BISTABLE SOLITONS IN SINGLE- AND MULTICHANNEL WAVEGUIDES WITH THE CUBIC-QUINTIC NONLINEARITY

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We consider spatial solitons in a channel waveguide or in a periodic array of rectangular potential wells (the Kronig–Penney (KP) model) in the presence of the uniform cubic-quintic (CQ) nonlinearity. Using the variational approximation and numerical methods, we find two branches of fundamental (single-humped) soliton solutions. The soliton characteristics, in the form of the integral power Q (or width w) vs. the propagation constant k , reveal a strong bistability with two different solutions found for a given k . Violating the known Vakhitov–Kolokolov criterion, the solution branches with $dQ/dk > 0$ and $dQ/dk < 0$ are simultaneously stable. Various families of higher-order solitons are also found in the KP version of the model: symmetric and antisymmetric double-humped solitons, three-peak solitons with and without the phase shift π between the peaks, etc. In a relatively shallow KP lattice, all the solitons belong to the semi-infinite gap beneath the linear band structure of the KP potential, while finite gaps between the bands remain empty (solitons in the finite gaps can be found if the lattice is much deeper). But in contrast to the situation known for the model combining a periodic potential and the self-focusing Kerr nonlinearity, the fundamental solitons fill only a finite zone near the top of the semi-infinite gap, which is a manifestation of the saturable character of the CQ nonlinearity.

Keywords: spatial solitons, Kronig–Penney model, Vakhitov–Kolokolov criterion

1. Introduction

Spatial solitons in nonlinear optical waveguides with diverse nonlinearities have attracted a great deal of attention [1]–[3]. In theoretical studies, the solitons are usually obtained as solutions of the nonlinear Schrödinger (NLS) equation with various nonlinearities. Starting from the early works [4], the NLS equations are derived from Maxwell’s equations for a nonlinear material medium using the paraxial approximation.

A related topic is the study of spatial solitons in channel-type waveguide structures in nonlinear planar media. In the case of the Kerr (cubic) nonlinearity, the solutions of the nonlinear equation are in one-to-one correspondence with the modal eigenfunctions of the linear channel waveguide and can be classified according to the number of zeros in the solution [5]. As in linear channel waveguides, a given number of zeros defines a single eigenfunction.

In the search for waveguide structures with enhanced functionality, it is natural to consider a combination of channel waveguides and saturable nonlinearities, among which the simplest is the cubic-quintic (CQ) one. The study of solitons in waveguides with saturable nonlinearities is also an interesting problem from the standpoint of general soliton theory. In particular, we demonstrate that the abovementioned one-to-one correspondence between the eigenmodes and the number of zeros, which is a quasilinear feature, does not

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hold in this case, and a much larger variety of soliton states exists in the CQ waveguide, including two different branches of stationary stable solitons and breathers.

Based on experimental observations, the CQ nonlinearity was proposed as an empirical description of the nonlinear dielectric response of chalcogenide glasses [6] and organic materials [7] (although considerable two-photon absorption occurs in the latter case). In addition to that, the CQ nonlinearity can be regarded as the simplest phenomenological model of photorefractive media, which provide a universal setting for experimental studies of spatial solitons (see, e.g., [3] and the references therein).

On the theoretical side, one-dimensional [8] and multidimensional [9] solitons in uniform CQ media have been studied in many works (see the review in [10]), including the case where the effective dispersion is generated not by the optical material but by a Bragg grating [11].

A combination of the CQ nonlinearity with waveguide structures was proposed very recently [12]. Our objective in the present work is to find various species of solitons in the CQ model combined with a single waveguide or with a periodic array of waveguides. We demonstrate that the guided CQ solitons are bistable: two distinct ones with different powers and widths can be found for a given value of the propagation constant. An intriguing feature of these solitons is that they apparently violate the well-known Vakhitov–Kolokolov (VK) stability criterion [13], which is often considered a universal necessary stability condition [14]: the criterion predicts that one branch of the trapped CQ solitons may be stable and the other must be unstable, but in direct simulations *both* branches appear to be *completely stable*. This property, obviously, is quite promising for applications, such as switching. Moreover, solitons of either type can easily self-trap from an arbitrary input beam, which demonstrates their true robustness.

Depending on the input parameters, still another possible outcome is self-trapping of stable breathers, while decay of the initial beam into radiation never occurs. In addition to the bistability, a feature of the trapped CQ solitons that is quite promising for applications is that the edges can be much sharper in them than in their free-space counterparts.

In the case of the multichannel waveguide, we produce several species of stable solitons in the CQ-nonlinear KP model: together with two branches of the fundamental (SH) solitons, as in the single-channel model, various types of higher-order stable soliton are found, including symmetric and antisymmetric double-humped (SDH and ASDH) ones, two kinds of three-humped (TH) solitons with the phase difference zero or π between the central and side peak, and more.

A principal difference from the single-channel model, in which the integral power Q of the SH soliton takes all the values from 0 to ∞ , is that the SH soliton exists up to the finite maximum value $Q_{\max}^{(\text{CQ})}$ in the KP system, above which only multihumped solitons can be found. In fact, the KP model features a unique beam-splitting property, which may be of obvious interest for applications: solitons of any type with a given number of the peaks are stable, but they exist in a finite range of Q ; as the integral power exceeds the abovementioned maximum value, additional peaks, corresponding to subpulses trapped in local channels of the periodic structure, consecutively appear.

A related distinctive feature of the CQ model with the periodic structure is the band structure of the soliton solutions. We demonstrate that as in the model combining the self-focusing Kerr nonlinearity and a periodic potential, all the solitons in a relatively weak lattice fall into a semi-infinite gap under the bottom of the linear band structure corresponding to the given periodic potential, while finite gaps between the linear bands remain empty (solitons appear in them if the lattice is much stronger). But in contrast to the situation in the Kerr model, the fundamental-soliton band is finite itself, being localized near the top of the semi-infinite gap.

2. The model

We start with the paraxial NLS equation that includes the CQ nonlinearity and a normalized potential

term describing a channel waveguide of the simplest step-index type:

$$i\frac{\partial\Psi}{\partial z} + \frac{\partial^2\Psi}{\partial x^2} = U(x)\Psi - 2|\Psi|^2\Psi + |\Psi|^4\Psi. \quad (1)$$

Here, Ψ is the local amplitude of the transverse electric wave with the electric field directed perpendicular to the plane (x, z) , z and x are the longitudinal and transverse coordinates, the cubic and quintic nonlinear coefficients can both be scaled as written in Eq. (1), and the effective waveguide potential is

$$U(x) = \begin{cases} -U, & |x| \leq d, \\ 0, & |x| > d, \end{cases} \quad (2)$$

where d and $U > 0$ are the respective waveguide half-width and strength (depth of the potential well, which is proportional to the difference of the squared refractive index values in the core and cladding). In the case of a multichannel waveguide, we assume the periodic modulation of $U(x)$ (with a period L) in the form of a periodic array of waveguide cores of the width D , separated by buffer layers of the width $L - D$,

$$U(x) = \begin{cases} -U, & LN + D < |x| < L(N + 1) - D, \\ 0, & LN < |x| < LN + D, \end{cases} \quad (3)$$

where N is a positive integer.

Stationary solutions are sought as $\Psi(x, z) = R(x)e^{ikz}$, where $R(x)$ is a real function and k is the propagation constant (k must be positive for soliton solutions). Substituting this waveform in Eq. (1) yields a stationary equation (where the prime denotes d/dx),

$$-kR + R'' = U(x)R - 2R^3 + R^5. \quad (4)$$

An exact solution for the CQ soliton in the free space ($U = 0$) is well known,

$$R^2 = \frac{2k}{1 + \sqrt{1 - 4k/3} \cosh(2x\sqrt{k})}, \quad (5)$$

where the propagation constant takes values from $0 < k < 3/4$. In the presence of the waveguide ($U > 0$), it can be found from the stationary version of Eq. (1) that the soliton peak power $R^2(x = 0)$ takes values in the interval

$$\frac{3}{2} \left[1 - \sqrt{1 - \frac{4}{3}(k - U)} \right] < kR^2(0) < \frac{3}{2} \left[1 + \sqrt{1 - \frac{4}{3}(k - U)} \right].$$

Important characteristics of the soliton are its integral power $Q = \int_{-\infty}^{+\infty} R^2(x) dx$ and its half-width, which we define as

$$w = Q^{-1} \int_0^\infty R^2 x dx. \quad (6)$$

In the absence of the waveguide, $Q(k)$ can be easily calculated for exact soliton solution (5),

$$Q_0(k) = \frac{\sqrt{3}}{2} \log \left(\frac{\sqrt{3} + 2\sqrt{k}}{\sqrt{3} - 2\sqrt{k}} \right). \quad (7)$$

The VK stability condition has the form $dQ/dk > 0$ [13], [14]. As follows from Eq. (7), this condition holds for the whole soliton family in the case where $U = 0$. In accordance with this, it is known that all these solitons are indeed stable.

Fig. 1. A typical example of the soliton pair found at $U = 0.7$, $D = 3$, and $k = 1$ in the single-channel model. The solitons (a) and (b) belong to the lower and upper solution branches in Fig. 2.

2.1. Stationary solitons in a single-channel waveguide. Using the shooting method to solve Eq. (4) numerically, we obtain a pair of soliton solutions for a given value of k . A typical example of the soliton shapes is shown in Fig. 1. For the global characteristic of the whole family of the soliton solutions, we show the integral power Q vs. k in Fig. 2. These figures clearly show the bistability of the CQ solitons: for a fixed value of the propagation constant, two different solitons with different widths and different integral powers can be found, which is impossible in the absence of the guiding structure. It is obvious that the bistability occurs precisely in the wavenumber region $k > 3/4$, where the free-space CQ solitons do not exist. The bistability may be of obvious interest for applications, such as switching, provided both branches of the solutions are stable, which is indeed the case (see below). Another noteworthy feature is that the maximum value of k is essentially larger in the trapped-soliton family than in the free space. Because the soliton field decays as $e^{-\sqrt{k}|x|}$ as $|x| \rightarrow \infty$, this implies that the trapped soliton can have sharper edges than the free one, which offers an advantage for designing multichannel systems using spatial solitons [15]. In this connection, it is also essential that the largest value of k is attained by the free soliton in the limit of $k = 3/4$, where its width diverges, while for the trapped solitons, the largest k is attained at a finite width.

The VK stability condition predicts that soliton families with $dQ/dk < 0$ are unstable [13], [14]. This criterion hence implies that the branches of the solution branch with negative slope in Fig. 2 are unstable. But it has not been proved that the VK condition is applicable to the present model, and the numerical results directly contradict the prediction: both branches turn out to be *completely stable* in simulations of their perturbed evolution and also in the sense that either of them can easily self-trap from an initial localized beam (a detailed analysis of the self-trapping can be found in [12]).

Actual simulations of the perturbed soliton evolution were performed using the split-step method [16]. These simulations show that all the perturbed solitons, irrespective of their belonging to the upper or lower branch in Fig. 2, remain stable after having passed at least the distance $z = 80$ in the present notation. This implies the stability of the propagation through 50 soliton periods, which is more than sufficient for observing the solitons experimentally and for using them in applications.

2.2. Solitons in a multichannel waveguide. In the single-channel model briefly outlined above, the trapped CQ solitons can be found not only numerically but also by using the variational approximation (see [12], [17]). Proceeding to the KP model (the one with periodic potential (3)), we chose to focus on numerical results because the KP potential makes the variational expressions too cumbersome.

As in the case of the single-channel model, solving stationary equation (4) numerically yields a single

Fig. 2. The soliton integral power Q vs. the propagation constant k for the fixed channel width $D = 3$ and different values of its depth U in the single-channel model. For $U = 0$ (free solitons), the dependence $Q(k)$ is given by exact expression (7) (the dashed line).

Fig. 3. Two typical examples of the fundamental soliton in the model with the periodic KP potential. Both solitons pertain to the propagation constant $k = 1$ (the solitons with the larger and smaller amplitudes pertain to the respective upper and lower branches of the curve $Q(k)$ in Fig. 4). The underlying KP potential $U(x)$ is shown in the lower part of the figure.

fundamental (SH) solution for a given k if it is less than a certain threshold value k_{thr} , there are two coexisting SH solitons in the region $k_{\text{thr}} < k < k_{\text{max}}$, and no solution is found for $k > k_{\text{max}}$. A typical example of a pair of the fundamental (SH) solitons in the bistability range with different amplitudes and widths for the same k are shown in Fig. 3, and an example of the curve $Q(k)$ for the SH soliton family is shown in Fig. 4.

It is obvious from Fig. 4 that the upper branch of the SH-soliton family *does not* satisfy the VK stability criterion $dQ/dk > 0$ [13], similar to what was observed above in the single-channel model. Nevertheless, both computing the stability eigenvalues (in the framework of linearized equation (1)) and simulating the evolution directly demonstrate clearly that the entire upper branch (as well as its lower counterpart) is stable. The VK criterion thus fails in the CQ model with both the single-channel and KP potentials. In this connection, we again stress that the criterion has never been proved valid for this type of system, to

Fig. 4. Generic examples of the curves $Q(k)/N$ showing the integral power per hump for the families of SDH and ASDH ($N=2$) solitons in the model with the KP potential are shown together with that for the fundamental SH ($N=1$) solitons. The parameters are $U = 0.7$ and $D = L - D = 3$.

the best of our knowledge. In any case, this result demonstrates real validity limits for the VK criterion.

An important issue is to identify the location of the solitons in the k space relative to the band structure of the linear KP model. The corresponding quasiperiodic Bloch modes have the usual form $R(x) = e^{iqx}P(x)$, where q is a real quasimomentum (in terms of quantum mechanics) and $P(x)$ is a periodic function with the period L . As is known [18], the band structure of KP potential (3), $k = k(q)$, is determined by an exact relation

$$\frac{k - \tilde{k}}{2\sqrt{k\tilde{k}}} \sinh((L - D)\sqrt{k}) \sin(D\sqrt{\tilde{k}}) + \cosh((L - D)\sqrt{k}) \cos(D\sqrt{\tilde{k}}) = \cos(Lq), \quad (8)$$

where $\tilde{k} \equiv U - k$. Comparing the numerical results, both for the fundamental SH solitons considered here and the higher-order ones (SDH, ASDH, and TH solitons) considered below, shows that if the lattice strength U is not too large, then none of them is located inside any of finite gaps generated by Eq. (8) between the Bloch bands. Instead, all the solitons lie in the semi-infinite gap under the bottom of the band structure (in particular, the region $k > U$ belongs to the semi-infinite band).

Two families of double-humped solitons, SDH and ASDH solitons, are easily found in the present model, corresponding to the respective even and odd solutions $R(x)$ of Eq. (4). The corresponding solution families can again be described by the corresponding dependences $Q(k)$, which are shown in Fig. 4 along with the $Q(k)$ curve for the SH (fundamental) solitons. Following the pattern of their SH (fundamental) counterparts, each family of the double-humped solitons also features the bistability in the sense of including two different solutions corresponding to the same k from the respective interval $k_{\text{thr}} < k < k_{\text{max}}$. Moreover, *four* different solitons of the SDH type can coexist in a narrow interval of k close to the turning point. As in the case of the SH solitons, the ones shown in Fig. 4 cannot be continued past the points at which they end, $Q = Q_{\text{max}}^{(\text{SDH})}$ and $Q = Q_{\text{max}}^{(\text{ASDH})}$. Typical examples of SDH and ASDH soliton pairs pertaining to a common value of k are shown in Fig. 5. We note that the $Q(k)$ curves for the SDH and ASDH solitons respectively end at $Q_{\text{max}}^{(\text{SDH})} \approx 19.3$ and $Q_{\text{max}}^{(\text{ASDH})} \approx 18.7$, which is almost exactly twice the abovementioned endpoint value $Q_{\text{max}}^{(\text{SH})} \approx 9$ for the SH-soliton branch (pertaining to the same values of the parameters of the KP potential). Together with the shapes of the SDH and ASDH solitons shown in Fig. 5, these observations strongly suggest that the double-humped solitons can be considered bound states of two SH ones with the phase shifts between them respectively 0 and π .

The stability of the double-humped solitons, as in the case of the SH ones, was studied by computing

Fig. 5. Generic examples of (a) SDH and (b) ASDH solitons in the KP model belonging to the families shown in Fig. 4. In each panel, a pair of solitons found at the same propagation constant value $k = 1$ is displayed.

the eigenvalues of small-perturbation modes and by simulating the evolution directly. Both methods showed that the families of SDH and ASDH solitons are completely stable in their entire existence region.

A manifestation of the beam-splitting property of the CQ model with the periodic potential is observed as follows. Above the integral power limit value $Q_{\max}^{(\text{SH})}$, the broadening SH soliton with a bounded amplitude starts to generate extra peaks (humps). For obvious symmetry reasons, this leads to a transition from the SH soliton to a TH one, of which an example is shown in Fig. 6. This example stresses that the formation of the TH soliton is a peculiarity of the periodic potential, while the CQ model with the single-channel potential supports only SH solitons with an arbitrarily large Q . The TH solitons exist in two modifications: with the phase shift π as in Fig. 6 or without the phase shift between the central and side peaks. We note that the corresponding integral power maximum value $Q_{\max}^{(\text{TH})} \approx 27$, compared with the values $Q_{\max}^{(\text{SH})}$ and $Q_{\max}^{(\text{SDH})}$, $Q_{\max}^{(\text{ASDH})}$ found above for the SH and SDH, ASDH solitons with the same KP potential parameters (see Fig. 4), is almost exactly equal to $3Q_{\max}^{(\text{SH})}$ and $(3/2)Q_{\max}^{(\text{SDH})}$, which confirms that the TH soliton is a bound state of three fundamental solitons.

To generate solitons in finite gaps of the KP band structure, it is necessary to increase the lattice strength by a factor ~ 10 compared with the above examples. In such a deep lattice, stable solitons of the SH, SDH, and ASDH types can be found in finite gaps, but they do not have bistability. The shapes of the finite-gap solitons can be drastically different from what was found above in the shallow lattices; as shown in Fig. 7, the solitons found in finite gaps are characterized by a strongly pronounced flat-top profile. Detailed results for this class of the gap solitons will be reported elsewhere.

3. Conclusion

We have demonstrated that properties of spatial solitons in the well-known model with the CQ non-linearity change drastically when a waveguide is introduced. The soliton family becomes bistable: two solitons with different integral powers and different widths can be found for a given value of the propagation constant k . The bistability occurs at precisely those values of k where the free-space solitons do not exist. Formally applying the VK criterion predicted that one of the two soliton branches must be unstable, but direct stability tests disproved this expectation, clearly showing that both branches are stable, which suggests a possibility of use for switching and other photonic-device applications.

In the case of multichannel waveguides (we considered the KP model for this case), the system supports

Fig. 6. A TH stable soliton found at $k = 1$ in the KP model with $U = 1$, $D = 3$, and the buffer layer thickness $L - D = 1$. For comparison, the dashed curve shows an SH soliton found at the same $k = 1$ in the single-channel model.

Fig. 7. An example of a “flat-top” soliton found inside a finite gap of a strong KP periodic potential. The parameters are $U = 6$, $D = 6$, $L = 12$, and $k = 2$.

many soliton states, including the fundamental SH solitons, SDH and ASDH solitons, and TH solitons with and without the phase shift π between the peaks. If the periodic potential is weak, all the solitons fall into the semi-infinite gap beneath the linear band structure of the KP potential, while finite gaps between the Bloch bands remain empty. But in contrast to what was found in the model combining a periodic potential and the self-focusing Kerr nonlinearity, only a finite zone near the top of the semi-infinite gap is filled by the solitons in the CQ model, which is a consequence of the saturable character of the nonlinearity. In the case where the KP potential is stronger (and/or buffer layers between the waveguide cores are broader), fundamental and double (symmetric and antisymmetric) stable solitons with a characteristic flat-top shape are found in the finite gaps separating the Bloch bands in the linear KP model.

Both computing stability eigenvalues and directly simulating perturbed solitons show that they are all stable in the KP model. As in the single-channel case and in contrast to the periodic model with the Kerr nonlinearity, the soliton families feature bistability in the form of two (sometimes even four) different states found for a given propagation constant k with different values of the integral power Q . Violating the VK

criterion, the solution branches with both $dQ/dk > 0$ and $dQ/dk < 0$ are *stable*.

Another distinctive feature of the KP model is the *beam-splitting* property: as Q increases past a critical value, the number of peaks in the soliton shape subsequently increases, each peak corresponding to a subbeam trapped in an individual channel of the periodic structure. This property is explained by an upper bound on the soliton amplitude in the CQ model, which allows increasing the beam power only by broadening the beam. The broadening leads to trapping new subbeams, which form additional peaks in the soliton. The beam-splitting and strong stability of the resulting multipeak solitons are effects of direct interest for applications. It is plausible that characteristic features demonstrated by the present model will be shared by more general models combining a saturable nonlinearity and a periodic potential.

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