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EDITORIAL

Virtue theory of mathematical practices: an introduction

Andrew Aberdein[1] · Colin Jakob Rittberg[2] · Fenner Stanley Tanswell[3]

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Until recently, discussion of virtues in the philosophy of mathematics has been fleeting and fragmentary at best. But in the last few years this has begun to change. As virtue theory has grown ever more influential, not just in ethics where virtues may seem most at home, but particularly in epistemology and the philosophy of science, some philosophers have sought to push virtues out into unexpected areas, including mathematics and its philosophy. But there are some mathematicians already there, ready to meet them, who have explicitly invoked virtues in discussing what is necessary for a mathematician to succeed.

In both ethics and epistemology, virtue theory tends to emphasize character virtues, the acquired excellences of people. But people are not the only sort of thing whose excellences may be identified as virtues. Theoretical virtues have attracted attention in the philosophy of science as components of an account of theory choice. Within the philosophy of mathematics, and mathematics itself, attention to virtues has emerged from a variety of disparate sources. Theoretical virtues have been put forward both to analyse the practice of proof and to justify axioms; intellectual virtues have found multiple applications in the epistemology of mathematics; and ethical virtues have been offered as a basis for understanding the social utility of mathematical practice. Indeed, some authors have advocated virtue epistemology as the correct epistemology for mathematics (and perhaps even as the basis for progress in the metaphysics of mathematics). This topical collection brings together several of the researchers who have begun to study mathematical practices from a virtue perspective with the intention of consolidating and encouraging this trend.

This article belongs to the topical collection "Virtue Theory of Mathematical Practices", edited by Andrew Aberdein, Colin Jakob Rittberg, & Fenner Stanley Tanswell.

✉ Andrew Aberdein
aberdein@fit.edu

1 School of Arts and Communication, Florida Institute of Technology, 150 West University Blvd, Melbourne, Florida 32901-6975, USA

2 Logic and Philosophy of the Cognitive Sciences, Vrije Universiteit Amsterdam, Amsterdam, The Netherlands

3 Centre for Mathematical Cognition, Loughborough University, Loughborough, UK

1 Virtues of mathematics: theoretical virtues

Theoretical virtues are a well-established component of the philosophy of science, although not always under this designation. But in recent decades, virtue terminology has been more widely embraced. Thus, while W. V. O. Quine was happy to identify conservatism, modesty, simplicity, generality, and refutability as "virtues that a hypothesis may enjoy in varying degrees" (Quine and Ullian 1978), Thomas Kuhn endorsed the properties of accuracy, consistency, scope, simplicity, and fruitfulness merely as "characteristics of a good scientific theory" (Kuhn 1977). But, forty years on, Samuel Schindler could embrace Kuhn's list while unabashedly referring to Kuhn's characteristics as virtues (and adding two more of his own, testability and coherence or non-ad-hocness) (Schindler 2018, p. 5).

While Kuhn and Quine are writing about science in general rather than mathematics in particular, Quine at least can be seen to employ his virtuistic methodology in the appraisal of mathematical theories. Or so Lieven Decock has argued. In his account of Penelope Maddy's critique of Quine's defence of $V = L$ as a set-theoretic axiom, Decock observes that the appeal to theoretical virtues is shared by Quine and Maddy: "Maddy has presented a whole inventory of rules of thumb, which are in fact epistemic virtues, of set-theorists" (Decock 2002, p. 12). This inventory includes limitation of size, iterative conception, one step back from disaster, maximize, realism, whimsical identity, inexhaustibility, uniformity, reflection, generalization, richness, and resemblance (Maddy 1988). Nowhere in this paper does Maddy herself explicitly denominate the properties she discusses as virtues. However, her specific examples often bear up this analysis. For example, even the superficially unpromising "whimsical identity" she illustrates with such arguments as "It would seem rather accidental if $\aleph_0$ can be characterized [thus and so] " (Maddy 1988, p. 502, quoting Kanamori and Magidor 1978). That makes it a close ally of non-ad-hocness. As this example indicates, Maddy's close attention to the reasoning of research mathematicians grounds her appeal to theoretical virtues in actual mathematical practice. This trend has become more overt in more recent work (Maddy 1998, 2011, 2019).

Some specific theoretical virtues have attracted sustained attention from mathematicians and philosophers of mathematics. For example, simplicity in mathematics has been the focus of substantial work: the provision of an unambiguous criterion for mathematical simplicity was to have been the twenty-fourth of David Hilbert's celebrated and influential list of twenty-three important open problems announced at the 1900 International Congress of Mathematicians in Paris (Thiele 2003). In recent years, simplicity has drawn the attention of mathematicians and philosophers (McLarty 2007; Nelson 2007), and been the subject of both an edited volume and a special issue of *Philosophical Transactions of The Royal Society* (Kossak and Ording 2017; Hipólito and Kahle 2019). Fruitfulness has also attracted significant attention (Tappenden 2008; Yap 2011; Carter 2019) and purity has been the subject of multiple studies (Detlefsen and Arana 2011; Baldwin 2013; Ferreirós 2016). Historians of mathematics have traced appeals to purity in the conceptual foundations of mathematics back to the eighteenth century and beyond (Ferraro and Panza 2012). Likewise, depth has lately been much discussed, notably in a special issue of *Philosophia Mathematica* (Ernst et al. 2015b). Therein the historian of mathematics Jeremy Gray traces the appreciation of mathematical depth to Gauss and his successors in the early nineteenth century (Gray 2015). Several recent studies in the philosophy of mathematical practice have made further appeal to depth as a theoretical virtue (Imocrante 2015; Waxman 2021; D'Alessandro xxx). Some theoretical properties are more ambiguous: for instance, Gillian Russell notes that logical strength has been presented variously either as a virtue or as a vice by competing logicians; she argues that it is neither (Russell 2019).

Other philosophers of mathematics have deployed whole calendars of theoretical virtues. For example, Marc Lange asserts that the "many virtues that a mathematical proof may exhibit … include accessibility to a given audience, beauty, brevity, depth, elegance, explanatory power, fruitfulness, generalizability, purity, and visualizability" (Lange 2016, p. 8 f.). Don Berry defends a set of what he calls practical virtues: permanence, reliability, autonomy, and consensus (or PRAC, for short). He contends that "these practical virtues facilitate the flourishing of mathematics as a discipline: the progress of mathematical enquiry and the enormous success of the field" (Berry 2018, p. 114). Daniel Waxman proposes that theoretical virtues provide a resolution for the puzzle of mathematics' unreasonable effectiveness posed by Eugene Wigner—that mathematical results are so useful in science, even when the mathematics was originally pursued with no such application in sight (Wigner 1960). [1] Waxman suggests that the solution to the puzzle lies in the fact that "simplicity, unificatory power, explanatory depth, epistemic tractability, surprisingness, the ability to forge connections between seemingly disparate subject-matters, fruitfulness, etc.—are precisely those often discussed in the philosophy of science, confirmation theory, and more recently within metaphysics too, under the heading of 'theoretical virtues' " (Waxman 2021, p. 15). In other words, since mathematicians and scientists seek to optimize with respect to the same theoretical virtues, it should be unsurprising if they produce structurally similar theories.

Accounts of theoretical virtues in mathematics can also appeal to some recent empirical work. Matthew Inglis and Andrew Aberdein have investigated some of the terms that mathematicians use to describe proofs (Inglis and Aberdein 2015). A sample of more than 250 professional mathematicians were invited to think of a proof and then presented with 80 adjectives known to be used to describe proofs and asked how their chosen proof ranked on a Likert scale for each term. Inglis and Aberdein's factor analysis of the resulting data supported four main factors, which they dubbed the aesthetics, intricacy, utility, and precision dimensions. The terms that loaded strongly onto the aesthetics factor included striking, ingenious, inspired, profound, creative, deep, sublime, innovative, beautiful, elegant, and charming. Dense, difficult, intricate, unpleasant, confusing, and tedious all loaded strongly onto the intricacy factor, whereas simple had a strong negative loading. Practical, informative, efficient, applicable, and useful loaded strongly onto the utility factor. The adjectives which loaded most strongly onto the precision factor included careful, precise, meticulous, and rigorous (Inglis and Aberdein 2015, p. 99 f.). Although it was no part of the study's

[1] A puzzle anticipated by, among others, Charles Babbage: "In mathematical science, more than in all others, it happens that truths which are at one period the most abstract, and apparently the most remote from all useful application, become in the next age the bases of profound physical inquiries, and in the succeeding one, perhaps, by proper simplification and reduction to tables, furnish their ready and daily aid to the artist and the sailor" (Babbage 1830, p. 17 f.).

design, it can be readily seen that many of these terms correspond directly to theoretical virtues discussed above, and others could plausibly be assimilated as additional virtues (or vices). This study thereby suggests a method for imposing a similar fourfold grouping on the diversity of theoretical virtues and vices in mathematics, with a basis in mathematical practice much broader than the intuitions of any individual practitioner. Furthermore, it is notable that several of these adjectives, such as creative, (un)pleasant, efficient, or meticulous, might as readily describe mathematicians as mathematics, thereby indicating an overlap with the topic of the next section.

2 Virtues of mathematicians: character virtues

Virtues may be manifested not only by theories, but also by people. This is, of course, the more familiar application of virtue talk. In philosophy it is a very ancient one, but within recent decades it has undergone a significant resurgence after centuries of neglect. This is true not only of virtue ethics, which seeks an account of right action in the moral virtues of actors, but also of virtue epistemology, which seeks an account of right belief in the intellectual virtues of believers. Both forms of virtue theory have found an application to science, and more specifically to mathematics.

Pierre Duhem is best known to philosophers of science as the originator of the problem of underdetermination: that no amount of data is sufficient to narrow down the choice of theory to exactly one candidate (Duhem 1954). Duhem's own suggested resolution to this problem has led to his identification as a pioneering virtue epistemologist of science. Duhem argued that scientists rely on *le bon sens*, common sense or good sense, in their final choice of theory. David Stump argues that this is best understood as an intellectual virtue possessed by successful scientists; a spirited debate has ensued (Stump 2007; Ivanova 2010, 2011, 2014; Kidd 2011).

While Duhem was a physicist not a mathematician, *le bon sens* would seem necessary for success in mathematics too. But much closer connections to mathematical practice may be drawn. For example, here is the celebrated mathematician and educator George Pólya in the first chapter of one of his two classic books, *Mathematics and Plausible Reasoning*, talking about the "moral qualities" required of a mathematician:

- First, we should be ready to revise any one of our beliefs.
- Second, we should change a belief when there is a compelling reason to change it.
- Third, we should not change a belief wantonly, without some good reason.

He says that "These points sound pretty trivial. Yet one needs rather unusual qualities to live up to them" (Pólya 1954, p. 8). Here are those unusual qualities:

- The first point needs "intellectual courage". You need courage to revise your beliefs. Galileo, challenging the prejudice of his contemporaries and the authority of Aristotle, is a great example of intellectual courage.
- The second point needs "intellectual honesty". To stick to my conjecture that has been clearly contradicted by experience just because it is *my* conjecture would be dishonest.

– The third point needs "wise restraint". To change a belief without serious examination, just for the sake of fashion, for example, would be foolish. Yet we have neither the time nor the strength to examine seriously all our beliefs. Therefore it is wise to reserve the day's work, our questions, and our active doubts for such beliefs as we can reasonably expect to amend. "Do not believe anything, but question only what is worth questioning" (Pólya 1954, p. 8).

So, although Pólya never uses the word "virtue", it is an obvious synonym for "moral quality", especially when you consider which moral qualities he has in mind. Courage and honesty are classic examples of character virtues. Wise restraint sounds much like what Aristotle would call phronesis, more frequently translated as practical wisdom or common sense. [2] Phronesis has a central role among Aristotle's intellectual virtues. Since Aristotle's ethical virtues are means between vices of excess and deficiency, a virtuous agent must have the faculty of reliably identifying such means, and phronesis is that faculty. So Pólya is at least echoing a virtue theory. Unfortunately he doesn't develop it. This passage is taken from the very start of the book; there is no subsequent reference to "moral qualities" anywhere else in either volume. The virtue talk is apparently intended as a sort of exhortation; perhaps it is implicit in the rest of the book, but it is not invoked directly.

Several specific issues in the philosophy of mathematics have also attracted a virtue theoretic treatment. Ernest Sosa has defended an account of a priori knowledge as grounded in a reliable epistemic virtue of rational intuition (Sosa 2007). Sosa's account has found explicit application to several issues in the philosophy of mathematics, including the explication of epistemically lucky mathematical statements (Miščević 2007) and the justification of mathematical axioms (Clemente 2016). More broadly, Fenner Tanswell has made the case that virtue epistemology provides a suitable account of mathematical epistemology, especially knowledge from mathematical proofs (Tanswell 2016). The central idea is that *mathematical rigour* is seen as a property that straddles the proof itself and the virtuous mathematical agent (in the same manner as qualities such as creative and meticulous, as mentioned above). Hence, to gain mathematical knowledge from a proof, we need to carry out the reasoning activity for which the proof provides a recipe in a suitably *rigorous* fashion.

Alasdair MacIntyre is a major figure in the contemporary revival of virtue theory. His work has been found useful by several researchers seeking to extend virtues to mathematics. While MacIntyre is primarily an ethicist, his work is informed by a deep engagement with the history of ideas, and in particular by accounts of theory change in the philosophy of science (MacIntyre 1977). That gives it a deeper relevance to the philosophy of mathematical practice than superficially more closely related work in virtue epistemology. Most centrally, MacIntyre has much to say about the nature of a practice. For MacIntyre, a practice is

> any coherent and complex form of socially established cooperative human activity through which goods internal to that form of activity are realized in the course

[2] Thereby perhaps suggesting a link to Duhem's *le bon sens*; although see (Estrada Olguin 2017) for an argument that the latter should rather be assimilated to another of Aristotle's intellectual virtues, noûs.

> of trying to achieve those standards of excellence which are appropriate to, and partially definitive of, that form of activity, with the result that human powers to achieve excellence, and human conceptions of the ends and goods involved, are systematically extended (MacIntyre 1984, p. 187).

Practices are embedded in the narratives of human lives, which in turn comprise multigenerational traditions. At the level of practice, a virtue may then be approximated as "an acquired human quality the possession and exercise of which tends to enable us to achieve those goods which are internal to practices and the lack of which effectively prevents us from achieving any such goods" (MacIntyre 1984, p. 191). However, such qualities will only ultimately count as virtues if they also contribute to the narrative and tradition stages.

MacIntyre draws a threefold distinction between encyclopaedic, genealogical, and tradition-constituted forms of enquiry (MacIntyre 1990). The encyclopaedist perceives the object of enquiry as a rational, objective structure of facts governed by laws, and thereby understands the task of enquiry as that of revealing these facts and laws. The genealogist rejects that Enlightenment picture as a smokescreen concealing a network of power relationships, the uncovering of which is the true task of enquiry. MacIntyre's preferred approach rejects both of these alternatives in favour of recovering a pre-Enlightenment perspective on intellectual enquiry as a congeries of overlapping practices, each with its own internal goods, to which the practitioner must be acculturated, characteristically by a process of apprenticeship.

David Corfield has argued that MacIntyre's account of a tradition of inquiry provides an effective normative framework for the analysis of mathematical practice (Corfield 2012). For a start, we may perceive an instructive analogy between MacIntyre's threefold account of enquiry and different schools of thought in contemporary philosophy of mathematics: foundational encyclopaedists; sociological genealogists; and tradition-constituted philosophers of mathematical practice (Corfield 2012, p. 250 f.). Most importantly, traditions supply their constituent practices with a telos, or goal. For Corfield, the telos of the mathematical tradition is understanding (Corfield 2012, p. 256). Only in that context, he suggests, do many individual mathematical practices, such as seeking out more explanatory proofs of already settled results, make sense.

The mathematician Michael Harris shares Corfield's enthusiasm for a MacIntyrean account of mathematical practice. He makes an overt contrast with what he perceives as an unsatisfying account of mathematics in terms of theoretical virtues:

> Pure research in mathematics as in other fields is *good* because it often leads to useful practical consequences; it is *true* because it offers a privileged access to certain truths; it is *beautiful*, an art form. To claim that these virtues are present in mathematics is not wrong, but it sheds little light on what is distinctively *mathematical* and even less about pure mathematicians' *intentions* (Harris 2015, ix f.).

Instead, he proposes a set of "virtues rather different from those usually invoked", including

> the sense of contributing to a meaningful *tradition*, which entails both an attention to past achievements and an orientation to the future that is particularly

> pronounced in the areas of number theory to which my work is devoted; the participation in what has been described, in other settings, as a *relaxed field*, not subject to the pressures of material gain and productivity; and the pursuit of *pleasure* of an elusive, but nevertheless specific, kind (Harris 2015, x f.).

MacIntyre's work has also found application in mathematics education (Thornton 2016, for example).

One core part of the recent trend towards examining the intersection between epistemology and ethics through the lens of virtue theory has been the rapidly growing literature on epistemic injustice (Fricker 2007; Kidd et al. 2017). Epistemic injustice is injustice along a specifically epistemic dimension, e.g. when someone's testimony is not trusted because of their race or gender. Miranda Fricker argues that one way to address this is to develop the virtue of epistemic justice. Work on epistemic injustice has been applied to mathematics (Rittberg et al. 2020) and mathematics education (Tanswell and Rittberg 2020). In the former, the authors explore cases across modern and historical mathematics where mathematical practices generate epistemic injustices, and the impact this has on the mathematics that is produced. In the latter paper, the authors look at how epistemic injustice fits with existing literature on mathematics teaching and social justice. They make use of the work of Max Weber (2009) on the conflicts of norms and values that arise from taking on different social roles, as framed in (Larvor 2020), with the idea that clashes between the roles of research mathematicians, teachers of mathematics, and mathematics students, can give rise to epistemic injustice, and that we can use tools from virtue theory to begin to address this. Today, epistemic injustices in mathematics are the focus of a dedicated research project (Rittberg 2020).

Justice is a very familiar virtue; much less familiar is the controversial Aristotelian virtue of *megaloprepeia*, or magnificence. But even this virtue has found mathematical application: Harris invokes it to characterize the increasingly intimate relationship between mathematical research and the financial institutions that rely on ever more sophisticated mathematical methods. As he sardonically observes, the relationship could be seen as

> an exponentially virtuous circle: academic mathematics departments host finance mathematics programs that generate the UHNWI [ultra-high-net-worth individuals] within financial institutions and they, in turn, provide the "external goods" necessary to maintain the practice of pure mathematics, a kind of perpetual *megaloprepeia* machine from which the Columbia math department even manages to extract a limitless cornucopia of fresh fruit (Harris 2015, 105).

Megaloprepeia has sometimes been perceived as out of place in Aristotle's system of virtues, not least since, by giving a disproportionate role to private philanthropy, it represents a "capture of the community in private hands" (Ward 2011, p. 275). While this clearly poses a risk, it is exactly the risk which Aristotle's virtue of magnificence is intended to allay; *megaloprepeia* should represent virtuously conducted philanthropy (Athanassoulis 2016, p. 790 ff.). And, as Harris concedes, mathematical philanthropists have exercised the virtue well, or at least have outperformed many public funding agencies: "The deeper irony is that the (ostensibly) democratically based social institutions of government are perceived as less sympathetic to the ‘internal goods’ of mathematical practice than the structures of *megaloprepeia* endowed by Powerful Beings like Clay, AIM, or Simons” (Harris 2015, p. 106).

For Aristotle, famously, human flourishing, or eudaimonia, is “an activity of the soul in accordance with virtue, or if there are more kinds of virtue than one, in accordance with the best and most perfect kind” (Aristotle 1976, 1098a). There is a long tradition identifying mathematics as contributing to such activity. Alan Nelson, for example, has argued that for Descartes it was these aspects of mathematics that were predominant—and not, as might be supposed, its applications to science (Nelson 2019); in her paper in this collection, Laura Kotevska makes a similar case for Arnauld and Nicole. The contemporary mathematician Francis Su has a similarly eudaimonic perspective on mathematics. His retiring presidential address to the Mathematical Association of America and the paper and book based upon it share the title, “Mathematics for Human Flourishing” (Su 2017, 2020). For Su, human flourishing comprises “a wholeness—of being and doing, of realizing one’s potential and helping others do the same, of acting with honor and treating others with dignity, of living with integrity even in challenging circumstances” (Su 2020, p. 10). Like his seventeenth-century predecessors, he argues that “the pursuit of math can, if grounded in human desires, build aspects of character and habits of mind that will allow you to live a more fully human life and experience the best of what life has to offer” (Su 2020, p. 12). Hence Su’s approach to mathematics is fundamentally virtue-theoretic: he claims “that the proper practice of mathematics cultivates virtues that help people flourish. These virtues serve you well no matter what profession you chose or where your life takes you. And the movement toward virtue is aroused by basic human desires—the universal longings that we all have—which fundamentally motivate everything we do” (Su 2020, p. 10). Su’s initial list of desires comprised play, beauty, truth, justice, and love (Su 2017); he later expanded it to include exploration, meaning, permanence, struggle, power, freedom, and community (Su 2020). This imposes a structure on his discussion of individual virtues: for example, he links the virtues of imagination, creativity, and expectation of enchantment to the desire for exploration; and the virtues of endurance, unflappable character, competence to solve new problems, self-confidence, and mastery to the desire for struggle. In total, Su associates more than sixty character virtues with aspects of mathematical practices: perhaps the most extensive such survey to date.

3 The papers

The contributors to this topical collection address many of the different issues discussed above: assessing the merit of set-theoretic axiom candidates in terms of theoretical virtues; reflecting on what we can learn from MacIntyre for the study of mathematical practices; exploring the virtue-theoretic thinking of early modern mathematicians; applying virtue theory to questions on testimony in mathematics; proposing “mathematizing” as a virtuous practice; and discussing specific epistemic virtues, such as intellectual generosity and intellectual humility. The contributions also open debates on some largely unexplored questions about the virtue-theoretic study of mathematical practices: how the core concepts of virtue theory relate to the study of mathematical practices; how specific virtues manifest in mathematical practices; and how mathematical practices connect with social responsibility. Together, the contributions show the breadth and indicate the depth of virtue-theoretic studies of mathematical practices. Below we give brief summaries of each contribution, ordered by the date they first appeared online.

In their paper, "Mathematical practice and epistemic virtue and vice", Fenner Stanley Tanswell and Ian James Kidd pose a series of foundational questions for any virtue theory of mathematics: What sorts of epistemic virtues are required for effective mathematical practice? Should these be virtues of individual or collective agents? What sorts of corresponding epistemic vices might interfere with mathematical practice? How do these virtues and vices of mathematics relate to the virtue-theoretic terminology used by philosophers? They address these questions in order to explore how the richness of mathematical practices is enhanced by thinking in terms of virtues and vices, and how the philosophical picture is challenged by the complexity of the case of mathematics. For example, within different social and interpersonal conditions, a trait often classified as a vice might be epistemically productive and vice versa. They illustrate that this occurs in mathematics by discussing an historical study of the aggressive adversarialism of the Gelfand seminar in post-war Moscow (Gerovitch 2016). They take this example to demonstrate that virtue epistemologies of mathematics should avoid pre-emptive judgements about the sorts of epistemic character traits that ought to be promoted and criticised.

Rebecca Lea Morris, in her "Intellectual generosity and the reward structure of mathematics", presents intellectual generosity as a means to ameliorate problems with the theorem-credit economy in mathematics. She argues that Roberts and Wood's (2007) account of intellectual generosity suitably captures the kind of generosity William Thurston manifested in his mathematical work where he willingly shared intrinsic and extrinsic intellectual goods with his fellow practitioners. In particular, intellectual generosity led him to produce expository work in mathematics. From this case study Morris draws valuable lessons about the benefits of the virtue to the practice. Her focus is the reward structure of mathematics, in which points are scored mainly by being the first to prove a theorem and much less through expository mathematical work. Morris makes the case that, because mathematics has become hyper-specialised whilst at the same time mathematical progress often involves cross-fertilisation between different mathematical fields, expository work is beneficial to progress in mathematics even though there is little reward for it. The Thurston case shows that intellectual generosity fosters expository work in mathematics. Thus, intellectual generosity may ameliorate a problem with the reward structure of mathematics.

In "The role of testimony in mathematics", Line Edslev Andersen, Hanne Andersen, and Henrik Kragh Sørensen provide an explanation for the common practice amongst mathematicians of basing one's beliefs about the correctness of a proof on the testimony of others. The paper builds on and expands earlier work by the first author (Andersen 2017, 2020) which shows that whilst mathematicians regard it as an ideal to check every proof before they rely on it in their own work, this epistemic autonomy is rarely attained. Rather, it is common practice to rely on the testimony of others about the correctness of certain proofs. This opens mathematicians up to the risk of relying on testifiers who have overlooked substantial errors in a proof. The authors argue that the likelihood that there is such a substantial error in the proof decreases with the number of truthful and conscientious experts who have engaged with it. The authors follow this up with an argument that truthfulness and conscientiousness are encouraged by mathematical proving practices. These points help to explain why many mathematicians will require that a number of experts have checked a proof before they rely on the proof without checking it themselves.

John Heron proposes to assess the merit of axiom-candidates in set theory in terms of theoretical virtues in his “Set-theoretic justification and the theoretical virtues”. He points out that contemporary discussions about the foundations of set theory focus on extrinsic evidence for an axiom, which is understood as the best explanation for some given mathematical data. However, no clear account of what is meant by ‘explanation’ is provided in these debates. Heron proposes a virtue-theoretic approach. This raises the question of whether the virtuousness of certain axiom-candidates resides “in the mathematics” or “in us” (Ernst et al. 2015a, b). Heron argues that even if one agrees with (Maddy 2011) that the virtuousness of these candidates resides “in the mathematics”, there remains the Kuhnian point that, because there are multiple virtues at play, they need to be weighed against each other. Since such weighing is done by agents, there is a subjectivity to axiom-choice according to Heron which he connects to the debate about absolutely undecidable set-theoretic propositions.

In “Prolegomena to virtue-theoretic studies in the philosophy of mathematics”, James V. Martin makes the case that a virtue-theoretic philosophy of mathematical practices needs to get a grip on how the virtue and the practice terminology connect. Martin points to the success of the MacIntyrean framework in establishing such a connection for the moral virtues and proposes to adapt the framework to the study of mathematics. He draws on Karin Knorr-Cetina’s account of mathematics as an epistemic objectual practice to recast MacIntyre’s three-tiered understanding of the virtues for the case of mathematics (Knorr-Cetina 2001). Martin’s account points to methodological questions of how mathematical practices ought to be investigated in light of his virtue-theoretic framework, and he offers a number of methodological principles for a realistic study of mathematical practices inspired by Wittgenstein.

Laura Kotevska’s historically minded “Moral improvement through mathematics: Antoine Arnauld and Pierre Nicole’s *Nouveaux éléments de géométrie*” traces seventeenth-century virtue-theoretic thinking about mathematics by mathematicians. She explores what the two Port-Royalists Arnauld and Nicole saw as the propaedeutic value of mathematics. Kotevska highlights the surprisingly critical views on practising mathematics for its own sake Arnauld and Nicole express in their treatise on geometry. So why write such a treatise? Because, so they argued, mathematics can achieve extra-mathematical goals. Mathematical practices cultivate proper reasoning, and proper reasoning is a moral imperative to the two Christian thinkers. Arnauld and Nicole framed their elaborations on these matters in terms of virtue and argued that mathematics fosters self-improvement, deepens piety, and cultivates epistemic virtues. Kotevska shows how Arnauld and Nicole sought to teach mathematics students how to use their studies for moral and spiritual improvement.

In “Mathematics, ethics, and purism: An application of MacIntyre’s virtue theory”, Paul Ernest squares the desire for unfettered research in pure mathematics with the social responsibilities of mathematics and its applications. He argues that the body of knowledge of mathematics is a-ethical, i.e. not subject to ethical considerations, because it lacks relevant agency. For Ernest, the goods internal to mathematical research practices are a-ethical in this sense but the practices themselves are not. They are social practices that place ethical imperatives of social interaction upon their practitioners. Ernest employs MacIntyre to argue that a virtuous research mathematician not only requires the qualities necessary to achieve goods internal to her practice, but (i) needs to strive for these goods in the context of other practices in which she partakes, and (ii) carries a responsibility towards the greater tradition that shapes her life. MacIntyre's three-stage account allows Ernest to form an argument which bridges the gap between the purist ideology of an essentially "harmless and innocent" mathematics (Hardy 1940, p. 44) with a social responsibility of the virtuous mathematician.

In "Mathematizing as a virtuous practice: Different narratives and their consequences for mathematics education and society", Deborah Kant and Deniz Sarikaya present the Freudenthalian notion of 'mathematizing' as a virtuous practice. Mathematizing is the ability to employ mathematics to render (worldly and mathematical) reality understandable (Freudenthal 1968). They propose to apply the notion to improve on popular narratives about mathematics. The authors engage with the narratives that (i) mathematics is useful; (ii) mathematics is beautiful; (iii) mathematicians aim at deep understanding; and (iv) mathematicians aim at theorem-credit. They highlight shortcomings of each of these narratives, point out how they resonate with the narrative that mathematics is about mathematizing, and indicate how the mathematizing narrative overcomes the highlighted shortcomings of the other narratives.

In "Intellectual humility in mathematics", Colin Jakob Rittberg employs accounts of intellectual humility proposed by virtue epistemologists to studies of mathematical practices. He argues that these accounts of the virtue are only partially successful at tracking manifestations of the virtue in mathematical practices, from which he draws the dual-conclusion that (i) virtue theorists of mathematics ought to adjust the accounts of the virtues provided by virtue epistemologists to their study of mathematics and (ii) that theoretical reflections on intellectual humility by virtue epistemologists have overlooked certain aspects of the virtues. The paper is centred around three accounts of intellectual humility (Kidd 2016; Roberts and Wood 2007; Whitcomb et al. 2017), which are employed in three case studies: the Erdős–Selberg debate; the disagreement about the epistemic status of the abc-conjecture; and Väänänen's proposed 'multiverse logic'. The upshot is a detailed study of how intellectual humility can manifest in mathematical practices which not only contributes to regulative virtue epistemologies but is a vital step towards establishing a virtue theory of mathematical practices.

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