $r_g = r_c$ everywhere, is unconditionally stable; i.e., LE perturbations that maintain local equilibrium always damp-out over time for such a foam model.
- As a corollary, uniform states in a population-balance model that postulates a fixed maximum value of n_D is unconditionally stable where that limit is enforced.
- In the model of Kam (2008), the LE strong-foam state is unstable. The instability is more visible for a long-duration simulation. However, the instability is more severe for slower foam kinetics. In particular, for foam kinetic parameters suggested to fit an entrance region of several cm as in Ashoori et al. (2011b) the strong-foam state is distinctly unstable.
- Diffusion can affect the stability of the strong-foam states. If foam kinetics is not too slow, the instability of the strong-foam state vanishes in presence of diffusion.
- The intermediate-foam state is mostly not stable, a finding that is consistent with experimental results (Gauglitz et al. 2002). The instability of these states are relatively strong and it still remains in the presence of diffusion.
- Some weak-foam LE states are not stable, especially at higher injection velocities.
- For some values of water fractional-flow, there is no stable LE state for this foam model.
- A numerical scheme can mask the intrinsic instability of strong-foam state. Specifically, the unstable strong-foam states are stable with an upwind-finite-difference scheme (if foam kinetics is not too slow). This probably reflects numerical diffusion in the finite-difference scheme.

Appendix A: Model of Kam (2008)

In the model of Kam (2008), lamellae creation (r_g) requires exceeding a minimum pressure gradient; above the threshold, creation increases rapidly with increasing pressure gradient, reflecting lamella mobilization and division, and at high-pressure gradients it reaches a plateau, on the assumption that lamella creation rate must have an upper bound:

$$r_g = \frac{C_g}{2} \left(\operatorname{erf} \left(\frac{\nabla p - \nabla p_o}{\sqrt{2}} \right) - \operatorname{erf} \left(\frac{-\nabla p_o}{\sqrt{2}} \right) \right). \quad (\text{A.1})$$

Table 1 Relative-permeability and capillary-pressure functions, model parameters and some properties used in foam model of Kam (2008)

<i>Relative-permeability functions</i>	
k_{rw}	$0.7888 [(S_w - S_{wc}) / (1 - S_{wc} - S_{gr})]^{1.9575}$
k_{rg}^0	$[(1 - S_w - S_{gr}) / (1 - S_{wc} - S_{gr})]^{2.2868}$
<i>Capillary-pressure function</i>	
P_c	$\sigma (\varphi / k)^{0.5} [(S_w - S_{wc}) / (1 - S_{wc} - S_{gr})]^{-0.2}$
<i>Foam model parameters</i>	
∇p_0	4.2 psi/ft
n	1
C_g / C_c	3.6×10^{16}
C_f	1.535×10^{-16}
C_c	1
<i>Other parameters</i>	
S_{wc}	0.04
S_{gr}	0.0
μ_w	0.001 Pa s
μ_g^0	0.00002 Pa s
k	$30.4 \times 10^{-12} \text{ m}^2$
φ	0.31
S_w^*	0.0585
$n_{\max}$	$8 \times 10^{13} \text{ m}^{-3}$
σ	0.03 N/m

^a Except for Figs. 5 and 16

We follow Kam who unconventionally uses ∇p to mean $|\partial p / \partial x|$ (a scalar, not a vector). Note that the foam model equations in this Appendix require quantities in the units specified in Table 1.

As in other population-balance models (Kovscek and Radke 1994; Kovscek et al. 1995), the rate of film breakage (r_c) is described by a function that approaches infinity as either water saturation or capillary pressure approach a limiting value (Khatib et al. 1988). We modify this function to include dimensionless foam texture $n_D \equiv n_f / n_{\max}$ where $n_{\max}$ is foam texture at minimum bubble size which is thought to correspond to average pore size (Alvarez et al. 2001):

$$r_c = C_c n_{\max} n_D \left(\frac{S_w}{S_w - S_w^*} \right)^n. \quad (\text{A.2})$$

Here $n_{\max}$ is the upper limit to foam texture reflecting a lower limit of bubble size, (Alvarez et al. 2001). We take $n_{\max} = 8 \times 10^{13} \text{ m}^{-3}$ (cf. Kam et al. 2007). Note that Eqs. A.1 and A.2 together with other equations apply for $n_D \leq 1$; if $n_D > 1$, n_D is reset to 1. Thus, n_D is a continuous function but its derivatives are discontinuous.

In Eqs. A.1 and A.2 ∇p_0 , n , C_g , and C_c are model parameters. As in other population-balance models (Hirasaki and Lawson 1985; Falls et al. 1988, 1989; Kovscek and Radke 1994) effective foam viscosity is a shear-thinning function of gas velocity:

$$\mu_g^f = \mu_g^0 + \frac{C_f n_{\max} n_D}{\left(\frac{u_g}{\varphi S_g} \right)^{1/3}} \quad (\text{A.3})$$

where μ_g^f is gas viscosity in the absence of foam and C_f is a model parameter.

Transport of liquid and gas is governed by Darcy's law. Liquid and gas relative-permeability functions and liquid-phase viscosity are assumed to be unaffected by foam. Hence, for

horizontal displacements, neglecting capillary-pressure gradients, gas velocity, and fractional flow of water can be evaluated from

$$u_g = \frac{kk_{rg}^0}{\mu_g^f} \nabla p \quad (\text{A.4})$$

$$f_w = \frac{1}{1 + \frac{\mu_w}{k_{rw}} \frac{k_{rg}^0}{\mu_g^f}}. \quad (\text{A.5})$$

The relative-permeability and capillary-pressure functions and other model parameters and fluid properties are given in Table 1.

Foam texture at local equilibrium (LE) can be obtained by equating Eqs. A.1 and A.2.

$$\begin{cases} \text{for } n_D \leq 1 & n_D^{\text{LE}} = \frac{C_g}{2C_c n_{\max}} \left(\frac{S_w - S_w^*}{S_w} \right)^n \left(\operatorname{erf} \left(\frac{\nabla P - \nabla P_0}{\sqrt{2}} \right) - \operatorname{erf} \left(\frac{-\nabla P_0}{\sqrt{2}} \right) \right) \\ \text{else} & n_D^{\text{LE}} = 1 \end{cases} \quad (\text{A.6})$$

At steady-state, one can determine μ_g^f , u_g and n_D by simultaneously solving Eqs. A.3–A.6, and then construct the fractional-flow curve using Eq. A.5. As shown by Kam (2008) the resulting fractional-flow curves can be quite complex.

Appendix B: Derivation of Linear-Stability Condition in the Presence of Artificial Diffusion

Equations B.1 and B.2 are the two conventional governing equations for foam flow in the presence of artificial diffusion:

$$\varphi \frac{\partial S_w}{\partial t} + u \frac{\partial f_w}{\partial x} + \frac{\partial}{\partial x} \left(\frac{\lambda_w \lambda_g}{\lambda_w + \lambda_g} \frac{dP_c}{dS_w} \frac{\partial S_w}{\partial x} - D \frac{\partial S_w}{\partial x} \right) = 0 \quad (\text{B.1})$$

$$\varphi \frac{\partial}{\partial t} (S_g n_D) + \frac{\partial}{\partial x} \left(u_g n_D - D \frac{\partial (S_g n_D)}{\partial x} \right) = \frac{\varphi}{n_{\max}} S_g (r_g - r_c) \quad (\text{B.2})$$

where D (m^2/s) is the diffusion coefficient.

The linearization procedure is the same as Eqs. 9 and 10 except that the terms $-D \frac{\partial^2 S_w}{\partial x^2} \approx -D \left(\frac{\partial^2 \Delta S_w}{\partial x^2} \right)$ and $-D \frac{\partial^2 (S_g n_D)}{\partial x^2} \approx -D(1 - S_w^0) \frac{\partial^2 \Delta n_D}{\partial x^2} + D n_D^0 \frac{\partial^2 \Delta S_w}{\partial x^2}$ appear in system (10). Therefore, the linear system in the presence of diffusion in terms of dependent variables ΔS_w and Δn_D is as follows

$$\begin{aligned} \frac{\partial}{\partial t} \begin{bmatrix} \Delta S_w \\ \Delta n_D \end{bmatrix} + \begin{bmatrix} A_{11}^0 & A_{12}^0 \\ 0 & A_{22}^0 \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} \Delta S_w \\ \Delta n_D \end{bmatrix} &= \frac{\partial}{\partial x} \left(\begin{bmatrix} B_{11}^0 + D & 0 \\ -D n_D^0 & D(1 - S_w^0) \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} \Delta S_w \\ \Delta n_D \end{bmatrix} \right) \\ &+ \begin{bmatrix} 0 & 0 \\ C_{12}^0 & C_{22}^0 \end{bmatrix} \begin{bmatrix} \Delta S_w \\ \Delta n_D \end{bmatrix}. \end{aligned} \quad (\text{B.3})$$

This system above has the same form of solution as in Eq. 12, but with the following eigenvalues and eigenvectors

$$v_{1,2} = \frac{C_{22}^0 - \kappa^2(B_{11}^0 + D) - \kappa^2 D(1 - S_w^0)}{2} - i\kappa \frac{A_{22}^0 + A_{11}^0}{2} \pm \frac{\sqrt{(C_{22}^0 + \kappa^2(B_{11}^0 + D) - \kappa^2 D(1 - S_w^0) + i\kappa(A_{11}^0 - A_{22}^0))^2 - 4i\kappa(C_{21}^0 C_{22}^0 + \kappa^2 D n_D^0)}}{2} \quad (\text{B.4})$$

$$r_{1,2} = \frac{-i\kappa A_{12}^0}{v_{1,2} + \kappa^2 B_{11}^0 + i\kappa A_{11}^0}. \quad (\text{B.5})$$

Notice that the eigenvectors have the same expression as in (14). Of course, eigenvalues and eigenvectors are identical to the ones in (13) and (14) for $D = 0$.

Appendix C: Derivation of Linear-Stability Condition in Finite-Difference Scheme

Kam (2008) analyzes the stability of some LE states, applying an upwind-finite-difference simulator. Here, we investigate if and how the numerical scheme can change the stability of LE states. As we discussed above, $U = e^{i\kappa x + \nu t} \begin{bmatrix} r \\ 1 \end{bmatrix}$ is the solution of the linearized system (10) rewritten here as

$$\frac{\partial U}{\partial t} + \begin{bmatrix} A_{11}^0 & A_{12}^0 \\ 0 & A_{22}^0 \end{bmatrix} \frac{\partial U}{\partial x} = \frac{\partial}{\partial x} \left(\begin{bmatrix} B_{11}^0 & 0 \\ 0 & 0 \end{bmatrix} \frac{\partial U}{\partial x} \right) + \begin{bmatrix} 0 & 0 \\ C_{21}^0 & C_{22}^0 \end{bmatrix} U \quad (\text{C.1})$$

Recasting this equation in the upwind-finite-difference scheme, the above system becomes

$$\frac{1}{\Delta t} (U_i^{n+1} - U_i^n) + \frac{1}{\Delta x} \begin{bmatrix} A_{11}^0 & A_{12}^0 \\ 0 & A_{22}^0 \end{bmatrix} (U_i^n - U_{i-1}^n) - \frac{1}{(\Delta x)^2} \begin{bmatrix} B_{11}^0 & 0 \\ 0 & 0 \end{bmatrix} (U_{i+1}^n - 2U_i^n + U_{i-1}^n) - \begin{bmatrix} 0 & 0 \\ C_{21}^0 & C_{22}^0 \end{bmatrix} U_i^n = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (\text{C.2})$$

where subscript i is grid-block counter in x direction and superscript n is time counter. The increments in x and t are Δx and Δt . Inserting the oscillatory solution U in Eq. C.2 we obtain

$$\left\{ \begin{aligned} & \frac{1}{\Delta t} (e^{i\kappa x + \nu(t+\Delta t)} - e^{i\kappa x + \nu t}) I + \frac{1}{\Delta x} (e^{i\kappa x + \nu t} - e^{i\kappa(x-\Delta x) + \nu t}) \begin{bmatrix} A_{11}^0 & A_{12}^0 \\ 0 & A_{22}^0 \end{bmatrix} \\ & - \frac{1}{(\Delta x)^2} (e^{i\kappa(x+\Delta x) + \nu t} - 2e^{i\kappa x + \nu t} + e^{i\kappa(x-\Delta x) + \nu t}) \begin{bmatrix} B_{11}^0 & 0 \\ 0 & 0 \end{bmatrix} - e^{i\kappa x + \nu t} \begin{bmatrix} 0 & 0 \\ C_{21}^0 & C_{22}^0 \end{bmatrix} \end{aligned} \right\} \times \begin{bmatrix} r \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (\text{C.3})$$

where I is a 2×2 unit matrix. Reordering the terms, Eq. C.3 reduces to

$$\left[\begin{aligned} & \frac{(e^{\nu \Delta t} - 1)}{\Delta t} + \frac{(1 - e^{-i\kappa \Delta x})}{\Delta x} A_{11}^0 - \frac{(e^{i\kappa \Delta x} - 2 + e^{-i\kappa \Delta x})}{(\Delta x)^2} B_{11}^0 \\ & - C_{21}^0 \end{aligned} \quad \begin{aligned} & \frac{(1 - e^{-i\kappa \Delta x})}{\Delta x} A_{12}^0 \\ & \frac{(e^{\nu \Delta t} - 1)}{\Delta t} + \frac{(1 - e^{-i\kappa \Delta x})}{\Delta x} A_{22}^0 - C_{22}^0 \end{aligned} \right] \times \begin{bmatrix} r \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (\text{C.4})$$

This system has a nontrivial solution if the determinant of the matrix of coefficients is zero. Solving that quadratic equation we find the two roots for ν .

$$\nu_{1,2} = \frac{1}{\Delta t} \ln \left(1 - \frac{1}{2} \left[\frac{(1-e^{-ik\Delta x})}{\Delta x} A_{22}^0 - C_{22}^0 + \frac{(1-e^{-ik\Delta x})}{\Delta x} A_{11}^0 - \frac{(e^{ik\Delta x}-2+e^{-ik\Delta x})}{(\Delta x)^2} B_{11}^0 \right] \pm \sqrt{\frac{\Delta t^2 \left[\frac{(1-e^{-ik\Delta x})}{\Delta x} A_{22}^0 - C_{22}^0 + \frac{(1-e^{-ik\Delta x})}{\Delta x} A_{11}^0 - \frac{(e^{ik\Delta x}-2+e^{-ik\Delta x})}{(\Delta x)^2} B_{11}^0 \right]^2}{-4 \left[\Delta t^2 \left[\frac{(1-e^{-ik\Delta x})}{\Delta x} A_{11}^0 - \frac{(e^{ik\Delta x}-2+e^{-ik\Delta x})}{(\Delta x)^2} B_{11}^0 \right] \left[\frac{(1-e^{-ik\Delta x})}{\Delta x} A_{22}^0 - C_{22}^0 \right] + C_{21}^0 \frac{(1-e^{-ik\Delta x})}{\Delta x} A_{12}^0 \Delta t^2 \right]} \right) \right) \quad (\text{C.5})$$

In the limit of infinitely small Δx and Δt with fixed ratio, the finite-difference eigenvalues approach the roots of Eq. 12.

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