On Economic Equilibrium Type Problems with Applications

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Abstract The existence solution for a class of economic equilibrium type problems in reflexive Banach space is considered. New results concerning the variational inequality approach to Arrow–Debreu model of economic equilibrium introduced in Naniewicz (Math Oper Res 32:436–466, 2007) are found and applied to ensure the existence of Pareto optimal solutions for a class of multiobjective optimization problems with so-called budget-like constraints. To achieve this goal, the theory of pseudo-monotone multivalued mappings combined with some fixed point technique for multivalued mappings with nonconvex values is used.

Keywords Economic equilibrium · Constrained multiobjective optimization · Variational inequalities · Pseudo-monotone multivalued operators · Ky Fan inequality · Fixed point

Mathematics Subject Classifications (2010) 49J40 · 91B50 · 90C46

1 Introduction

We study the inclusion problem

$$0 \in \mathcal{R}(x) + \partial f_0(x) \tag{1}$$

for a nonmonotone multivalued mapping $\mathcal{R} : \mathcal{K} \rightarrow 2^{\mathcal{K}^*}$ of the form

$$\mathcal{R}(x) = \sum_{j=1}^m \partial f_j(\Lambda_j(x)x), \quad x \in \mathcal{K}$$

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where $\pi \in \mathcal{R}(x)$ iff there exist $\pi_j \in X^*$, $\alpha_j \in \mathbb{R}$ such that $\pi = \sum_{j=1}^m \pi_j$, $\pi_j \in \partial f_j(\alpha_j x)$ and $-\langle x, \pi_j \rangle - \phi_j(x) \in \partial \text{ind}_{\geq 0}(\alpha_j)$. $\Lambda_j : \mathcal{K} \rightarrow 2^{\mathbb{R}_+}$ stands for a multivalued mapping given by

$$\alpha_j \in \Lambda_j(x) = \{ \beta_j \in \mathbb{R}_+ : \exists \tau_j \in \partial f_j(\beta_j x) \text{ such that } -\langle x, \tau_j \rangle - \phi_j(x) \in \partial \text{ind}_{\geq 0}(\beta_j) \}$$

$j = 1, \dots, m$. Here $\mathcal{K} \subset X$ is a closed convex cone in a reflexive Banach space X , X^* is the dual of X , $\mathcal{K}^* \subset X^*$ is the negative polar cone of $\mathcal{K}$. It is worth pointing out that we do not require the cone $\mathcal{K}$ having nonempty interior.

The following assumptions will be needed throughout the paper: $f_j : X \rightarrow \mathbb{R} \cup \{+\infty\}$, $j = 0, 1, \dots, m$, are convex, lower semicontinuous functions such that $\text{Dom}(f_0) \subset \mathcal{K}$ and $\text{Dom}(f_j^*) \subset \mathcal{K}^*$, $j = 1, \dots, m$, while $\phi_j : \mathcal{K} \rightarrow \mathbb{R}_+$, $j = 1, \dots, m$, are continuous, nonnegative valued functions on $\mathcal{K}$. “ ∂ ” stands for the subdifferential operator in the sense of Convex Analysis, $\text{ind}_{\geq 0} : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$ is the indicator function of the segment $[0, +\infty)$. A symbol “Dom” is used to denote the effective domain of the corresponding function, $\mathbb{R}_+ = \{t \in \mathbb{R} : t \geq 0\}$.

The problem (1) can be reformulated to the system of subdifferential inclusions

$$\left. \begin{aligned} \pi_j &\in \partial f_j(\alpha_j x), \quad j = 1, \dots, m \\ -\sum_{j=1}^m \pi_j &\in \partial f_0(x) \\ -\langle x, \pi_j \rangle - \phi_j(x) &\in \partial \text{ind}_{\geq 0}(\alpha_j), \quad j = 1, \dots, m \end{aligned} \right\} \quad (2)$$

with the unknowns $x \in \mathcal{K}$, $(\pi_j) \in (\mathcal{K}^*)^m$, $(\alpha_j) \in \mathbb{R}_+^m$ to be found. In what follows, it will be referred to as the economic equilibrium type problem. Our main result on this issue can be formulated as follows.

Theorem 1 Assume that for each $j = 1, \dots, m$ the hypotheses below hold:

- (H₁) $0 \in \text{cl}(\text{Dom}(f_j^*))$ and $\forall y \in \mathcal{K} \setminus \{0\} \exists \beta \geq 0: \beta y \in \text{Int}(\text{Dom}(f_j))$
- (H₂) $\exists \rho_j > 0, \exists M_j > 0: \{\tau \in \mathcal{K}^*: \langle \partial f_j^*(\tau) \tau \rangle \cap (\mathbb{R}_- + \rho_j) = \emptyset\} \subset B(0, M_j)$
- (H₃) $\phi_j(x) \geq 0 \quad \forall x \in \mathcal{K}$
- (H₄) $\partial f_j(0)$ is locally compact
- (H₅) $\phi_j : \mathcal{K} \rightarrow \mathbb{R}_+$ is weakly continuous on $\mathcal{K}$
- (H₆) $\exists \rho > 0, \exists M > 0: \{y \in \mathcal{K}: f_0(y) \leq \sum_{j=1}^m \phi_j(y) + f_0(0) + \rho\} \subset B(0, M)$
- (H₇) $(-\sum_{j=1}^m \partial f_j(0)) \cap \partial f_0(0) = \emptyset$
- (H₉) $f_j^\infty(y) + \phi_j(y) > 0, \forall y \in \mathcal{K}$.

Then there exist disjoint sets $J, J_0 \subset \{1, \dots, m\}$ with $J \cup J_0 = \{1, \dots, m\}$ and $x \in \mathcal{K} \setminus 0$, $(\pi_j) \in (\mathcal{K}^*)^m$, $(\alpha_j)_{j \in J} \in \text{Int}(\mathbb{R}_+)^{|J|}$, $(\alpha_j)_{j \in J_0} = 0 \in (\mathbb{R}_+)^{|J_0|}$, such that

$$\left. \begin{aligned} \pi_j &\in \partial f_j(\alpha_j x), \quad \langle x, \pi_j \rangle + \phi_j(x) = 0, \quad j \in J \\ \pi_j &\in \partial f_j(0), \quad \langle x, \pi_j \rangle + \phi_j(x) \geq 0, \quad j \in J_0 \\ -\sum_{j=1}^m \pi_j &\in \partial f_0(x). \end{aligned} \right\} \quad (3)$$

Two types of applications of the aforementioned theorem will be presented in this paper:

- first, applications related to economic equilibrium problems,
- second, applications related to a class of multiobjective optimization problems with, what will be called here, “budget-like” constraints.

If $\phi_j \geq 0$ $j = 1, \dots, m$, are convex, positive homogeneous functions and $f_0(y) = \sum_{j=1}^m \phi_j(y)$, $y \in \mathcal{K}$, then (2) can be seen as the variational formulation of the Arrow–Debreu model of economic equilibrium. This formulation has been introduced and primarily studied in [34].

$f_0(y) = \sup_{\tau \in \Delta^*} \langle y, \tau \rangle$, $y \in X$, $\Delta^* \subset X^*$ being a convex, closed, nonempty set of X^* , characterizes the total budget function while f_j^* (the conjugate of f_j) and ϕ_j are dis-utility and budget functions corresponding to j 's consumer, respectively. $-\pi_j \in -\mathcal{K}^*$, $j = 1, \dots, m$, are the commodity bundles related to j 's consumer and $x \in \mathcal{K}$ represents the price vector.

Indeed, if $\alpha_j > 0$, $j = 1, \dots, m$, then (2) ensures $(x, (\pi_j)) \in \mathcal{K} \times (\mathcal{K}^*)^m$ to be a solution of the problem

$$\left. \begin{aligned} f_j^*(\pi_j) &= \inf \{ f_j^*(\tau) : -\langle x, \tau \rangle \leq \phi_j(x) \text{ and } \tau \in \mathcal{K}^* \}, \quad j = 1, \dots, m \\ f_0(x) &= -\sum_{j=1}^m \langle x, \pi_j \rangle, \end{aligned} \right\}$$

i.e., $(x, (\pi_j))$ is the Walrasian Equilibrium for the corresponding economic equilibrium problem (cf. [28]). There are numerous literature references related to general equilibrium problem. For this issue we refer the reader to [1–6, 14–16, 19, 22, 24, 26–29, 33, 36–39, 45, 46] and the references therein.

Our main result related to the economic equilibrium issues generalizes that of [34] where X has been assumed to be separable. It reads as follows.

Theorem 2 *Let X be a reflexive Banach space and $\mathcal{K}^*$ be normal. Moreover, assume that for each $j \in \{1, \dots, m\}$ the hypotheses below hold:*

- (H'_1) $0 \in \text{cl}(\text{Dom}(f_j^*))$
- (H'_3) $\exists \gamma > 0 : \sum_{j=1}^m \phi_j(y) \geq \gamma \|y\| \quad \forall y \in \mathcal{K}$
- (H_4) $\partial f_j(0)$ is locally compact
- (H'_7) $(-\sum_{j=1}^m \partial f_j(0)) \cap \Delta^* = \emptyset$
- (H'_8) $\exists C > 0 : \Delta^* \cap (-\mathcal{K}^*) \subset B(0, C)$
- (H'_9) $\phi_j(y) + f_j^\infty(y) > 0 \quad \forall y \in \mathcal{K} \setminus \{0\}$
- (H'_{10}) $0 \notin \partial f_j(0)$.

Then there exist $(x, (\pi_j), (\alpha_j)) \in \mathcal{K} \times (\mathcal{K}^*)^m \times (\mathbb{R}_+)^m$, $\|x\| = 1$, and $r \in (0, 1]$ such that

$$\left. \begin{aligned} \pi_j &\in \partial f_j(\alpha_j x), \quad j = 1, \dots, m \\ -\sum_{j=1}^m \pi_j &\in r \partial f_0(x) \\ -\langle x, \pi_j \rangle - \phi_j(x) &\in \partial \text{ind}_{\geq 0}(\alpha_j), \quad j = 1, \dots, m. \end{aligned} \right\}$$

Note that x is then a solution of the inclusion

$$0 \in \mathcal{R}(x) + r \partial f_0(x).$$

Concerning the second group of applications related to multiobjective optimization issues we replace f_j by $f_j^\beta(y) = f_j(\frac{1}{\beta_j}y)$, $y \in X$, $\beta = (\beta_j) \in \text{Int}(\mathbb{R}_+^m)$ and obtain the existence of solution to the problem

$$\left. \begin{aligned} \pi_j^\beta &\in \frac{1}{\beta_j} \partial f_j \left(\frac{\alpha_j^\beta}{\beta_j} x^\beta \right) \\ - \sum_{j=1}^m \pi_j^\beta &\in \partial f_0(x^\beta) \\ - \langle x^\beta, \pi_j^\beta \rangle - \phi_j(x^\beta) &= 0 \end{aligned} \right\} \quad j = 1, \dots, m.$$

Next, we show that the multivalued mapping $\Gamma(\beta) = \{\alpha^\beta\}$ with nonconvex values admits a fixed point. This leads to the existence of $x^\beta \in \mathcal{K}$ such that

$$\left. \begin{aligned} \sum_{j=1}^m \frac{1}{\beta_j} f_j(x^\beta) + f_0(x^\beta) &\leq \sum_{j=1}^m \frac{1}{\beta_j} f_j(y) + f_0(y) \quad \forall y \in \mathcal{K} \\ - \langle x^\beta, \pi_j^\beta \rangle - \phi_j(x^\beta) &= 0, \quad j = 1, \dots, m. \end{aligned} \right\} \quad (4)$$

Therefore we are allowed to conclude that a multiobjective optimization problem related to the vector objective $(f_0, f_1, \dots, f_m)$ possesses at least one Pareto optimal solution fulfilling the “budget-like” constraints (4)₂.

Our result extends those obtained in [35] and reads as follows.

Theorem 3 Assume that $\mathcal{K}^*$ is normal and $f_0(y) = \sum_{j=1}^m \phi_j(y)$, $y \in \mathcal{K}$, where $\phi_j(\cdot)$ are convex, lower semicontinuous, positive homogeneous, nonnegative valued. Suppose that for each $j = 1, \dots, m$ the hypotheses below hold:

- (H'_1) $0 \in \text{cl}(\text{Dom}(f_j^*))$
- (H'_3) $\exists \gamma_j > 0: \phi_j(y) \geq \gamma_j \|y\| \quad \forall y \in \mathcal{K}$
- (H'_8) $\exists C > 0: \Delta^* \cap (-\mathcal{K}^*) \subset B(0, C)$
- (H_{13}) $\exists G > 0: \sum_{j=1}^m \phi_j(y) \leq G \|y\| \quad \forall y \in \mathcal{K}$.

Moreover, let one of the two assumptions below hold:

- (H''_{10}) $\partial f_j(0) = \partial f_j^*(0) = \emptyset, \quad j = 1, \dots, m,$

or

- (H_{12}) $\limsup_{\|y\| \rightarrow \infty} \frac{\langle y, \partial f_j(y) \rangle}{\phi_j(y)} < 0, \quad j = 1, \dots, m.$

Then there exist $(x^\beta, (\pi_j^\beta), (\beta_j))$ such that (4) holds.

There is a vast literature on multiobjective optimization problems. We refer to [10, 11, 13, 18, 20, 21, 23, 25, 30–32, 40, 44, 47–50] (see also [1, 28]) for a variety approaches, results and discussion.

2 Preliminaries

Let X be a reflexive Banach space with its dual X^* . The pairing over $X \times X^*$ will be denoted by $\langle \cdot, \cdot \rangle$. Assume $f_j: X \rightarrow \mathbb{R} \cup \{+\infty\}$, $j = 0, 1, \dots, m$, to be convex, lower

semicontinuous, proper functions. Their conjugates in the sense of Convex Analysis will be denoted by $f_j^* : X^* \rightarrow \mathbb{R} \cup \{+\infty\}$, respectively. Moreover, assume that there exists a closed, convex cone $\mathcal{K} \subset X$ such that $\text{Dom}(f_0) \subset \mathcal{K}$, $\text{Dom}(f_j^*) \subset \mathcal{K}^*$, $j = 1, \dots, m$, where $\mathcal{K}^* = \{\tau \in X^* : \langle y, \tau \rangle \leq 0 \ \forall y \in \mathcal{K}\}$ is the negative polar cone of $\mathcal{K}$. In the presented approach we do not assume that $\mathcal{K}$ has nonempty interior. Let $\phi_j : \mathcal{K} \rightarrow \mathbb{R}_+$, $j = 1, \dots, m$, be continuous functions on $\mathcal{K}$ with nonnegative values.

Fix $x \in \mathcal{K}$, $x \neq 0$, and consider the collection of minimization problems:

Problem $(\bar{P}_j^x)$ $\bar{v}_j^x := \inf\{f_j^*(\tau) + \text{ind}_{\leq \phi_j(x)}(A_x \tau) : \tau \in X^*\}$, $j = 1, \dots, m$,

where $A_x : X^* \rightarrow \mathbb{R}$ is a linear continuous operator given by

$$A_x \tau = -\langle x, \tau \rangle, \quad \tau \in X^*, \quad (5)$$

and $\text{ind}_{\leq \phi_j(x)}$ stands for the indicator of $\{t \in \mathbb{R} : t \leq \phi_j(x)\}$. It is easy to check that the adjoint of A_x takes the form $A_x^* \alpha = -\alpha x$, $\alpha \in \mathbb{R}$, and the conjugate of $\text{ind}_{\leq \phi_j(x)}$ is given by $\text{ind}_{\leq \phi_j(x)}^*(\alpha) = \alpha \phi_j(x) + \text{ind}_{\mathbb{R}_+}(\alpha)$, $\alpha \in \mathbb{R}$. According to the Fenchel duality theory (cf. [7, 8]) the dual of $(\bar{P}_j^x)$ reads

Problem $(\underline{P}_j^x)$ $\underline{v}_j^x := \inf\{f_j(\alpha x) + \alpha \phi_j(x) + \text{ind}_{\geq 0}(\alpha) : \alpha \in \mathbb{R}\}$, $j = 1, \dots, m$.

From the Fenchel duality theory we get the result.

Proposition 1 Suppose that for any $j = 1, \dots, m$ the assumptions below hold:

(H₁) $0 \in \text{cl}(\text{Dom}(f_j^*))$ and $\forall y \in \mathcal{K} \setminus \{0\} \exists \beta \geq 0: \beta y \in \text{Int}(\text{Dom}(f_j))$

(H₂) $\exists M_j > 0: \{\tau \in \mathcal{K}^* : \langle \partial f_j^*(\tau), \tau \rangle \cap \mathbb{R}_- \neq \emptyset\} \subset B(0, M_j)$

(H₃) $\exists \delta_j > 0: \phi_j(x) \geq \delta_j \ \forall x \in \mathcal{K}$.

Then for each $x \in \mathcal{K} \setminus \{0\}$ the optimization problems $(\bar{P}_j^x)$ and $(\underline{P}_j^x)$ have solutions $\pi_j^x \in \mathcal{K}^*$ and $\alpha_j^x \geq 0$, respectively, which are interrelated by

$$\pi_j^x \in \partial f_j(\alpha_j^x x) \Leftrightarrow \alpha_j^x x \in \partial f_j^*(\pi_j^x) \quad (6)$$

$$-\langle x, \pi_j^x \rangle - \phi_j(x) \in \partial \text{ind}_{\geq 0}(\alpha_j^x) \Leftrightarrow \alpha_j^x \in \partial \text{ind}_{\leq 0}(-\langle x, \pi_j^x \rangle - \phi_j(x)) \quad (7)$$

$$\bar{v}_j^x + \underline{v}_j^x = 0. \quad (8)$$

Moreover, if we define $\Lambda_j : \mathcal{K} \rightarrow 2^{\mathbb{R}_+}$ by

$$\Lambda_j(x) = \begin{cases} \{\beta_j \in \mathbb{R} : \exists \tau_j \in \partial f_j(\beta_j x) \text{ with } -\langle x, \tau_j \rangle - \phi_j(x) \in \partial \text{ind}_{\geq 0}(\beta_j)\} & x \in \mathcal{K} \setminus 0 \\ 0 & x = 0, \end{cases}$$

$j = 1, \dots, m$, then the graph of multivalued mapping Λ_j is closed and Λ_j has nonempty, bounded, closed and convex values.

Proof This is an easy task to check that for $x \in \mathcal{K} \setminus \{0\}$, (H_1) implies

$$0 \in \text{Int}[A_x \text{Dom}(f_j^*) - \text{Dom}(\text{ind}_{\leq \phi_j(x)})]$$

and

$$0 \in \text{Int}[A_x^* \text{Dom}(\text{ind}_{\leq \phi_j(x)}^*) + \text{Dom}(f_j)],$$

which due to the theorem of Fenchel [42] (see also [7]) there exist $\pi_j^x \in \mathcal{K}^*$ and $\alpha_j^x \geq 0$ such that (6), (7) and (8) hold. Accordingly, we conclude that for any $x \in \mathcal{K} \setminus \{0\}$, $\Lambda_j(x)$ is non empty because $\alpha_j^x \in \Lambda_j(x)$.

Furthermore, one can easily check that $\Lambda_j(x)$ consists of all solutions of variational inequality

$$\langle x, \partial f_j(\alpha_j^x x) \rangle (t - \alpha_j^x) + \phi_j(x)(t - \alpha_j^x) \geq 0, \quad \forall t \geq 0, \quad (9)$$

involving maximal monotone mapping $G_j^x(t) := \langle x, \partial f_j(tx) \rangle$, $t \geq 0$. Thus, it is a convex and closed set (see [17]).

Now we claim that $\Lambda_j(x)$, $x \in \mathcal{K} \setminus \{0\}$ is bounded. To show this let $\alpha_j^x \in \Lambda_j(x)$. From $\alpha_j^x x \in \partial f_j^*(\pi_j^x)$ it follows that

$$0 \geq \alpha_j^x \langle x, \pi_j^x \rangle \in \langle \partial f_j^*(\pi_j^x), \pi_j^x \rangle.$$

Due to (H_2) this yields $\|\pi_j^x\| \leq M_j$ for some $M_j > 0$. This fact, by the lower semicontinuity and convexity of f_j^* , implies

$$f_j^*(\pi_j^x) \geq -a_j \|\pi_j^x\| - b_j \geq -a_j M_j - b_j := -m_j, \quad \text{for some } a_j, b_j \geq 0. \quad (10)$$

Since $-\langle x, \pi_j^x \rangle - \phi_j(x) \in \partial \text{ind}_{\geq 0}(\alpha_j^x)$, we are led to

$$-\alpha_j^x \langle x, \pi_j^x \rangle = \alpha_j^x \phi_j(x), \quad (11)$$

which in view of $\alpha_j^x x \in \partial f_j^*(\pi_j^x)$ implies

$$\alpha_j^x \phi_j(x) \leq m_j + f_j^*(\tau) - \alpha_j^x \langle x, \tau \rangle, \quad \forall \tau \in \text{Dom}(f_j^*). \quad (12)$$

Now, on the contrary, suppose that $\Lambda_j(x)$ is unbounded. There exists a sequence $\{\alpha_j^n\} \subset \Lambda_j(x)$ with $\alpha_j^n \rightarrow \infty$ as $n \rightarrow \infty$ and such that

$$\phi_j(x) \leq \frac{m_j}{\alpha_j^n} + \frac{1}{\alpha_j^n} f_j^*(\tau) + \|\tau\| \|x\|, \quad \forall \tau \in \text{Dom}(f_j^*).$$

Letting $n \rightarrow \infty$ yields

$$\phi_j(x) \leq \|\tau\| \|x\|, \quad \forall \tau \in \text{Dom}(f_j^*).$$

Since 0 is a cluster point of $\text{Dom}(f_j^*)$ (due to (H_1)), one obtains $\phi_j(x) \leq 0$, contrary to (H_3) . The boundedness of $\Lambda_j(x)$ has been established.

Now let us notice that for sufficiently small $\epsilon > 0$ the condition $\|x\| \leq \epsilon$ implies $-\langle x, \pi_j^x \rangle \leq \|x\| \|\pi_j^x\| \leq \epsilon M_j < \delta_j \leq \phi_j(x)$. Thus by (11) we deduce that $\alpha_j^x = 0$. Therefore in some neighborhood of 0 in $\mathcal{K}$, say $O(0)$, we have $\Lambda_j(x) = \{0\}$ for any $x \in O(0)$, which means that $\Lambda_j: \mathcal{K} \rightarrow 2^{\mathbb{R}_+}$ is continuous at $0 \in \mathcal{K}$.

In order to establish the closedness of the graph of $\Lambda_j(\cdot)$ on $\mathcal{K} \times \mathbb{R}_+$ it remains to check whether from the assumptions: $\{x^n\} \subset \mathcal{K}$, $\alpha_j^n \in \Lambda_j(x^n)$, $x^n \rightarrow x^*$ in X , $\alpha_j^n \rightarrow \alpha_j^*$ in $\mathbb{R}$ for some $x^* \in \mathcal{K}$, $x^* \neq 0$, and $\alpha_j^* \in \mathbb{R}_+$, respectively, one can deduce that $\alpha_j^* \in \Lambda_j(x^*)$.

Since there exists $\pi_j^n \in \mathcal{K}^*$ with $\alpha_j^n x^n \in \partial f_j^*(\pi_j^n)$, we get

$$\alpha_j^n \langle x^n, \pi_j^n \rangle \in \langle \partial f_j^*(\pi_j^n), \pi_j^n \rangle.$$

But the left hand side of this inclusion is non-positive. Therefore, by (H_2) , $\{\pi_j^n\}$ is bounded. Consequently, one can suppose that $\pi_j^n \rightharpoonup \pi_j^*$ weakly in X^* for some $\pi_j^* \in \mathcal{K}^*$ (by passing to a subsequence, if necessary). Taking into account the conditions

$$\begin{aligned} \alpha_j^n x^n &\in \partial f_j^*(\pi_j^n) \\ -\langle x^n, \pi_j^n \rangle - \phi_j(x^n) &\in \partial \text{ind}_{\geq 0}(\alpha_j^n), \end{aligned}$$

we are allowed to pass to the limit as $n \rightarrow \infty$. By the continuity of $\phi_j(\cdot)$, the maximal monotonicity of $\partial f_j^*(\cdot)$ and $\partial \text{ind}_{\geq 0}(\cdot)$, we get

$$\begin{aligned} -\alpha_j^* x^* &\in \partial f_j^*(\pi_j^*) \\ -\langle x^*, \pi_j^* \rangle - \phi_j(x^*) &\in \partial \text{ind}_{\geq 0}(\alpha_j^*) \end{aligned}$$

from which it follows $\alpha_j^* \in \Lambda_j(x^*)$, as desired. The proof is complete. $\square$

Now we introduce a multivalued mapping $\mathcal{R}: \mathcal{K} \rightarrow 2^{\mathcal{K}^*}$ by setting

$$\begin{aligned} \mathcal{R}(x) &= \sum_{j=1}^m \partial f_j(\Lambda_j(x)x) \\ &= \left\{ \pi \in X^*: \exists \alpha_j \in \mathbb{R}_+, \exists \pi_j \in \partial f_j(\alpha_j x) \text{ with } \pi = \sum_{j=1}^m \pi_j \text{ and} \right. \\ &\quad \left. -\langle x, \pi_j \rangle - \phi_j(x) \in \partial \text{ind}_{\geq 0}(\alpha_j), \quad j = 1, \dots, m \right\}, \quad x \in \mathcal{K}. \end{aligned} \quad (13)$$

Note that $\mathcal{R}(\cdot)$ is not monotone due to the dependence of Λ_j on x .

Proposition 2 *Under the assumptions of Proposition 1, $\mathcal{R}$ given by (13) is a uniformly bounded multivalued mapping with nonempty, convex, closed values.*

Proof To begin with let us notice that if $\tau \in \mathcal{R}(y)$, $\tau \in \mathcal{K}$, then $\tau = \sum_{j=1}^m \tau_j$ with $\tau_j \in \partial f_j(\beta_j y)$ and $\beta_j \in \Lambda_j(y)$. Hence $\langle \partial f_j^*(\tau_j), \tau_j \rangle \ni \beta_j \langle y, \tau_j \rangle \leq 0$ which by (H_2) implies $\|\tau_j\| \leq M_j$. Consequently, $\|\tau\| \leq \sum_{j=1}^m M_j$. The boundedness of $\mathcal{R}$ follows.

Next we show that $\partial f_j(\alpha_1 x) = \partial f_j(\alpha_2 x)$ for any $\alpha_1, \alpha_2 \in \Lambda_j(x)$, $x \in \mathcal{K}$. To this end, assume that $\tilde{\pi}_1, \tilde{\pi}_2 \in \partial f_j(\Lambda_j(x))$. There exist $\alpha_1, \alpha_2 \in \Lambda_j(x)$ with the property that

$$\tilde{\pi}_1 \in \partial f_j(\alpha_1 x) \quad \text{and} \quad \tilde{\pi}_2 \in \partial f_j(\alpha_2 x).$$

For $t \in [0, 1]$ set $\tilde{\pi}_t = t\tilde{\pi}_1 + (1-t)\tilde{\pi}_2$. Since $\tilde{\pi}_1$ and $\tilde{\pi}_2$ are solutions of $(\bar{P}_j^x)$, we have

$$\begin{aligned} f_j^*(\tilde{\pi}_1) &= f_j^*(\tilde{\pi}_2) = f_j^*(\tilde{\pi}_t), \\ f_j^*(\tilde{\pi}_n) + f_j(\alpha_n x) &= \alpha_n \langle x, \tilde{\pi}_n \rangle, \\ \alpha_n (\langle x, \tilde{\pi}_n \rangle + \phi_j(x)) &= 0, \quad n = 1, 2. \end{aligned}$$

Case 1 $\alpha_1, \alpha_2 > 0$.

Taking into account that $\alpha_n x \in \partial f_j^*(\tilde{\pi}_n)$, $n = 1, 2$, we obtain

$$-\langle \alpha_n x, \tau - \tilde{\pi}_n \rangle + f_j^*(\tau) - f_j^*(\tilde{\pi}_n) \geq 0, \quad \forall \tau \in \mathcal{K}^*, \quad n = 1, 2.$$

Since in such a case $\langle x, \tilde{\pi}_n \rangle + \phi_j(x) = 0$, $n = 1, 2$, we conclude that $\langle x, \tilde{\pi}_t \rangle + \phi_j(x) = 0$, and consequently

$$-\langle \alpha_n x, \tau - \tilde{\pi}_t \rangle + f_j^*(\tau) - f_j^*(\tilde{\pi}_t) \geq 0, \quad \forall \tau \in \mathcal{K}^*, \quad n = 1, 2.$$

Hence, by adding these inequalities multiplied by t^* and $1 - t^*$, $t^* \in [0, 1]$, respectively, we easily arrive at

$$-\langle \alpha_{t^*} x, \tau - \tilde{\pi}_t \rangle + f_j^*(\tau) - f_j^*(\tilde{\pi}_t) \geq 0, \quad \forall \tau \in \mathcal{K}^*,$$

where $\alpha_{t^*} := t^* \alpha_1 + (1 - t^*) \alpha_2$. Hence $\alpha_{t^*} x \in \partial f_j^*(\tilde{\pi}_t)$ for any $t, t^* \in [0, 1]$ which is equivalent to $\tilde{\pi}_t \in \partial f_j(\alpha_{t^*} x)$. But this means that $\partial f_j(\alpha x)$ does not depend on the choice of $\alpha \in \Lambda_j(x)$, as claimed.

Case 2 $\alpha_1 = 0, \alpha_2 > 0$.

In this case we have

$$\tilde{\pi}_1 \in \partial f_j(0) \quad \text{and} \quad \tilde{\pi}_2 \in \partial f_j(\alpha_2 x),$$

and

$$-\langle x, \tilde{\pi}_1 \rangle - \phi_j(x) \leq 0, \quad \langle x, \tilde{\pi}_2 \rangle + \phi_j(x) = 0.$$

From the monotonicity of $\partial f_j(\cdot)$ it follows

$$\langle \alpha_2 x, \tilde{\pi}_2 - \tilde{\pi}_1 \rangle \geq 0.$$

Hence

$$-\alpha_2 \phi_j(x) = \langle \alpha_2 x, \tilde{\pi}_2 \rangle \geq \langle \alpha_2 x, \tilde{\pi}_1 \rangle$$

and consequently, $-\langle x, \tilde{\pi}_1 \rangle = \phi_j(x)$. Thus we have arrived at the case considered previously. The assertion follows.

To complete the proof it remains to invoke the well known properties of maximal monotone mappings $\partial f_j(\cdot)$ and combine them with the properties of $\Lambda_j(\cdot)$ mentioned in Proposition 1. $\square$

3 Pseudo-Monotonicity of $\mathcal{R}$

Let us recall some definitions and properties of the theory of monotone type mappings.

Definition 1 [12] An operator $T: \mathcal{K} \rightarrow 2^{X^*}$ is said to be pseudo-monotone from $\mathcal{K}$ into 2^{X^*} if the following conditions hold:

- (1) For any $x \in \mathcal{K}$, $T(x)$ is nonempty, bounded, closed and convex;
- (2) T is upper semicontinuous from each $F \cap \mathcal{K}$ (F - finite dimensional subspace of X) to the weak topology of X^* ;
- (3) For any $\{x^n\} \subset \mathcal{K}$, the conditions $x^n \rightharpoonup x$ in X and

$$\limsup_{n \rightarrow \infty} \langle x^n - x, \tau^n \rangle \leq 0$$

with $\tau^n \in T(x^n)$ imply that for any $y \in \mathcal{K}$ there exists $\tau(y) \in T(x)$ such that

$$\liminf_{n \rightarrow \infty} \langle x^n - y, \tau^n \rangle \geq \langle x - y, \tau(y) \rangle.$$

Now we are ready to discuss the pseudo-monotone properties of $\mathcal{R}$.

Proposition 3 Suppose that for any $j \in \{1, \dots, m\}$ the following conditions hold:

- (H_1) $0 \in \text{cl}(\text{Dom}(f_j^*))$ and $\forall y \in \mathcal{K} \setminus \{0\} \exists \beta \geq 0: \beta y \in \text{Int}(\text{Dom}(f_j))$
- (H_2) $\exists M_j > 0: \{\tau \in \mathcal{K}^*: \langle \partial f_j^*(\tau), \tau \rangle \cap \mathbb{R}_- \neq \emptyset\} \subset B(0, M_j)$
- (H_3) $\exists \delta_j > 0: \phi_j(x) \geq \delta_j \quad \forall x \in \mathcal{K}$
- (H_4) $\partial f_j(0)$ is locally compact
- (H_5) $\phi_j: \mathcal{K} \rightarrow \mathbb{R}_+$ is weakly continuous on $\mathcal{K}$.

Then $\mathcal{R}: \mathcal{K} \rightarrow 2^{X^*}$ is a pseudo-monotone mapping on $\mathcal{K}$.

Proof In order to show that $\mathcal{R}$ is pseudo-monotone on $\mathcal{K}$ we have to check that (1)–(3) of Definition 1 hold.

- (1) See Proposition 2.
- (3) Suppose that $x^n \rightharpoonup x$ weakly in X , $\{x^n\} \subset \mathcal{K}$, $\pi^n \in \mathcal{R}(x^n)$ and

$$\limsup_{n \rightarrow \infty} \langle x^n - x, \pi^n \rangle \leq 0. \quad (14)$$

Our task is to show that for each $y \in \mathcal{K}$ there exists an element $\pi(y) \in X^*$ such that

$$\pi(y) \in \mathcal{R}(x) \quad (15)$$

$$\liminf_{n \rightarrow \infty} \langle x^n - y, \pi^n \rangle \geq \langle x - y, \pi(y) \rangle. \quad (16)$$

Since $\pi^n \in \mathcal{R}(x^n)$, π^n admits the representation

$$\pi^n = \sum_{j=1}^m \pi_j^n, \quad (17)$$

for some

$$\pi_j^n \in \partial f_j(\alpha_j^n x^n) \quad (18)$$

(or equivalently: $\alpha_j^n x^n \in \partial f_j^*(\pi_j^n)$), where $\alpha_j^n \in \Lambda_j(x^n)$. By the definition of Λ_j ,

$$-\langle x^n, \pi_j^n \rangle - \phi_j(x^n) \in \partial \text{ind}_{\geq 0}(\alpha_j^n). \quad (19)$$

As it has been already shown, (H_2) implies the boundedness of $\{\pi_j^n\}_{n=1}^\infty \subset \mathcal{K}^*$, namely $\|\pi_j^n\| \leq M_j$, $j = 1, \dots, m$. For the boundedness of $\{\alpha_j^n\} \subset \mathbb{R}_+$ we proceed like in the proof of Proposition 1. Namely, analogously to (12) it follows

$$\alpha_j^n \phi_j(x^n) \leq m_j + f_j^*(\tau) - \alpha_j^n \langle x^n, \tau \rangle, \quad \forall \tau \in \text{Dom}(f_j^*). \quad (20)$$

Hence we get

$$\phi_j(x^n) \leq \frac{m_j}{\alpha_j^n} + \frac{1}{\alpha_j^n} f_j^*(\tau) + \|\tau\| \|x^n\|, \quad \forall \tau \in \text{Dom}(f_j^*).$$

If $\alpha_j^n \rightarrow \infty$ then we are led to the contradiction. Indeed, by using the weak continuity of ϕ_j and boundedness of $\{\|x^n\|\}$, $\|x^n\| \leq C$, it follows

$$\phi_j(x) \leq C\|\tau\|, \quad \forall \tau \in \text{Dom}(f_j^*).$$

By (H_1) , 0 is a cluster point of $\text{Dom}(f_j^*)$ which implies $\phi_j(x) \leq 0$, contrary to (H_3) . The assertion follows.

Therefore, without loss of generality it can be assumed that

$$\pi_j^n \rightharpoonup \pi_j \quad \text{weakly in } X^* \text{ as } n \rightarrow \infty \quad (21)$$

$$\alpha_j^n \rightarrow \alpha_j \quad \text{in } \mathbb{R} \text{ as } n \rightarrow \infty \quad (22)$$

for some $\pi_j \in \mathcal{K}^*$ and $\alpha_j \in \mathbb{R}_+$. From (14) it follows

$$\limsup_{n \rightarrow \infty} \left\langle x^n - x, \sum_{j=1}^m \pi_j^n \right\rangle \leq 0. \quad (23)$$

On the other hand, by virtue of (18),

$$f_j(\alpha_j x) - f_j(\alpha_j^n x^n) \geq \langle \alpha_j x - \alpha_j^n x^n, \pi_j^n \rangle.$$

Since $\alpha_j^n x^n \rightharpoonup \alpha_j x$ weakly in X , the weak lower semicontinuity of f_j yields

$$\liminf_{n \rightarrow \infty} \langle \alpha_j^n x^n, \pi_j^n \rangle \geq \langle \alpha_j x, \pi_j \rangle. \quad (24)$$

Now we claim that for any $j \in \{1, \dots, m\}$,

$$\liminf_{n \rightarrow \infty} \langle x^n, \pi_j^n \rangle \geq \langle x, \pi_j \rangle. \quad (25)$$

Indeed, if $\alpha_j > 0$, the assertion can be deduced directly from (24). If $\alpha_j = 0$ then $\alpha_j^n x^n \rightarrow 0$ strongly in X . Hence, by the maximal monotonicity of ∂f_j , $\pi_j \in \partial f_j(0)$. Since

∂f_j is monotone, we have $\langle \alpha_j^n x^n, \pi_j^n - \pi_j \rangle \geq 0$. It immediately implies (25) provided that $\alpha_j^n > 0$ (for infinitely many n). If $\alpha_j^n = 0$ for any n then $\pi_j^n \in \partial f_j(0)$ which by (H_4) implies that $\pi_j^n \rightarrow \pi_j$ strongly in X^* . Hence $\langle x^n, \pi_j^n \rangle \rightarrow \langle x, \pi_j \rangle$ as $n \rightarrow \infty$ and (25) follows.

Now combining (25) and (23) yields

$$\lim_{n \rightarrow \infty} \langle x^n, \pi_j^n \rangle = \langle x, \pi_j \rangle, \quad \forall j = 1, \dots, m. \quad (26)$$

Consequently,

$$\lim_{n \rightarrow \infty} \langle x^n - y, \pi^n \rangle = \langle x - y, \pi \rangle$$

for every $y \in X$, where $\pi = \sum_{j=1}^m \pi_j$, i.e. (16) holds with $\pi(y) := \pi$.

Now we assert that $\pi \in \mathcal{R}(x)$, or equivalently,

$$\alpha_j x \in \partial f_j^*(\pi_j), \quad \forall j = 1, \dots, m \quad (27)$$

$$-\langle x, \pi_j \rangle - \phi_j(x) \in \partial \text{ind}_{\geq 0}(\alpha_j), \quad \forall j = 1, \dots, m. \quad (28)$$

For (27) we need (18). Namely, taking into account that $\alpha_j^n x^n \rightharpoonup \alpha_j x$ and $\pi_j^n \rightharpoonup \pi_j$ weakly in X and X^* , respectively, by (26) we get

$$\limsup_{n \rightarrow \infty} \langle \alpha_j^n x^n - \alpha_j x, \pi_j^n \rangle \leq 0. \quad (29)$$

Hence the maximal monotonicity of ∂f_j^* yields (27).

In order to complete the proof we have to show (28). This in view of (26) and (H_5) can be easily deduced by passing to the limit as $n \rightarrow \infty$ in (19). The proof of condition (2) in the definition of pseudo-monotonicity is complete.

- (2) Suppose that $x^n \rightarrow x$ in $F \cap \mathcal{K}$, F being the finite dimensional subspace of X , $\pi^n \in \mathcal{R}(x^n)$. Since $\mathcal{R}$ is uniformly bounded, $\{\pi^n\}$ is bounded and therefore (14) is fulfilled. As in (3) we show that every weak limit of $\{\pi^n\}$ lies in $\mathcal{R}(x)$. This ensures the desired upper semicontinuity.

The proof of Proposition 3 is complete. $\square$

4 Existence Results

Consider the problem of finding $x \in \mathcal{K}$ such that for a certain $\pi \in \mathcal{R}(x)$ the following variational inequality holds:

$$\langle y - x, \pi \rangle + f_0(y) - f_0(x) \geq 0, \quad \forall y \in \mathcal{K}. \quad (30)$$

Recall that $\text{Dom } f_0 \subset \mathcal{K}$. Thus this problem can be written equivalently as

$$0 \in \mathcal{R}(x) + \partial f_0(x). \quad (31)$$

The important point to note here is the fact that both $\mathcal{K}$ and $\mathcal{K}^*$ are allowed to have empty interiors. The reason is that we are going to use the theory of generalized pseudo-monotone mappings developed by Browder–Hess in [12, pp. 288–293]. In this theory the corresponding cone having internal points is not required.

Proposition 4 Suppose that for any $j = 1, \dots, m$ the assumptions below are fulfilled:

- (H₁) $0 \in \text{cl}(\text{Dom}(f_j^*))$ and $\forall y \in \mathcal{K} \setminus \{0\} \exists \beta \geq 0: \beta y \in \text{Int}(\text{Dom}(f_j))$
- (H₂) $\exists M_j: \{\tau \in \mathcal{K}^*: \langle \partial f_j^*(\tau), \tau \rangle \cap \mathbb{R}_- \neq \emptyset\} \subset B(0, M_j)$
- (H₃) $\exists \delta_j > 0: \phi_j(x) \geq \delta_j \quad \forall x \in \mathcal{K}$
- (H₄) $\partial f_j(0)$ is locally compact
- (H₅) $\phi_j: \mathcal{K} \rightarrow \mathbb{R}_+$ is weakly continuous on $\mathcal{K}$
- (H₆) $\exists y_0 \in \text{Dom}(f_0), \exists M > 0: \{y \in \mathcal{K}: f_0(y) \leq \sum_{j=1}^m \phi_j(y) + f_0(y_0)\} \subset B(0, M)$.

Then the problem: Find $x \in \mathcal{K}$ such that

$$0 \in \partial f_0(x) + \mathcal{R}(x) \quad (32)$$

admits at least one solution. Equivalently, there exists $(x, (\pi_j), (\alpha_j)) \in \mathcal{K} \times (\mathcal{K}^*)^m \times (\mathbb{R}_+)^m$ such that

$$\left. \begin{aligned} \pi_j &\in \partial f_j(\alpha_j x) \\ -\sum_{j=1}^m \pi_j &\in \partial f_0(x) \\ -\langle x, \pi_j \rangle - \phi_j(x) &\in \partial \text{ind}_{\geq 0}(\alpha_j). \end{aligned} \right\} \quad (33)$$

Proof Introduce the duality mapping $J: X \rightarrow 2^{X^*}$ by $J(y) = \{\tau \in X^*: \langle y, \tau \rangle = \|\tau\| \|y\|, \|y\| = \|\tau\|\}$ (cf. [12], p. 280). Recall that $J(\cdot) = \partial(\frac{1}{2}\|\cdot\|^2)$.

For a maximal monotone mapping $T: X \rightarrow 2^{X^*}$ define its Yosida approximation $T_\lambda: X \rightarrow 2^{X^*}, \lambda > 0$, by $T_\lambda := (T^{-1} + \lambda J^{-1})^{-1}$. It is well known that for any $\lambda > 0$, T_λ is a maximal monotone, bounded operator with $\text{Dom } T_\lambda = X$. Thus it is pseudo-monotone on X (cf. [12], Proposition 8, p. 266).

Now, let us set $T = \partial f_0$ and $T_\lambda = (\partial f_0)_\lambda$. Proceeding like in the proof of ([12], Proposition 9, p. 267) we ensure that $\mathcal{R} + (\partial f_0)_\lambda$ is a pseudo-monotone mapping on $\mathcal{K}$ for any $\lambda > 0$. Thus applying ([12], Theorem 15, p. 289) we deduce the existence of $x^\lambda \in B(0, 2M) \cap \mathcal{K}$ such that

$$\langle y - x^\lambda, \mathcal{R}(x^\lambda) + (\partial f_0)_\lambda(x^\lambda) \rangle \geq 0, \quad \forall y \in \mathcal{K} \cap B(0, 2M), \quad (34)$$

with M as in (H₆). Thus there exist $\pi_j^\lambda \in \mathcal{K}^* \cap B(0, M_j), \tau^\lambda \in (\partial f_0)_\lambda(x^\lambda)$ and $z^\lambda \in X$ with the properties that

$$\sum_{j=1}^m \pi_j^\lambda \in \mathcal{R}(x^\lambda), \quad \tau^\lambda \in \partial f_0(z^\lambda), \quad \lambda \tau^\lambda \in J(x^\lambda - z^\lambda)$$

and

$$\left\langle y - x^\lambda, \sum_{j=1}^m \pi_j^\lambda + \tau^\lambda \right\rangle \geq 0, \quad \forall y \in \mathcal{K} \cap B(0, 2M). \quad (35)$$

By the definition of J it follows

$$\langle x^\lambda - z^\lambda, \lambda \tau^\lambda \rangle = \|x^\lambda - z^\lambda\| \|\lambda \tau^\lambda\|, \quad \|\lambda \tau^\lambda\| = \|x^\lambda - z^\lambda\|.$$

Now, substituting $y = y_0$ into (35) leads to the estimate

$$-\left\langle x^\lambda, \sum_{j=1}^m \pi_j^\lambda \right\rangle \geq \langle x^\lambda - z^\lambda, \tau^\lambda \rangle + \langle z^\lambda - y_0, \tau^\lambda \rangle,$$

from which we get

$$\sum_{j=1}^m \phi_j(x^\lambda) \geq \frac{\|x^\lambda - z^\lambda\|^2}{\lambda} + f_0(z^\lambda) - f_0(y_0).$$

Taking into account that $f_0(z^\lambda) \geq -a\|z^\lambda\| - b$ for some $a, b > 0$ (f_0 is proper, lower semicontinuous convex) and that $x^\lambda \in B(0, 2M)$, one gets

$$C \geq \sup_{\|y\| \leq 2M} \left\{ \sum_{j=1}^m \phi_j(y) \right\} + f_0(y_0) + 2aM \geq \frac{\|x^\lambda - z^\lambda\|^2}{\lambda} - a\|x^\lambda - z^\lambda\|$$

for some constant $C > 0$. Thus $\|x^\lambda - z^\lambda\| \rightarrow 0$ as $\lambda \rightarrow 0$. Since $\{x^\lambda\} \subset B(0, 2M) \cap \mathcal{K}$ and $\mathcal{R}$ is uniformly bounded, one can extract a sequence $\lambda_n \rightarrow 0$ such that

$$x^{\lambda_n} \rightharpoonup x^*, \quad \text{weakly in } X \text{ as } n \rightarrow \infty,$$

$$\pi_j^{\lambda_n} \rightharpoonup \pi_j^* \quad \text{weakly in } X^* \text{ as } n \rightarrow \infty,$$

Notice that $z^{\lambda_n} \rightharpoonup x^*$ as well. By substituting $y = x^*$ into (35) we obtain

$$f_0(x^*) - f_0(z^{\lambda_n}) - \frac{\|x^{\lambda_n} - z^{\lambda_n}\|^2}{\lambda_n} \geq \left\langle x^{\lambda_n} - x^*, \sum_{j=1}^m \pi_j^{\lambda_n} \right\rangle$$

from which, by the weak lower semicontinuity of f_0 , it follows

$$0 \geq \limsup_{n \rightarrow \infty} \left[\left\langle x^{\lambda_n} - x^*, \sum_{j=1}^m \pi_j^{\lambda_n} \right\rangle \right],$$

Now applying the pseudo-monotonicity of $\mathcal{R}$ yields

$$\sum_{j=1}^m \pi_j^* \in \mathcal{R}(x^*), \quad \lim_{n \rightarrow \infty} \left\langle x^{\lambda_n} - x^*, \sum_{j=1}^m \pi_j^{\lambda_n} \right\rangle = 0.$$

Finally, for any $y \in B(0, 2M) \cap \mathcal{K} \cap \text{Dom}(f_0)$, from (35) we obtain

$$\begin{aligned} 0 &\leq \left\langle y - x^{\lambda_n}, \sum_{j=1}^m \pi_j^{\lambda_n} + \tau^{\lambda_n} \right\rangle \\ &= \left\langle y - x^{\lambda_n}, \sum_{j=1}^m \pi_j^{\lambda_n} \right\rangle + \langle y - z^{\lambda_n}, \tau^{\lambda_n} \rangle + \langle z^{\lambda_n} - x^{\lambda_n}, \tau^{\lambda_n} \rangle \\ &\leq \left\langle y - x^{\lambda_n}, \sum_{j=1}^m \pi_j^{\lambda_n} \right\rangle + f_0(y) - f_0(z^{\lambda_n}) - \frac{\|x^{\lambda_n} - z^{\lambda_n}\|^2}{\lambda_n} \\ &\leq \left\langle y - x^{\lambda_n}, \sum_{j=1}^m \pi_j^{\lambda_n} \right\rangle + f_0(y) - f_0(z^{\lambda_n}). \end{aligned}$$

Passing again to the limit with $n \rightarrow \infty$ yields

$$\left\langle y - x^*, \sum_{j=1}^m \pi_j^* \right\rangle + f_0(y) - f_0(x^*) \geq 0, \quad \forall y \in B(0, 2M) \cap \mathcal{K}. \quad (36)$$

Substituting $y = y_0$ leads easily to the inequality

$$f_0(x^*) \leq \sum_{j=1}^m \phi_j(x^*) + f_0(y_0),$$

which thanks to (H_6) implies $\|x^*\| \leq M$. Having in mind the validity of (36) for any $y \in B(0, 2M) \cap \mathcal{K}$, we easily deduce (32). The proof is complete. $\square$

The problem (33) will be referred to as the economic equilibrium type problem. If additionally ϕ_j are positive homogeneous and $f_0 = \sum_{j=1}^m \phi_j$ then (33) represents the variational formulation of the Arrow–Debreu model of economic equilibrium (cf. [1, 28]). This variational inequality approach has been first introduced and studied in the setting of reflexive, separable Banach spaces in [34].

5 Case $\phi_j \geq 0$

In this section we improve the results of Proposition 4 by assuming that $\phi_j \geq 0$.

For any $\varepsilon > 0$ introduce the data system on $(X \times \mathbb{R}) \times (X^* \times \mathbb{R})$ by

$$\left. \begin{aligned} \tilde{f}_j^\varepsilon(\tau, q) &= f_j^*(\tau) + \varphi^\varepsilon(q), \quad \tau \in \mathcal{K}^*, \quad q \in \mathbb{R}_- \\ \tilde{f}_0^\varepsilon(y, z) &= f_0(y) + z + \text{ind}_{[\varepsilon, \infty)}(z), \quad y \in \mathcal{K}, \quad z \in \mathbb{R}_+ \\ \tilde{\phi}_j^\varepsilon(y, z) &= \phi_j(y) + s \max \left\{ \left(z - \frac{1}{2}\varepsilon \right), \frac{\varepsilon}{2} \right\}, \quad y \in \mathcal{K}, \quad z \in \mathbb{R}_+, \end{aligned} \right\} \quad (37)$$

where $\varphi^\varepsilon(q) = -\varepsilon(-q)^{1-\varepsilon} + \text{ind}_{[-s, 0]}(q)$, $q \in \mathbb{R}_-$, $s > 0$ is a positive constant with $s \in (0, \frac{1}{m})$.

Proposition 5 Assume that for each $j = 1, \dots, m$ the hypotheses below hold:

- (H_1) $0 \in \text{cl}(\text{Dom}(f_j^*))$ and $\forall y \in \mathcal{K} \setminus \{0\} \exists \beta \geq 0: \beta y \in \text{Int}(\text{Dom}(f_j))$
- (H_2') $\exists \rho_j > 0, \exists M_j > 0: \{\tau \in \mathcal{K}^*: \langle \partial f_j^*(\tau), \tau \rangle \cap (\mathbb{R}_- + \rho_j) = \emptyset\} \subset B(0, M_j)$
- (H_3') $\phi_j(x) \geq 0 \quad \forall x \in \mathcal{K}$
- (H_4) $\partial f_j(0)$ is locally compact
- (H_5) $\phi_j: \mathcal{K} \rightarrow \mathbb{R}_+$ is weakly continuous on $\mathcal{K}$
- (H_6') $\exists \rho > 0, \exists M > 0: \{y \in \mathcal{K}: f_0(y) \leq \sum_{j=1}^m \phi_j(y) + f_0(0) + \rho\} \subset B(0, M)$
- (H_7) $(-\sum_{j=1}^m \partial f_j(0)) \cap \partial f_0(0) = \emptyset$.

Then for each $\varepsilon > 0$ there exists $(x^\varepsilon, (\pi_j^\varepsilon), (r_j^\varepsilon), (\alpha_j^\varepsilon)) \in (\mathcal{K} \setminus \{0\}) \times (\mathcal{K}^*)^m \times (\mathbb{R}_+)^m \times (\mathbb{R}_+)^m$ such that

$$\left. \begin{aligned} & \pi_j^\varepsilon \in \partial f_j(\alpha_j^\varepsilon x^\varepsilon) \\ & - \sum_{j=1}^m \pi_j^\varepsilon \in \partial f_0(x^\varepsilon) \\ & - \langle x^\varepsilon, \pi_j^\varepsilon \rangle - \phi_j(x^\varepsilon) - \varepsilon \left(\frac{s}{2} + q_j^\varepsilon \right) \in \partial \text{ind}_{\geq 0}(\alpha_j^\varepsilon) \\ & -q_j^\varepsilon = (1 - \varepsilon)^{1/\varepsilon} \left(\frac{1}{\alpha_j^\varepsilon + r_j^\varepsilon} \right)^{1/\varepsilon}, \quad q_j^\varepsilon \in [-s, 0), \quad -r_j^\varepsilon \in \partial \text{ind}_{[-s, 0]}(q_j^\varepsilon), \quad r_j^\varepsilon \geq 0. \end{aligned} \right\} \quad (38)$$

Proof Now we show that all the requirements of Proposition 4 are fulfilled when replacing (f_j^*, f_0, ϕ_j) by $(\tilde{f}_j^\varepsilon, \tilde{f}_0^\varepsilon, \tilde{\phi}_j^\varepsilon)$.

Let us begin with (H_3) . It is easy to check that

$$\tilde{\phi}_j^\varepsilon(y, z) = \phi_j(y) + s \max \left\{ \left(z - \frac{1}{2}\varepsilon \right), \frac{\varepsilon}{2} \right\} \geq \frac{s}{2}\varepsilon > 0, \quad \forall (y, z) \in \mathcal{K} \times \mathbb{R}_+,$$

what desired.

To check that (H_2) holds, let $y \in \partial f_j^*(\tau)$ and $z \in \partial \varphi^\varepsilon(q)$. Hence $z = \varepsilon(1 - \varepsilon)(-q)^{-\varepsilon} - r$, $-r \in \partial \text{ind}_{[-s, 0]}(q)$, $r \geq 0$, $q \in [-s, 0)$. From the condition

$$\begin{aligned} 0 & \geq \left\langle \partial \tilde{f}_j^\varepsilon(\tau, q), (\tau, q) \right\rangle = \left\langle \partial f_j^*(\tau), \tau \right\rangle + \partial \varphi^\varepsilon(q)q \\ & \geq \left\langle \partial f_j^*(\tau), \tau \right\rangle + q[\varepsilon(1 - \varepsilon)(-q^\varepsilon)^{-\varepsilon} - r] \geq \left\langle \partial f_j^*(\tau), \tau \right\rangle - \varepsilon s \left(\frac{1}{s} \right)^\varepsilon \end{aligned}$$

we are led to

$$\varepsilon s \left(\frac{1}{s} \right)^\varepsilon \geq \left\langle \partial f_j^*(\tau), \tau \right\rangle.$$

Hence, by (H'_2) it follows that $\|\tau\| \leq M_j$ provided $\varepsilon > 0$ is small enough. The boundedness of $q \in [-s, 0]$ is obvious.

Concerning (H_6) , we wish to deduce that

$$\left\{ (y, z) \in \mathcal{K} \times (\mathbb{R}_+ + \varepsilon) : \tilde{f}_0^\varepsilon(y, z) \leq \sum_{j=1}^m \tilde{\phi}_j^\varepsilon(y, z) + \tilde{f}_0^\varepsilon(0, \varepsilon) \right\}$$

is bounded. Notice that this set can be rewritten equivalently as

$$\left\{ (y, z) \in \mathcal{K} \times (\mathbb{R}_+ + \varepsilon) : f_0(y) + z(1 - sm) \leq \sum_{j=1}^m \phi_j(y) + f_0(0) + \varepsilon \left(1 - \frac{sm}{2} \right) \right\}.$$

So, by (H'_6) we get $\|y\| \leq M$. This in turn leads to

$$z \leq \frac{1}{1-sm} \sup_{\substack{\|y\| \leq M \\ y \in \mathcal{K}}} \left\{ \sum_{j=1}^m \phi_j(y) - f_0(y) + f_0(0) + \varepsilon \right\},$$

as desired. The remaining assumptions of Proposition 4 are obviously satisfied. Therefore, we conclude the existence of

$$\left((x^\varepsilon, z^\varepsilon), (\pi_j^\varepsilon, q_j^\varepsilon), (\alpha_j^\varepsilon) \right) \in ((\mathcal{K} \times \mathbb{R}_+ \times ((\mathcal{K}^*)^m \times (\mathbb{R}_-)^m) \times (\mathbb{R}_+)^m)$$

such that

$$\left. \begin{aligned} \alpha_j^\varepsilon(x^\varepsilon, z^\varepsilon) &\in \partial \tilde{f}_j^\varepsilon(\pi_j^\varepsilon, q_j^\varepsilon) \\ -\langle (x^\varepsilon, z^\varepsilon), (\pi_j^\varepsilon, q_j^\varepsilon) \rangle - \tilde{\phi}_j^\varepsilon(x^\varepsilon, z^\varepsilon) &\in \partial \text{ind}_{\geq 0}(\alpha_j^\varepsilon) \\ -\sum_{j=1}^m (\pi_j^\varepsilon, q_j^\varepsilon) &\in \partial \tilde{f}_0^\varepsilon(x^\varepsilon, z^\varepsilon), \end{aligned} \right\} \quad (39)$$

or equivalently,

$$\left. \begin{aligned} \alpha_j^\varepsilon x^\varepsilon &\in \partial f_j^\star(\pi_j^\varepsilon), \quad \alpha_j^\varepsilon z^\varepsilon \in \partial \varphi^\varepsilon(q_j^\varepsilon) = \varepsilon(1-\varepsilon)(-q_j^\varepsilon)^{-\varepsilon} + \partial \text{ind}_{[-s,0]}(q_j^\varepsilon) \\ -\langle x^\varepsilon, \pi_j^\varepsilon \rangle - \phi_j(x^\varepsilon) - z^\varepsilon(s + q_j^\varepsilon) + \frac{s}{2}\varepsilon &\in \partial \text{ind}_{\geq 0}(\alpha_j^\varepsilon) \\ -\sum_{j=1}^m \pi_j^\varepsilon &\in \partial f_0(x^\varepsilon), \quad -\sum_{j=1}^m q_j^\varepsilon \in 1 + \partial \text{ind}_{[\varepsilon, +\infty)}(z^\varepsilon). \end{aligned} \right\} \quad (40)$$

From (40)₁ it follows that $-\sum_{j=1}^m q_j^\varepsilon \leq sm < 1$. In view of (40)₃ it is only possible when $z^\varepsilon = \varepsilon$. Hence we obtain

$$-q_j^\varepsilon = (1-\varepsilon)^{1/\varepsilon} \left(\frac{1}{\alpha_j^\varepsilon + r_j^\varepsilon} \right)^{1/\varepsilon}, \quad q_j^\varepsilon \in [-s, 0), \quad -r_j^\varepsilon \in \partial \text{ind}_{[-s,0]}(q_j^\varepsilon), \quad r_j^\varepsilon \geq 0, \quad (41)$$

which completes the proof. $\square$

Another version of this result is useful when studying economic equilibrium problems. Namely, if we assume that $\text{Dom}(f_0^\star) \cap (-\mathcal{K}^*)$ is bounded and the cone $\mathcal{K}^*$ is normal (for the characterization of normal cones we refer to [41]), i.e. there exists $\rho > 0$ such that

$$\|\tau + \pi\| \geq \rho(\|\tau\| + \|\pi\|), \quad \forall \tau, \pi \in \mathcal{K}^*, \quad (1) \quad (42)$$

then in Proposition 5 the assumption (H_1) can be considerably simplified and (H_2) is redundant. Namely, the following result is available.

¹This condition is equivalent to the well known normality property of $\mathcal{K}^*$ defined as the boundedness of $(B_{X^*}(0, 1) + \mathcal{K}^*) \cap (B_{X^*}(0, 1) - \mathcal{K}^*)$ in X^* .

Proposition 6 Suppose that $\mathcal{K}^*$ is normal and for any $j = 1, \dots, m$, the assumptions below are fulfilled:

- (H₁) $0 \in \text{cl}(\text{Dom}(f_j^*))$
- (H₂) $\phi_j(x) \geq 0 \quad \forall x \in \mathcal{K}$
- (H₃) $\partial f_j(0)$ is locally compact
- (H₄) $\phi_j: \mathcal{K} \rightarrow \mathbb{R}_+$ is weakly continuous on $\mathcal{K}$
- (H₅) $\exists \rho > 0, \exists M > 0: \{y \in \mathcal{K}: f_0(y) \leq \sum_{j=1}^m \phi_j(y) + f_0(0) + \rho\} \subset B(0, M)$
- (H₆) $(-\sum_{j=1}^m \partial f_j(0)) \cap \partial f_0(0) = \emptyset$
- (H₇) $\text{Dom}(f_0^*) \cap (-\mathcal{K}^*) \subset B(0, C)$ for some $C > 0$.

Then for each $\varepsilon > 0$ there exists $(x^\varepsilon, (\pi_j^\varepsilon), (\alpha_j^\varepsilon)) \in \mathcal{K} \setminus \{0\} \times (\mathcal{K}^*)^m \times (\mathbb{R}_+)^m$ such that (38) holds.

Proof Let $R = \frac{C}{\rho^m} + 1$ and $f_{jR}^*(\cdot) := f_j^*(\cdot) + \text{ind}_{\overline{B(0, R)}}(\cdot)$, $j = 1, \dots, m$. It is easy to show that f_0, f_{jR}^* and ϕ_j satisfy all the hypotheses of Proposition 5. Indeed, since the effective domain of f_{jR}^* is bounded, the effective domain of its conjugate $(f_{jR}^*)^*$ coincides with the whole X . Consequently, (H₁) and (H₂) hold immediately while (H₄) is obviously fulfilled. Accordingly, by Proposition 5 there exists $(x^R, (\pi_j^R), (\alpha_j^R)) \in \mathcal{K} \times (\mathcal{K}^*)^m \times (\mathbb{R}_+)^m$, such that (38) holds with ∂f_j^* replaced by ∂f_{jR}^* .

Now, from (H₈) it follows that $\|\sum_{j=1}^m \pi_j^R\| \leq C$ which by (42) implies that $\|\pi_j^R\| \leq \frac{C}{\rho^m}$. Hence $\|\pi_j^R\| < R$ for each $j = 1, \dots, m$. Therefore the inclusion (38)₂ takes the form $\alpha_j^R x^R \in \partial f_{jR}^*(\pi_j^R) = \partial f_j^*(\pi_j^R)$, $j = 1, \dots, m$. But this means that $(x^R, (\pi_j^R), (\alpha_j^R))$ is a solution of (38) in which only the basic data f_0, f_j, ϕ_j are involved. $\square$

6 Economic Equilibrium Type Problem

Now we are going to study the behavior of $(x^\varepsilon, (\pi_j^\varepsilon), (\alpha_j^\varepsilon))$ when $\varepsilon \rightarrow 0$ in the circumstances of Propositions 5 and 6.

Theorem 4 Assume that for each $j = 1, \dots, m$ the hypotheses below hold:

- (H₁) $0 \in \text{cl}(\text{Dom}(f_j^*))$ and $\forall y \in \mathcal{K} \setminus \{0\} \exists \beta \geq 0: \beta y \in \text{Int}(\text{Dom}(f_j))$
- (H₂) $\exists \rho_j > 0, \exists M_j > 0: \{\tau \in \mathcal{K}^*: \langle \partial f_j^*(\tau), \tau \rangle \cap (\mathbb{R}_- + \rho_j) = \emptyset\} \subset B(0, M_j)$
- (H₃) $\phi_j(x) \geq 0 \quad \forall x \in \mathcal{K}$
- (H₄) $\partial f_j(0)$ is locally compact
- (H₅) $\phi_j: \mathcal{K} \rightarrow \mathbb{R}_+$ is weakly continuous on $\mathcal{K}$
- (H₆) $\exists \rho > 0, \exists M > 0: \{y \in \mathcal{K}: f_0(y) \leq \sum_{j=1}^m \phi_j(y) + f_0(0) + \rho\} \subset B(0, M)$
- (H₇) $(-\sum_{j=1}^m \partial f_j(0)) \cap \partial f_0(0) = \emptyset$
- (H₉) $f_j^\infty(y) + \phi_j(y) > 0, \forall y \in \mathcal{K}$.

Then there exist disjoint $J, J_0 \subset \{1, \dots, m\}$ with $J \cup J_0 = \{1, \dots, m\}$ and $x \in \mathcal{K} \setminus 0$, $(\pi_j) \in (\mathcal{K}^*)^m$, $(\alpha_j)_{j \in J} \in \text{Int}(\mathbb{R}_+)^{|J|}$, $(\alpha_j)_{j \in J_0} = 0 \in (\mathbb{R}_+)^{|J_0|}$, such that

$$\left. \begin{aligned} \pi_j &\in \partial f_j(\alpha_j x), \quad \langle x, \pi_j \rangle + \phi_j(x) = 0, \quad j \in J \\ \pi_j &\in \partial f_j(0), \quad \langle x, \pi_j \rangle + \phi_j(x) \geq 0, \quad j \in J_0 \\ -\sum_{j=1}^m \pi_j &\in \partial f_0(x). \end{aligned} \right\} \quad (43)$$

Proof To establish the result, Proposition 5 will be used. By the boundedness of $\{x^\varepsilon\}$ and $\{\pi_j^\varepsilon\}$ one can extract sequences $\{x^{\varepsilon_n}\}$ and $\{\pi_j^{\varepsilon_n}\}$, $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$, such that

$$\begin{aligned} x^{\varepsilon_n} &\rightharpoonup x \quad \text{weakly in } X \\ \pi_j^{\varepsilon_n} &\rightharpoonup \pi_j \quad \text{weakly in } X^*, \quad j = 1, \dots, m. \end{aligned}$$

Set $J, J_0, J_\infty \subset \{1, \dots, m\}$ as follows

$$\begin{aligned} j \in J &\Leftrightarrow \lim_{n \rightarrow \infty} \alpha_j^{\varepsilon_n} = \alpha_j \in (0, \infty) \\ j \in J_0 &\Leftrightarrow \lim_{n \rightarrow \infty} \alpha_j^{\varepsilon_n} = 0 \\ j \in J_\infty &\Leftrightarrow \lim_{n \rightarrow \infty} \alpha_j^{\varepsilon_n} = +\infty. \end{aligned}$$

Proceeding like in the proof of Proposition 3 leads to

$$\lim_{n \rightarrow \infty} \langle x^{\varepsilon_n}, \pi_j^{\varepsilon_n} \rangle = \langle x, \pi_j \rangle, \quad j = 1, \dots, m. \quad (44)$$

Taking into account (38)₃ we get

$$\langle x, \pi_j \rangle + \phi_j(x) = 0, \quad j \in J \cup J_\infty \quad (45)$$

$$\langle x, \pi_j \rangle + \phi_j(x) \geq 0, \quad j \in J_0. \quad (46)$$

The maximal monotonicity of ∂f_j easily ensures (43) for $j \in J \cup J_0$.

It remains to show that $J_\infty = \emptyset$. Recall that the recession functional f_j^∞ of f_j is defined by [9, 43]

$$f_j^\infty(y) = \lim_{\lambda \rightarrow +\infty} \frac{1}{\lambda} f_j(\lambda y + y_0), \quad y \in X,$$

where y_0 is an arbitrarily chosen element of $\text{Dom}(f_j)$. Consider the case $\alpha_j^{\varepsilon_n} \rightarrow \infty$, i.e. $j \in J_\infty$. According to (38)₂ we have

$$f_j(y) - f_j(\alpha_j^{\varepsilon_n} x^{\varepsilon_n}) \geq \langle y - \alpha_j^{\varepsilon_n} x^{\varepsilon_n}, \pi_j^{\varepsilon_n} \rangle \quad \forall y \in X.$$

Substituting $y \mapsto \alpha_j^{\varepsilon_n} y + y_0$ with $y_0 \in \text{Dom}(f_j)$ leads to

$$\frac{1}{\alpha_j^{\varepsilon_n}} f_j(\alpha_j^{\varepsilon_n} y + y_0) - \frac{1}{\alpha_j^{\varepsilon_n}} f_j(\alpha_j^{\varepsilon_n} x^{\varepsilon_n}) \geq \langle y - x^{\varepsilon_n}, \pi_j^{\varepsilon_n} \rangle + \frac{1}{\alpha_j^{\varepsilon_n}} \langle y_0, \pi_j^{\varepsilon_n} \rangle \quad \forall y \in X.$$

In view of

$$\liminf_{n \rightarrow \infty} \frac{1}{\alpha_j^{\varepsilon_n}} f_j(\alpha_j^{\varepsilon_n} x^{\varepsilon_n}) \geq f_j^\infty(x)$$

[9], passing to the limit with $n \rightarrow \infty$ allows the conclusion

$$f_j^\infty(y) - f_j^\infty(x) \geq \langle y - x, \pi_j \rangle \quad \forall y \in X,$$

or equivalently,

$$\pi_j \in \partial f_j^\infty(x).$$

Since f_j^∞ is positive homogeneous, convex, lower semicontinuous, we have $f_j^\infty(x) = \langle x, \pi_j \rangle$ which implies $f_j^\infty(x) + \phi_j(x) = 0$ in the contrary to (H_9) . So, $J_\infty = \emptyset$, as desired. Observe that (H_7) ensures that $x \neq 0$. This completes the proof of Theorem 4. $\square$

Proceeding like in the proof of Proposition 6 one can formulate another version of Theorem 4 in which $\text{Dom}(f_0^*) \cap (-\mathcal{K}^*)$ is bounded and the cone $\mathcal{K}^*$ is normal. Then (H_1) can be weakened and replaced by (H'_1) . Moreover, (H'_2) is redundant.

Theorem 5 Assume that $\mathcal{K}^*$ is normal, i.e. there exists $\rho > 0$ such that

$$\|\tau + \pi\| \geq \rho(\|\tau\| + \|\pi\|), \quad \forall \tau, \pi \in \mathcal{K}^* \quad (47)$$

Moreover, assume that for each $j = 1, \dots, m$ the hypotheses below hold:

- (H'_1) $0 \in \text{cl}(\text{Dom}(f_j^*))$
- (H'_3) $\phi_j(x) \geq 0 \quad \forall x \in \mathcal{K}$
- (H_4) $\partial f_j(0)$ is locally compact
- (H_5) $\phi_j: \mathcal{K} \rightarrow \mathbb{R}_+$ is weakly continuous on $\mathcal{K}$
- (H'_6) $\exists \rho > 0, \exists M > 0: \{y \in \mathcal{K}: f_0(y) \leq \sum_{j=1}^m \phi_j(y) + f_0(0) + \rho\} \subset B(0, M)$
- (H_7) $(-\sum_{j=1}^m \partial f_j(0)) \cap \partial f_0(0) = \emptyset$
- (H_8) $\text{Dom}(f_0^*) \cap (-\mathcal{K}^*) \subset B(0, C)$ for some $C > 0$
- (H_9) $f_j^\infty(y) + \phi_j(y) > 0, \forall y \in \mathcal{K}$.

Then there exist disjoint $J, J_0 \subset \{1, \dots, m\}$ with $J \cup J_0 = \{1, \dots, m\}$ and $x \in \mathcal{K} \setminus 0$, $(\pi_j) \in (\mathcal{K}^*)^m$, $(\alpha_j)_{j \in J} \in \text{Int}(\mathbb{R}_+)^{|J|}$, $(\alpha_j)_{j \in J_0} = 0 \in (\mathbb{R}_+)^{|J_0|}$, such that (43) holds.

7 Economic Equilibrium Problem

Throughout this chapter it will be assumed that $\mathcal{K}^*$ is normal, i.e., (42) holds, $\phi_j \geq 0$ $j = 1, \dots, m$, are convex, positive homogeneous functions and

$$f_0(y) = \sum_{j=1}^m \phi_j(y) + \text{ind}_{\mathcal{K}}(y), \quad y \in \mathcal{K}. \quad (48)$$

Since f_0 is positive homogeneous, lower semicontinuous and convex, there exists a convex, closed, nonempty set $\Delta^* \subset X^*$, $0 \in \Delta^*$, $\Delta^* \supset \Delta^* + \mathcal{K}^*$, such that $f_0(y) = \sup_{\tau \in \Delta^*} \langle y, \tau \rangle$, $y \in X$.

Under the foregoing assumptions (43) can be seen as the variational formulation of the economic equilibrium problem as introduced in [34]. f_0 characterizes the total budget function while f_j^* and ϕ_j are dis-utility and budget functions corresponding to j 's consumer, respectively. In (43), $-\pi_j \in -\mathcal{K}^*$, $j = 1, \dots, m$, and $x \in \mathcal{K}$ are the commodity bundles and the price vector in the generalized Walrasian equilibrium,

respectively. Equation (43)₃ can be referred to as the market clearing condition because for strictly positive price vector x the classical formulation of this condition can be easily deduced (cf. [34]). Obviously, we shall be looking for nontrivial price vector x , i.e., $x \neq 0$.

Unfortunately, Theorem 5 does not cover the case of economic equilibrium problem as introduced above. Indeed, (48) does not imply (H'_6) . Moreover, due to $\phi_j(0) = 0$ and $f_j^\infty(0) = 0$, condition (H_9) cannot be satisfied. It must be also emphasized that (H_4) and (H_5) are too strong in many important infinite dimensional applications.

To overcome the aforementioned inconveniences we start our investigations with the approximate model in which f_0 is replaced by f_0^μ where

$$f_0^\mu(y) = \left(\sum_{j=1}^m \phi_j(y) \right)^{1+\mu} + \text{ind}_{\mathcal{K}}(y), \quad y \in X, \quad \mu > 0. \quad (49)$$

Our main result of this section proceeds in several steps formulated in the form of propositions.

First result in this direction is based on Proposition 5.

Proposition 7 *Let $\mathcal{K}^*$ be normal. Moreover, assume that for each $j = 1, \dots, m$ the hypotheses below hold:*

- (H_1) $0 \in \text{cl}(\text{Dom}(f_j^*))$ and $\forall y \in \mathcal{K} \setminus \{0\} \exists \beta \geq 0: \beta y \in \text{Int}(\text{Dom}(f_j))$
- (H'_2) $\exists \rho_j > 0, \exists M_j > 0: \{\tau \in \mathcal{K}^*: \langle \partial f_j^*(\tau), \tau \rangle \cap (\mathbb{R}_- + \rho_j) = \emptyset\} \subset B(0, M_j)$
- (H''_3) $\exists \gamma_j > 0: \phi_j(y) \geq \gamma_j \|y\| \quad \forall y \in \mathcal{K}$
- (H_4) $\partial f_j(0)$ is locally compact
- (H_5) $\phi_j: \mathcal{K} \rightarrow \mathbb{R}_+$ is weakly continuous on $\mathcal{K}$
- (H'_{10}) $0 \notin \partial f_j(0)$.

Then for each $\mu > 0$ there exist $(x, (\pi_j), (\alpha_j)) \in \mathcal{K} \setminus \{0\} \times (\mathcal{K}^*)^m \times (\mathbb{R}_+)^m$ such that

$$\left. \begin{aligned} \pi_j &\in \partial f_j(\alpha_j x) \\ -\sum_{j=1}^m \pi_j &\in \partial f_0^\mu(x) \\ -(x, \pi_j) - \phi_j(x) &\in \partial \text{ind}_{\geq 0}(\alpha_j) \end{aligned} \right\} \quad (50)$$

Proof First we check that under the foregoing assumptions, (H'_6) of Proposition 5 is satisfied. By (H''_3) we get $\sum_{j=1}^m \phi_j(y) \geq \gamma \|y\|$, $\forall y \in K$, $\gamma = \sum_{j=1}^m \gamma_j > 0$. Hence from $(\sum_{j=1}^m \phi_j(y))^{1+\mu} \leq \sum_{j=1}^m \phi_j(y) + \varepsilon$ we obtain the estimate $\|y\| \leq \frac{1}{\gamma}$ provided that $\sum_{j=1}^m \phi_j(y) \leq 1$. For $\sum_{j=1}^m \phi_j(y) > 1$ it follows

$$\sum_{j=1}^m \phi_j(y) \leq \left[(1 + \varepsilon)^{\frac{1}{\varepsilon}} \right]^\mu \leq e$$

giving rise to $\|y\| \leq \frac{\varepsilon}{\gamma}$, $\varepsilon \in (0, \mu]$, as required.

To check that (H_7) of Proposition 5 holds let us notice that $\partial f_0^\mu(0) \subset \mathcal{K}^*$. Assuming that $\tau \in (-\sum_{j=1}^m \partial f_j(0)) \cap \partial f_0^\mu(0) \subset (-\mathcal{K}^*) \cap \mathcal{K}^* = \{0\}$ we have $\tau = \sum_{j=1}^m \tau_j$ where $\tau_j \in \partial f_j(0) \subset \mathcal{K}^*$ and $\sum_{j=1}^m \tau_j = 0$. In view of (42) it follows that $\tau_j = 0$. Consequently, $0 \in \partial f_j(0)$ for each $j = 1, \dots, m$, in contrary to (H'_{10}) . Hence (H_7) holds.

Since the remaining assumptions of Proposition 5 are satisfied immediately, there exists $(x^\varepsilon, (\pi_j^\varepsilon), (\alpha_j^\varepsilon))$ with $x^\varepsilon \neq 0$, such that

$$\left. \begin{aligned} \pi_j^\varepsilon &\in \partial f_j(\alpha_j^\varepsilon x^\varepsilon) \\ -\sum_{j=1}^m \pi_j^\varepsilon &\in \partial f_0^\mu(x^\varepsilon) \\ -\langle x^\varepsilon, \pi_j^\varepsilon \rangle - \phi_j(x^\varepsilon) - \varepsilon \left(\frac{s}{2} + q_j^\varepsilon \right) &\in \partial \text{ind}_{\geq 0}(\alpha_j^\varepsilon) \\ -q_j^\varepsilon = (1 - \varepsilon)^{1/\varepsilon} \left(\frac{1}{\alpha_j^\varepsilon + r_j^\varepsilon} \right)^{1/\varepsilon}, \quad q_j^\varepsilon &\in [-s, 0), \quad -r_j^\varepsilon \in \partial \text{ind}_{[-s, 0]}(q_j^\varepsilon), \quad r_j^\varepsilon \geq 0. \end{aligned} \right\} \quad (51)$$

Now we proceed to show the existence of a nontrivial solution $x \neq 0$ of (50). By the boundedness argument we can suppose that for a subsequence $\varepsilon_n \rightarrow 0$, there exist $x \in \mathcal{K}$ and $\pi_j \in \mathcal{K}^*$ with $x^{\varepsilon_n} \rightharpoonup x$ weakly in X and $\pi_j^{\varepsilon_n} \rightharpoonup \pi_j$ weakly in X^* , $j = 1, \dots, m$.

First we claim that $\|x^{\varepsilon_n}\| \geq \delta$ for some positive $\delta > 0$. Indeed, assuming on the contrary that $x^{\varepsilon_n} \rightarrow 0$ strongly in X , by the maximal monotonicity of ∂f_0^μ we are led to $-\sum_{j=1}^m \pi_j \in \partial f_0^\mu(0) \subset \mathcal{K}^*$ which implies $\pi_j = 0$, $j = 1, \dots, m$.

Set $\tilde{x}^{\varepsilon_n} = \frac{x^{\varepsilon_n}}{\|x^{\varepsilon_n}\|}$, $\tilde{\alpha}_j^{\varepsilon_n} = \|x^{\varepsilon_n}\| \alpha_j^{\varepsilon_n}$.

Case $\tilde{\alpha}_j^{\varepsilon_n} \rightarrow \infty$. This implies $\alpha_j^{\varepsilon_n} \rightarrow \infty$ and by (51)₄ we are led to $q_j^{\varepsilon_n} \rightarrow 0$. By (H''_3) we then obtain

$$\begin{aligned} 0 &< \frac{1}{2} \gamma_j \leq \phi_j(\tilde{x}^{\varepsilon_n}) + \frac{\varepsilon_n}{\|x^{\varepsilon_n}\|} \left(\frac{s}{2} + q_j^{\varepsilon_n} \right) = -\langle \tilde{x}^{\varepsilon_n}, \pi_j^{\varepsilon_n} \rangle \\ &\leq \frac{m_j}{\tilde{\alpha}_j^{\varepsilon_n}} + \frac{1}{\tilde{\alpha}_j^{\varepsilon_n}} f_j^*(\tau) + \|\tau\| \|\tilde{x}^{\varepsilon_n}\|, \quad \forall \tau \in \text{Dom}(f_j^*) \end{aligned} \quad (52)$$

for sufficiently large n . Letting $n \rightarrow \infty$ implies $\frac{1}{2} \gamma_j \leq 0$. The contradiction.

Case $\tilde{\alpha}_j^{\varepsilon_n} \rightarrow 0$. In such a case we are led to $0 \in \partial f_j(0)$ which contradicts (H'_{10}) .

Case $\lim_{n \rightarrow \infty} \tilde{\alpha}_j^{\varepsilon_n} = \tilde{\alpha}_j \in (0, +\infty)$ for each $j = 1, \dots, m$. From $\pi_j^{\varepsilon_n} \in \partial f_j(\tilde{\alpha}_j^{\varepsilon_n} \tilde{x}^{\varepsilon_n})$ we get easily $\liminf_{n \rightarrow \infty} \langle \tilde{x}^{\varepsilon_n}, \pi_j^{\varepsilon_n} \rangle \geq 0$ which leads to $\lim_{n \rightarrow \infty} \langle \tilde{x}^{\varepsilon_n}, \pi_j^{\varepsilon_n} \rangle = 0$, $j = 1, \dots, m$. Further on, according to (51)₂ it follows

$$\|x^{\varepsilon_n}\|^\mu f_0^\mu(\tilde{x}^{\varepsilon_n}) + \frac{1}{\|x^{\varepsilon_n}\|} (f_0^\mu)^* \left(-\sum_{j=1}^m \pi_j^{\varepsilon_n} \right) = \langle \tilde{x}^{\varepsilon_n}, -\sum_{j=1}^m \pi_j^{\varepsilon_n} \rangle$$

giving rise to

$$\lim_{n \rightarrow \infty} \frac{1}{\|x^{\varepsilon_n}\|} (f_0^\mu)^* \left(-\sum_{j=1}^m \pi_j^{\varepsilon_n} \right) = 0.$$

On the other hand, by (51)₃ and (51)₄,

$$\begin{aligned} & \|x^{\varepsilon_n}\|^{\mu} f_0^{\mu}(\tilde{x}^{\varepsilon}) + \frac{1}{\|x^{\varepsilon_n}\|} (f_0^{\mu})^* \left(-\sum_{j=1}^m \pi_j^{\varepsilon_n} \right) \\ &= \left\langle \tilde{x}^{\varepsilon_n}, -\sum_{j=1}^m \pi_j^{\varepsilon_n} \right\rangle = \sum_{j=1}^m \phi_j(\tilde{x}^{\varepsilon_n}) + \frac{\varepsilon_n}{\|x^{\varepsilon_n}\|} \left(m \frac{S}{2} + \sum_{j=1}^m q_j^{\varepsilon_n} \right). \end{aligned} \quad (53)$$

Since $\tilde{\alpha}_j > 0$, so $\alpha_j^{\varepsilon_n} \rightarrow +\infty$, $j = 1, \dots, m$. In view of (51)₄ this implies $\sum_{j=1}^m q_j^{\varepsilon_n} \rightarrow 0$ which thanks to (H_3'') leads to

$$\begin{aligned} & \|x^{\varepsilon_n}\|^{\mu} f_0^{\mu}(\tilde{x}^{\varepsilon}) + \frac{1}{\|x^{\varepsilon_n}\|} (f_0^{\mu})^* \left(-\sum_{j=1}^m \pi_j^{\varepsilon_n} \right) = \left\langle \tilde{x}^{\varepsilon_n}, -\sum_{j=1}^m \pi_j^{\varepsilon_n} \right\rangle \\ &= \sum_{j=1}^m \phi_j(\tilde{x}^{\varepsilon_n}) + \frac{\varepsilon_n}{\|x^{\varepsilon_n}\|} \left(m \frac{S}{2} + \sum_{j=1}^m q_j^{\varepsilon_n} \right) \\ &\geq \sum_{j=1}^m \gamma_j. \end{aligned} \quad (54)$$

Therefore, letting $n \rightarrow \infty$ leads again to the contradiction $0 \geq \sum_{j=1}^m \gamma_j$. The proof of the claim that $\|x^{\varepsilon_n}\| \geq \delta$ is complete.

The condition $\|x^{\varepsilon_n}\| \geq \delta$ allows to establish (50) by repeating the reasoning already presented. Therefore the details will be omitted here. The proof of Proposition 7 is complete. $\square$

Now we are ready to examine the case $\mu \rightarrow 0$.

Proposition 8 *Let $\mathcal{K}^*$ be normal. Moreover, assume that for $j = 1, \dots, m$, the hypotheses below hold:*

- (H_1) $0 \in \text{cl}(\text{Dom}(f_j^*))$ and $\forall y \in \mathcal{K} \setminus \{0\} \exists \beta \geq 0: \beta y \in \text{Int}(\text{Dom}(f_j))$
- (H_2') $\exists \rho_j > 0, \exists M_j > 0: \{\tau \in \mathcal{K}^*: \langle \partial f_j^*(\tau), \tau \rangle \cap (\mathbb{R}_- + \rho_j) = \emptyset\} \subset B(0, M_j)$
- (H_3') $\exists \gamma_j > 0: \phi_j(y) \geq \gamma_j \|y\| \quad \forall y \in \mathcal{K}$
- (H_4) $\partial f_j(0)$ is locally compact
- (H_5) $\phi_j: \mathcal{K} \rightarrow \mathbb{R}_+$ is weakly continuous on $\mathcal{K}$
- (H_7') $(-\sum_{j=1}^m \partial f_j(0)) \cap \Delta^* = \emptyset$
- (H_{10}') $0 \notin \partial f_j(0)$.

Then there exist $(x, (\pi_j), (\alpha_j)) \in \mathcal{K} \setminus \{0\} \times (\mathcal{K}^*)^m \times (\mathbb{R}_+)^m$ and $r \in (0, 1]$ such that

$$\left. \begin{aligned} & \pi_j \in \partial f_j(\alpha_j x), \quad j = 1, \dots, m \\ & -\sum_{j=1}^m \pi_j \in r \partial f_0(x) \\ & -\langle x, \pi_j \rangle - \phi_j(x) \in \partial \text{ind}_{\geq 0}(\alpha_j), \quad j = 1, \dots, m. \end{aligned} \right\} \quad (55)$$

Proof By Proposition 7 we are allowed to suppose that for a sequence $\mu_n \rightarrow 0$ there exists $(x^{\mu_n}, (\pi_j^{\mu_n}), (\alpha_j^{\mu_n})) \in \mathcal{K} \setminus \{0\} \times (\mathcal{K}^*)^m \times (\mathbb{R}_+)^m$ such that

$$\left. \begin{aligned} 0 &\in \partial f_0^{\mu_n}(x^{\mu_n}) + \sum_{j=1}^m \partial f_j(\alpha_j^{\mu_n} x^{\mu_n}) \\ \pi_j^{\mu_n} &\in \partial f_j(\alpha_j^{\mu_n} x^{\mu_n}) \\ -\sum_{j=1}^m \pi_j^{\mu_n} &\in \partial f_0^{\mu_n}(x^{\mu_n}) \\ -\langle x^{\mu_n}, \pi_j^{\mu_n} \rangle - \phi_j(x^{\mu_n}) &\in \partial \text{ind}_{\geq 0}(\alpha_j^{\mu_n}) \end{aligned} \right\} \quad (56)$$

and $x^{\mu_n} \rightharpoonup x$, $\pi_j^{\mu_n} \rightharpoonup \pi_j$ weakly in X and X^* , respectively.

Consider two cases.

Case 1 $\|x^{\mu_n}\| \geq \delta > 0$. By (H_3'') we ensure the boundedness of $(\alpha_j^{\mu_n})$ and therefore we are allowed to assume that $\alpha_j^{\mu_n} \rightarrow \alpha_j \in \mathbb{R}_+$, $j = 1, \dots, m$. The method that has been already explained allows the conclusion $\lim_{n \rightarrow \infty} \langle x^{\mu_n}, \pi_j^{\mu_n} \rangle = \langle x, \pi_j \rangle$, $j = 1, \dots, m$, which by the maximal monotonicity argument leads to $-\sum_{j=1}^m \pi_j \in \partial f_0(x) \subset \Delta^*$ and $\pi_j \in \partial f_j(\alpha_j x)$ for each $j = 1, \dots, m$. In view of (H_7') it follows that $x \neq 0$ and that for at least one $j \in \{1, \dots, m\}$, $\alpha_j > 0$. (H_5) ensures the validity of (55)₃. Finally we conclude that $(x, (\pi_j), (\alpha_j)) \in \mathcal{K} \setminus \{0\} \times (\mathcal{K}^*)^m \times (\mathbb{R}_+)^m$ fulfills (55) with $r = 1$.

Case 2 $\|x^{\mu_n}\| \rightarrow 0$ as $n \rightarrow \infty$. Introduce $\tilde{x}^{\mu_n} = \frac{1}{\|x^{\mu_n}\|} x^{\mu_n}$ and $\tilde{\alpha}_j^{\mu_n} = \alpha_j^{\mu_n} \|x^{\mu_n}\|$, $j = 1, \dots, m$. By the boundedness argument one can suppose that $\tilde{x}^{\mu_n} \rightharpoonup \tilde{x}$ weakly in X and $(\|\tilde{x}^{\mu_n}\|)^{\mu_n} \rightarrow r$ with $r \in [0, 1]$.

The boundedness of $(\tilde{\alpha}_j^{\mu_n})$ follows from the estimate

$$0 < \gamma_j \leq \phi_j(\tilde{x}^{\mu_n}) = -\langle \tilde{x}^{\mu_n}, \pi_j^{\mu_n} \rangle \leq \frac{m_j}{\tilde{\alpha}_j^{\mu_n}} + \frac{1}{\tilde{\alpha}_j^{\mu_n}} f_j^*(\tau) + \|\tau\| \|\tilde{x}^{\mu_n}\|, \quad \forall \tau \in \text{Dom}(f_j^*). \quad (57)$$

Indeed, assuming that $\tilde{\alpha}_j^{\mu_n} \rightarrow +\infty$ as $n \rightarrow \infty$ leads to

$$0 < \gamma_j \leq \|\tau\|, \quad \forall \tau \in \text{Dom}(f_j^*).$$

which thanks to (H_1) implies the contradiction $\gamma_j \leq 0$. Therefore one can suppose that $\tilde{\alpha}_j^{\mu_n} \rightarrow \tilde{\alpha}_j$ for each $j = 1, \dots, m$.

According to (56)₃ it follows

$$(\|x^{\mu_n}\|)^{\mu_n} \left[(f_0(y))^{1+\mu_n} - (f_0(\tilde{x}^{\mu_n}))^{1+\mu_n} \right] \geq \left\langle y - \tilde{x}^{\mu_n}, -\sum_{j=1}^m \pi_j^{\mu_n} \right\rangle, \quad \forall y \in \mathcal{K}, \quad (58)$$

which by substituting $y \mapsto \tilde{x}$ and letting $n \rightarrow \infty$ yields

$$\limsup_{n \rightarrow \infty} \left\langle \tilde{x}^{\mu_n}, \sum_{j=1}^m \pi_j^{\mu_n} \right\rangle \leq \left\langle \tilde{x}, \sum_{j=1}^m \pi_j \right\rangle. \quad (59)$$

As in the proof of Proposition 6 from $\pi_j^{\mu_n} \in \partial f_j(\tilde{\alpha}_j \tilde{x}^{\mu_n})$ we get

$$\liminf_{n \rightarrow \infty} \langle \tilde{x}^{\mu_n}, \pi_j^{\mu_n} \rangle \geq \langle \tilde{x}, \pi_j \rangle, \quad j = 1, \dots, m.$$

Combining this with (59) yields

$$\lim_{n \rightarrow \infty} \langle \tilde{x}^{\mu_n}, \pi_j^{\mu_n} \rangle = \langle \tilde{x}, \pi_j \rangle, \quad j = 1, \dots, m. \quad (60)$$

Thus maximal monotonicity of ∂f_j allows the conclusion that $\pi_j \in \partial f_j(\tilde{\alpha}_j \tilde{x})$ $j = 1, \dots, m$.

Now passing to the limit in (58) leads to

$$r(f_0(y) - f_0(\tilde{x})) \geq \left\langle y - \tilde{x}, -\sum_{j=1}^m \pi_j \right\rangle, \quad \forall y \in \mathcal{K}. \quad (61)$$

Thus the existence of the desired $\tilde{x}$ and r has been established. It remains to show that $\tilde{x} \neq 0$ and $r > 0$.

Indeed, if $r = 0$ then $-\sum_{j=1}^m \pi_j \in \mathcal{K}^*$. On the other hand, $\sum_{j=1}^m \pi_j \in \mathcal{K}^*$ which in view of $\mathcal{K}^* \cap (-\mathcal{K}^*) = \{0\}$ yields $\sum_{j=1}^m \pi_j = 0$ and consequently, because $\mathcal{K}^*$ is normal we have $\pi_j = 0$ for each $j = 1, \dots, m$. Thus $0 \in \partial f_j(\tilde{\alpha}_j \tilde{x})$, $j = 1, \dots, m$, which by (H'_{10}) means that $\tilde{x} \neq 0$ and $\tilde{\alpha}_j > 0$ for each $j = 1, \dots, m$. Further on, from (56)₄ we get

$$0 < \gamma_j \leq \phi_j(\tilde{x}^{\mu_n}) = -\langle \tilde{x}^{\mu_n}, \pi_j^{\mu_n} \rangle$$

which by passing to the limit leads to the contradiction $\gamma_j \leq 0$. This contradiction ensures that $r > 0$.

Let $\tilde{x} = 0$. According to (61), $f_0(y) \geq \left\langle y, -\frac{1}{r} \sum_{j=1}^m \pi_j \right\rangle$ for all $y \in X$ which is equivalent to $-\frac{1}{r} \sum_{j=1}^m \pi_j \in \Delta^*$. But $0 \in \Delta^*$, so $-\sum_{j=1}^m \partial f_j(0) \ni -\sum_{j=1}^m \pi_j \in r\Delta^* \subset \Delta^*$ in the contrary to (H'_7) . The claim follows.

Accordingly we arrive at (55)₂. The validity of (55)₃ is obvious in view of (H_5) and (60). This completes the proof of Proposition 8. $\square$

Remark 1 Since $f_0^* = \text{ind}_{\Delta^*}$ the assumption (H_8) of Proposition 6 takes the form

$$(H'_8) \quad \exists C > 0: \Delta^* \cap (-\mathcal{K}^*) \subset B(0, C).$$

By proceeding like in the proof of Proposition 6 we arrive at the result.

Proposition 9 *Let $\mathcal{K}^*$ be normal. Moreover, let for each $j = 1, \dots, m$, the hypotheses below hold:*

- (H'_1) $0 \in \text{cl}(\text{Dom}(f_j^*))$
- (H'_3) $\exists \gamma_j > 0: \phi_j(y) \geq \gamma_j \|y\| \quad \forall y \in \mathcal{K}$
- (H_4) $\partial f_j(0)$ is locally compact
- (H_5) $\phi_j: \mathcal{K} \rightarrow \mathbb{R}_+$ is weakly continuous on $\mathcal{K}$
- (H'_7) $(-\sum_{j=1}^m \partial f_j(0)) \cap \Delta^* = \emptyset$
- (H'_8) $\exists C > 0: \Delta^* \cap (-\mathcal{K}^*) \subset B(0, C)$
- (H'_{10}) $0 \notin \partial f_j(0)$

Then there exist $(x, (\pi_j), (\alpha_j)) \in \mathcal{K} \setminus \{0\} \times (\mathcal{K}^)^m \times (\mathbb{R}_+)^m$ and $r \in (0, 1]$ such that (55) holds.*

Remark 2 Thanks to the positive homogeneity of $\phi_j(\cdot)$, $j = 1, \dots, m$, and $f_0(\cdot)$ we are allowed to suppose that x in Proposition 9 lies on a sphere with a radius $c > 0$ which may be specified arbitrarily when necessary.

Remark 3 The assumption (H_3'') is too restrictive in many applications. It can be replaced by a weaker one

$$(H_3''') \quad \exists \gamma > 0: \sum_{j=1}^m \phi_j(y) \geq \gamma \|y\| \quad \forall y \in \mathcal{K}.$$

To have the Lagrange multipliers (α_j) finite (i.e. $\alpha_j < +\infty$) we shall need a weaker version of (H_9) , namely

$$(H_9') \quad \phi_j(y) + f_j^\infty(y) > 0 \quad \forall y \in \mathcal{K} \setminus \{0\}.$$

We summarize foregoing remarks in the form of the following result.

Proposition 10 *Let $\mathcal{K}^*$ be normal. Moreover, assume that for each $j \in \{1, \dots, m\}$ the hypotheses below hold:*

$$\begin{aligned} (H_1') & \quad 0 \in \text{cl}(\text{Dom}(f_j^*)) \\ (H_3''') & \quad \exists \gamma > 0: \sum_{j=1}^m \phi_j(y) \geq \gamma \|y\| \quad \forall y \in \mathcal{K} \\ (H_4) & \quad \partial f_j(0) \text{ is locally compact} \\ (H_5) & \quad \phi_j: \mathcal{K} \rightarrow \mathbb{R}_+ \text{ is weakly continuous on } \mathcal{K} \\ (H_7') & \quad \left(-\sum_{j=1}^m \partial f_j(0)\right) \cap \Delta^* = \emptyset \\ (H_8') & \quad \exists C > 0: \Delta^* \cap (-\mathcal{K}^*) \subset B(0, C) \\ (H_9') & \quad \phi_j(y) + f_j^\infty(y) > 0 \quad \forall y \in \mathcal{K} \setminus \{0\} \\ (H_{10}') & \quad 0 \notin \partial f_j(0). \end{aligned}$$

Then there exist $(x, (\pi_j), (\alpha_j)) \in \mathcal{K} \times (\mathcal{K}^*)^m \times (\mathbb{R}_+)^m$, $\|x\| = 1$, and $r \in (0, 1]$ such that

$$\left. \begin{aligned} \pi_j &\in \partial f_j(\alpha_j x), \quad j = 1, \dots, m \\ -\sum_{j=1}^m \pi_j &\in r \partial f_0(x) \\ -\langle x, \pi_j \rangle - \phi_j(x) &\in \partial \text{ind}_{\geq 0}(\alpha_j), \quad j = 1, \dots, m. \end{aligned} \right\} \quad (62)$$

Proof To apply Proposition 9 replace $\phi_j(\cdot)$ by $\phi_j^\varepsilon(y) = \phi_j(y) + \varepsilon \|y\|$, $y \in \mathcal{K}$. Then there exist $(x^\varepsilon, (\pi_j^\varepsilon), (\alpha_j^\varepsilon)) \in \mathcal{K} \times (\mathcal{K}^*)^m \times (\mathbb{R}_+)^m$, $\|x^\varepsilon\| = 1$ and $r^\varepsilon \in (0, 1]$ such that (55) holds, i.e.

$$\left. \begin{aligned} \pi_j^\varepsilon &\in \partial f_j(\alpha_j^\varepsilon x), \quad j = 1, \dots, m \\ -\sum_{j=1}^m \pi_j^\varepsilon &\in r^\varepsilon \partial f_0^\varepsilon(x^\varepsilon) \\ -\langle x^\varepsilon, \pi_j^\varepsilon \rangle - \phi_j(x^\varepsilon) - \varepsilon \|x^\varepsilon\| &\in \partial \text{ind}_{\geq 0}(\alpha_j^\varepsilon), \quad j = 1, \dots, m, \end{aligned} \right\} \quad (63)$$

where $f_0^\varepsilon(\cdot) = f_0(\cdot) + m\varepsilon \|\cdot\|$. By the boundedness we can assume that for a sequence $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$, $x^{\varepsilon_n} \rightharpoonup x$ and $\pi_j^{\varepsilon_n} \rightharpoonup \pi_j$ weakly in X and X^* , respectively,

and $r^{\varepsilon_n} \rightarrow r$. On the way already presented we arrive at $\lim_{n \rightarrow \infty} \langle x^{\varepsilon_n}, \pi_j^{\varepsilon_n} \rangle = \langle x, \pi_j \rangle$, $j = 1, \dots, m$. Thus $\pi_j \in \partial f_j(\alpha_j x)$ provided that $j \in J = \{i \in \{1, \dots, m\} : \alpha_i^{\varepsilon_n} \rightarrow \alpha_i \in \mathbb{R}_+\}$. If $j \in J_\infty = \{i \in \{1, \dots, m\} : \alpha_i^{\varepsilon_n} \rightarrow +\infty\}$ then from

$$0 \geq -\varepsilon_n \|x^{\varepsilon_n}\| = \phi_j(x^{\varepsilon_n}) + \frac{1}{\alpha_j^{\varepsilon_n}} f_j(\alpha_j^{\varepsilon_n} x^{\varepsilon_n}) + \frac{1}{\alpha_j^{\varepsilon_n}} f_j^*(\pi_j^{\varepsilon_n}), \quad j = 1, \dots, m,$$

it results

$$0 \geq \phi_j(x) + f_j^\infty(x). \quad (64)$$

Moreover, passing to the limit in (63)₂ yields

$$r f_0(y) - r f_0(x) \geq \left\langle y - x, -\sum_{j=1}^m \pi_j \right\rangle \quad \forall y \in \mathcal{K}. \quad (65)$$

Now consider two possible cases.

Case 1 $J_\infty \neq \emptyset$. This condition by (H'_9) and (64) implies $x = 0$, and consequently, $\pi_j \in \partial f_j(0)$ for each $j \in J$.

If $r = 0$ then from (65) we get $-\sum_{j=1}^m \pi_j \in \mathcal{K}^*$. Since $\sum_{j=1}^m \pi_j \in \mathcal{K}^*$, in view of $\mathcal{K}^* \cap (-\mathcal{K}^*) = \{0\}$ it follows $\sum_{j=1}^m \pi_j = 0$ and consequently, ($\mathcal{K}^*$ is normal), $\pi_j = 0$ for each $j = 1, \dots, m$. Hence, $0 \in \partial f_j(0)$ for each $j \in J$ which in view of (H'_{10}) is impossible if J is not empty. Thus $J = \emptyset$. But this implies that $\alpha_j^{\varepsilon_n} > 0$ for sufficiently large n and each $j = 1, \dots, m$. However, this can happen when $r^{\varepsilon_n} = 1$. Therefore, from (H'_3) it follows

$$\gamma \leq \sum_{j=1}^m \phi_j(x^{\varepsilon_n}) + m\varepsilon_n = f_0(x^{\varepsilon_n}) + m\varepsilon_n = \left\langle x^{\varepsilon_n}, -\sum_{j=1}^m \pi_j^{\varepsilon_n} \right\rangle.$$

Letting $n \rightarrow \infty$ we are led to a contradiction $\gamma \leq 0$.

If $r > 0$ we proceed similarly. Namely, according to (63)₃,

$$r^{\varepsilon_n} \gamma \leq r^{\varepsilon_n} \sum_{j=1}^m \phi_j(x^{\varepsilon_n}) + r^{\varepsilon_n} m\varepsilon_n = r^{\varepsilon_n} f_0(x^{\varepsilon_n}) + r^{\varepsilon_n} m\varepsilon_n = \left\langle x^{\varepsilon_n}, -\sum_{j=1}^m \pi_j^{\varepsilon_n} \right\rangle.$$

Passing to the limit as $n \rightarrow \infty$ yields again the contradiction $r\gamma \leq 0$.

Case 2 $J_\infty = \emptyset$. In such a case our task is to show that $x \neq 0$ and $r > 0$. As already established, $\pi_j \in \partial f_j(\alpha_j x)$ for each $j = 1, \dots, m$, and (65) holds.

If $r = 0$ then it is easy to arrive at $\pi_j = 0$, $j = 1, \dots, m$, which due to (H'_{10}) makes $x \neq 0$ and $\alpha_j > 0$ for each $j = 1, \dots, m$. But this is only possible when $r = 1$. The contradiction.

If $r > 0$ and $x = 0$ then we get $0 \in r\partial f_0(0) + \sum_{j=1}^m \partial f_j(0)$ which contradicts (H'_7) because $0 \in r\Delta^* + \sum_{j=1}^m \partial f_j(0) \subset \Delta^* + \sum_{j=1}^m \partial f_j(0)$.

The proof of Proposition 10 is complete. □

The assumption (H_4) and (H_5) seem to be too restrictive in many important infinite dimensional applications. Now we show that the equality $f_0(\cdot) = \sum_{j=1}^m \phi_j(\cdot)$ makes this assumption redundant. Recall that $\phi_j(\cdot)$ has been assumed to be convex, lower semicontinuous, positive homogeneous, nonnegative valued.

Now we are ready to formulate our main result concerning the economic equilibrium problem.

Theorem 6 *Let X be a reflexive Banach space and let $\mathcal{K}^*$ be normal. Moreover, assume that for each $j \in \{1, \dots, m\}$ the hypotheses below hold:*

- (H'_1) $0 \in \text{cl}(\text{Dom}(f_j^*))$
- (H'''_3) $\exists \gamma > 0: \sum_{j=1}^m \phi_j(y) \geq \gamma \|y\| \quad \forall y \in \mathcal{K}$
- (H_4) $\partial f_j(0)$ is locally compact
- (H'_7) $(-\sum_{j=1}^m \partial f_j(0)) \cap \Delta^* = \emptyset$
- (H'_8) $\exists C > 0: \Delta^* \cap (-\mathcal{K}^*) \subset B(0, C)$
- (H'_9) $\phi_j(y) + f_j^\infty(y) > 0 \quad \forall y \in \mathcal{K} \setminus \{0\}$
- (H'_{10}) $0 \notin \partial f_j(0)$.

Then there exist $(x, (\pi_j), (\alpha_j)) \in \mathcal{K} \times (\mathcal{K}^*)^m \times (\mathbb{R}_+)^m$, $\|x\| = 1$, and $r \in (0, 1]$ such that

$$\left. \begin{aligned} \pi_j &\in \partial f_j(\alpha_j x), \quad j = 1, \dots, m \\ -\sum_{j=1}^m \pi_j &\in r \partial f_0(x) \\ -\langle x, \pi_j \rangle - \phi_j(x) &\in \partial \text{ind}_{\geq 0}(\alpha_j), \quad j = 1, \dots, m, \end{aligned} \right\} \quad (66)$$

or equivalently, there exists $x \in \mathcal{K}$, $x \neq 0$ and $r \in (0, 1]$ such that

$$0 \in \mathcal{R}(x) + r \partial f_0(x).$$

Proof To show this result a method based on the finite intersection property will be used (cf. [12]). Let Θ be the family of all finite dimensional subspaces of X , ordered by inclusion. Let $F \in \Theta$, $i^F: F \rightarrow X$ be the injection of F into X and $i^{F*}: X^* \rightarrow F^*$ be its adjoint. Set $f_j^F = f_j \circ i^F: F \rightarrow \overline{\mathbb{R}}$, $j = 0, 1, \dots, m$, $\mathcal{K}^F = \{y \in X: i^F(y) \in \mathcal{K}\} = \mathcal{K} \cap F$, $\phi_j^F = \phi_j \circ i^F: \mathcal{K}^F \rightarrow \mathbb{R}$, $j = 1, \dots, m$. Since i^{F*} is surjective we get $\mathcal{K}^{F*} = i^{F*}(\mathcal{K}^*)$ where $\mathcal{K}^{F*}$ is the negative polar of $\mathcal{K}^F$. Set $\Delta^{*F} = i^{F*}(\Delta^*)$.

First we consider the case when (H'''_3) is replaced by (H''_3) .

Notice that (H_4) and (H_5) are redundant in the setting of finite dimensional spaces. From (42) it follows that $\|\tau^F + \kappa^F\|_{F^*} \geq \rho(\|\tau^F\|_{F^*} + \|\kappa^F\|_{F^*}) \quad \forall \tau^F, \kappa^F \in \mathcal{K}^{F*}$, $\|\cdot\|_{F^*}$ being the norm in F^* defined by $\|i^{F*}(\tau)\|_{F^*} = \inf_{\kappa \in \text{Ker } i^{F*}} \|\tau + \kappa\|$, $\forall \tau \in X$. This allows to check that $\Delta^{*F} \cap (-\mathcal{K}^{F*}) \subset B_{F^*}(0, C)$ with the same constant $C > 0$ as in (H'_8) (not depending on F).

Further, set $F_0 \in \Theta$ “large” enough to fulfill the conditions: $0 \notin \partial f_j^{F_0}(0)$, $j = 1, \dots, m$, and $i^{F_0*}(\sum_{j=1}^m \partial f_j(0) + \Delta^*) \neq \emptyset$ which is possible due to (H'_{10}) and (H'_7) , respectively.

Without loss of generality we can suppose that $i^{F*}(\partial f_j(0)) = \partial f_j^F(0)$, $j = 1, \dots, m$, $F \in \Theta$. Indeed, following the technique of Proposition 9 we can suppose that $\text{Dom } f_j^*$, $j = 1, \dots, m$, are bounded which implies that $\text{Dom } f_j = X$, $j = 1, \dots, m$, and the assertion follows.

Now, if we denote by Θ_{F_0} the set of all finite dimensional subspaces of X containing F_0 then for each $F \in \Theta_{F_0}$ all the assumptions of Proposition 9 are fulfilled. Consequently, the existence of $x^F \in F$, $\|x^F\| = 1$, $(\pi_j^F) \in (\mathcal{K}^*)^m$, $(\alpha_j^F) \in \mathbb{R}_+^m$ and $r^F \in (0, 1]$ such that

$$\left. \begin{aligned} i^{F*}(\pi_j^F) &\in \partial f_j^F(\alpha_j^F x^F), \quad \langle x^F, \pi_j^F \rangle + \phi_j(x^F) = 0, \quad \alpha_j^F > 0 \\ i^{F*}(\pi_j^F) &\in \partial f_j^F(0), \quad \langle x^F, \pi_j^F \rangle + \phi_j(x^F) \geq 0, \quad \alpha_j^F = 0 \\ &-\sum_{j=1}^m i^{F*}(\pi_j^F) \in r^F \partial f_0^F(x^F), \end{aligned} \right\} \quad (67)$$

follows. Hence,

$$\left. \begin{aligned} f_j(y) - f_j(\alpha_j^F x^F) &\geq \langle y - \alpha_j^F x^F, \pi_j^F \rangle \quad \forall y \in F, \quad \alpha_j^F > 0 \\ f_j(y) - f_j(0) &\geq \langle y, \pi_j^F \rangle \quad \forall y \in F, \quad \alpha_j^F = 0 \\ f_0(y) - f_0(x^F) &\geq \left\langle y - x^F, -\frac{1}{r^F} \sum_{j=1}^m \pi_j^F \right\rangle \quad \forall y \in F. \\ \langle x^F, \pi_j^F \rangle + \phi_j(x^F) &= 0, \quad \alpha_j^F > 0 \\ \langle x^F, \pi_j^F \rangle + \phi_j(x^F) &\geq 0, \quad \alpha_j^F = 0. \end{aligned} \right\} \quad (68)$$

By (H'_8) and (42) we can suppose that $\{\pi_j^F\}_{F \in \Theta_{F_0}}$ is bounded in X^* . The boundedness of $\{\alpha_j^F\}_{F \in \Theta_{F_0}}$ follows from (H'_3) . Recall that $\{x^F\}_{F \in \Theta_{F_0}}$ lies on the unit sphere and $r^F \in (0, 1]$. Therefore

$$\Omega_F = \bigcup_{\substack{F' \in \Theta_{F_0} \\ F' \supset F}} \left\{ (x^{F'}, (\pi_j^{F'}), (\alpha_j^{F'}), r^{F'}) \right\}, \quad F \in \Theta_{F_0}$$

is contained in the closed ball $B_{X \times (X^*)^m \times \mathbb{R}_+^m \times \mathbb{R}}(0, R)$ of $X \times (X^*)^m \times \mathbb{R}_+^m \times \mathbb{R}$ with the radius R not depending on $F \in \Theta_{F_0}$. Since $B_{X \times (X^*)^m \times \mathbb{R}_+^m \times \mathbb{R}}(0, R)$ is weakly compact and the family $\{\text{weakcl}(\Omega_F)\}_{F \in \Theta_{F_0}}$ has the finite intersection property (easy to be checked), the intersection

$$\bigcap_{F \in \Theta_{F_0}} \{\text{weakcl}(\Omega_F)\} \neq \emptyset$$

is not empty, say

$$(x, (\pi_j), (\alpha_j), r) \in \bigcap_{F \in \Theta_{F_0}} \{\text{weakcl}(\Omega_F)\}. \quad (69)$$

Let $z \in X$ be arbitrarily given. Choose $F \in \Theta_{F_0}$ containing x and z . Taking into account (69) we are allowed to extract a sequence $(x^{F_n}, (\pi_j^{F_n}), (\alpha_j^{F_n}), r^{F_n})$ in Ω_F such that $x^{F_n} \rightharpoonup x$ weakly in X , $\pi_j^{F_n} \rightharpoonup \pi_j$ weakly in X^* , $(\alpha_j^{F_n}) \rightarrow (\alpha_j)$ in $\mathbb{R}^m$ and $r^{F_n} \rightarrow r$ in $\mathbb{R}$ as $n \rightarrow \infty$.

By substituting $y \mapsto x$ into (68) we show by the way already presented that

$$\lim_{n \rightarrow \infty} \langle x^{F_n}, \pi_j^{F_n} \rangle = \langle x, \pi_j \rangle, \quad j = 1, \dots, m. \quad (70)$$

Now letting $n \rightarrow \infty$ in (68) leads to

$$\left. \begin{aligned} f_j(z) - f_j(\alpha_j x) &\geq \langle z - \alpha_j x, \pi_j \rangle, \quad \alpha_j > 0 \\ f_j(z) - f_j(0) &\geq \langle z, \pi_j \rangle, \quad \alpha_j = 0 \\ rf_0(z) - rf_0(x) &\geq \left\langle z - x, -\sum_{j=1}^m \pi_j \right\rangle. \end{aligned} \right\} \quad (71)$$

Since $z \in X$ has been chosen arbitrarily,

$$\left. \begin{aligned} f_j(z) - f_j(\alpha_j x) &\geq \langle z - \alpha_j x, \pi_j \rangle, \quad \forall z \in \mathcal{K}. \\ rf_0(z) - rf_0(x) &\geq \left\langle z - x, -\sum_{j=1}^m \pi_j \right\rangle, \quad \forall z \in \mathcal{K}, \end{aligned} \right\} \quad (72)$$

i.e. $\pi_j \in \partial f_j(\alpha_j x)$, $j = 1, \dots, m$, and $-\sum_{j=1}^m \pi_j \in r\partial f_0(x)$.

We claim that $x \neq 0$ and $r > 0$.

Indeed, if $r = 0$ then from (72)₂ it follows that $-\sum_{j=1}^m \pi_j \in \mathcal{K}^*$. On the other hand, $\sum_{j=1}^m \pi_j \in \mathcal{K}^*$ which in view of $\mathcal{K}^* \cap (-\mathcal{K}^*) = \{0\}$ yields $\sum_{j=1}^m \pi_j = 0$ and consequently, $\pi_j = 0$ for each $j = 1, \dots, m$ ($\mathcal{K}^*$ is normal). Thus $0 \in \partial f_j(\alpha_j x)$, $j = 1, \dots, m$, which by (H_{10}) means that $x \neq 0$ and $\alpha_j > 0$ for each $j = 1, \dots, m$. Further on, from (68)₃ we get

$$0 < \gamma_j \leq \phi_j(x^{F_n}) = -\langle x^{F_n}, \pi_j^{F_n} \rangle$$

which by passing to the limit leads to the contradiction $\gamma_j \leq 0$. This contradiction ensures that $r > 0$.

Let $x = 0$. According to (72)₂, $f_0(z) \geq \langle z, -\frac{1}{r} \sum_{j=1}^m \pi_j \rangle$ for all $z \in \mathcal{K}$, which is equivalent to $-\frac{1}{r} \sum_{j=1}^m \pi_j \in \Delta^*$. But $0 \in \Delta^*$, so $-\sum_{j=1}^m \pi_j \in r\Delta^* \subset \Delta^*$ in the contrary to (H_7) . The assertion follows.

Since f_0 is weakly lower semicontinuous, $\liminf_{n \rightarrow \infty} f_0(x^{F_n}) \geq f_0(x)$. But from (68)₃ and (70) it follows that $\limsup_{n \rightarrow \infty} f_0(x^{F_n}) \leq f_0(x)$ which implies

$$\lim_{n \rightarrow \infty} f_0(x^{F_n}) = f_0(x).$$

Taking into account that $f_0(\cdot) = \sum_{j=1}^m \phi_j(\cdot)$ and that each of $\phi_j(\cdot)$'s is weakly lower semicontinuous, we obtain $\lim_{n \rightarrow \infty} \phi_j(x^{F_n}) = \phi_j(x)$, $j = 1, \dots, m$. Having in mind (70), thanks to (68)₄₋₅ we are led to (62)₃. Summing up, with the use of (H_3'') we have established (62). It must be emphasized that (H_5) has not been applied in the aforementioned procedure.

In the last step we return to (H_3''') . The idea is to consider the regularization $\phi_j(\cdot) + \varepsilon \|\cdot\|$, $\varepsilon > 0$, instead of $\phi_j(\cdot)$. Then for sufficiently small $\varepsilon > 0$ the existence of $(x^\varepsilon, (\pi_j^\varepsilon), (\alpha_j^\varepsilon), r^\varepsilon)$, $\|x^\varepsilon\| = 1$, follows such that

$$\left. \begin{aligned} \pi_j^\varepsilon &\in \partial f_j(\alpha_j^\varepsilon x), \quad j = 1, \dots, m \\ -\sum_{j=1}^m \pi_j^\varepsilon &\in r^\varepsilon \partial f_0^\varepsilon(x^\varepsilon) \\ -\langle x^\varepsilon, \pi_j^\varepsilon \rangle - \phi_j(x^\varepsilon) - \varepsilon \|x^\varepsilon\| &\in \partial \text{ind}_{\geq 0}(\alpha_j^\varepsilon), \quad j = 1, \dots, m, \end{aligned} \right\} \quad (73)$$

where $f_0^\varepsilon(\cdot) = f_0(\cdot) + \varepsilon m \|\cdot\|$. By the boundedness arguments we can assume that for a sequence $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$, $x^{\varepsilon_n} \rightharpoonup x$ and $\pi_j^{\varepsilon_n} \rightharpoonup \pi_j$ weakly in X and X^* , respectively, and $r^{\varepsilon_n} \rightarrow r$. On the way already presented we arrive at $\lim_{n \rightarrow \infty} \langle x^{\varepsilon_n}, \pi_j^{\varepsilon_n} \rangle = \langle x, \pi_j \rangle$, $j = 1, \dots, m$. Thus $\pi_j \in \partial f_j(\alpha_j x)$ provided that $j \in J = \{i \in \{1, \dots, m\} : \alpha_i^{\varepsilon_n} \rightarrow \alpha_i \in \mathbb{R}_+\}$. If $j \in J_\infty = \{i \in \{1, \dots, m\} : \alpha_i^{\varepsilon_n} \rightarrow +\infty\}$ then from

$$0 \geq -\varepsilon_n \|x^{\varepsilon_n}\| = \phi_j(x^{\varepsilon_n}) + \frac{1}{\alpha_j^{\varepsilon_n}} f_j(\alpha_j^{\varepsilon_n} x^{\varepsilon_n}) + \frac{1}{\alpha_j^{\varepsilon_n}} f_j^*(\pi_j^{\varepsilon_n}), \quad j = 1, \dots, m,$$

it results

$$0 \geq \phi_j(x) + f_j^\infty(x). \quad (74)$$

Moreover, passing to the limit in (73)₃ yields

$$r f_0(y) - r f_0(x) \geq \left\langle y - x, -\sum_{j=1}^m \pi_j \right\rangle \quad \forall y \in \mathcal{K}. \quad (75)$$

Proceeding analogously as in the proof of Proposition 10 we establish (62)₁₋₂ with $x \neq 0$ and $r > 0$.

It remains to confirm (62)₃. Since f_0 is weakly lower semicontinuous,

$$\liminf_{n \rightarrow \infty} f_0(x^{\varepsilon_n}) \geq f_0(x).$$

But from (73)₃ and (70) it follows that $\limsup_{n \rightarrow \infty} f_0(x^{\varepsilon_n}) \leq f_0(x)$ which implies

$$\lim_{n \rightarrow \infty} f_0(x^{\varepsilon_n}) = f_0(x).$$

Taking into account that $f_0(\cdot) = \sum_{j=1}^m \phi_j(\cdot)$ and that each of $\phi_j(\cdot)$'s is weakly lower semicontinuous, we obtain

$$\lim_{n \rightarrow \infty} \phi_j(x^{\varepsilon_n}) = \phi_j(x), \quad j = 1, \dots, m.$$

Having in mind (70), thanks to (73)₃ we are led to (62)₃. This completes the proof of Theorem 6. $\square$

Corollary 1 *Under the assumptions of Theorem 6, $(x, (\pi_j), r) \in \mathcal{K} \times (\mathcal{K}^*)^m \times (0, 1]$ is a solution of the following economic equilibrium problem:*

Find $\pi_j \in \mathcal{K}^$, $j = 1, \dots, m$, and $x \in \mathcal{K}$, $\|x\| = 1$, such that*

$$\left. \begin{aligned} f_j^*(\pi_j) &= \inf \{ f_j^*(\tau) : -\langle x, \tau \rangle \leq \phi_j(x) \text{ and } \tau \in \mathcal{K}^* \}, \quad j = 1, \dots, m \\ \sum_{j=1}^m \phi_j(x) &= -\frac{1}{r} \sum_{j=1}^m \langle x, \pi_j \rangle. \end{aligned} \right\}$$

If for each $j = 1, \dots, m$ the inequality constraints $-\langle x, \tau \rangle \leq \phi_j(x)$ hold as equalities then $r = 1$.

In the case $\partial f_j(0) = \emptyset$, $j = 1, \dots, m$, from Theorem 6 we get the result.

Corollary 2 *Let $\mathcal{K}^*$ be normal. Moreover, assume that for each $j \in \{1, \dots, m\}$ the hypotheses below hold:*

$$\begin{aligned} (H'_1) \quad & 0 \in \text{cl}(\text{Dom}(f_j^*)) \\ (H'''_3) \quad & \exists \gamma > 0: \sum_{j=1}^m \phi_j(y) \geq \gamma \|y\| \quad \forall y \in \mathcal{K} \\ (H'_8) \quad & \exists C > 0: \Delta^* \cap (-\mathcal{K}^*) \subset B(0, C) \\ (H'_9) \quad & \phi_j(y) + f_j^\infty(y) > 0 \quad \forall y \in \mathcal{K} \setminus \{0\} \\ (H'_{10}) \quad & \partial f_j(0) = \emptyset, \end{aligned}$$

there exist $(x, (\pi_j), (\alpha_j)) \in \mathcal{K} \times (\mathcal{K}^*)^m \times \text{Int}(\mathbb{R}_+^m)$, $\|x\| = 1$, such that

$$\left. \begin{aligned} \pi_j &\in \partial f_j(\alpha_j x), \quad j = 1, \dots, m \\ -\sum_{j=1}^m \pi_j &\in \partial f_0(x) \\ \langle x, \pi_j \rangle + \phi_j(x) &= 0, \quad j = 1, \dots, m. \end{aligned} \right\} \quad (76)$$

Moreover, $\pi_j \in \mathcal{K}^*$, $j = 1, \dots, m$, and $x \in \mathcal{K}$, $\|x\| = 1$, are solutions of the problem

$$\left. \begin{aligned} f_j^*(\pi_j) &= \inf \{ f_j^*(\tau): -\langle x, \tau \rangle \leq \phi_j(x) \text{ and } \tau \in \mathcal{K}^* \}, \quad j = 1, \dots, m \\ \sum_{j=1}^m \phi_j(x) &= -\sum_{j=1}^m \langle x, \pi_j \rangle. \end{aligned} \right\}$$

If (H'''_3) is replaced by (H'''_3) then (H'_9) is redundant and the following result follows.

Corollary 3 *Let $\mathcal{K}^*$ be normal. Moreover, assume that for each $j \in \{1, \dots, m\}$ the hypotheses below hold:*

$$\begin{aligned} (H'_1) \quad & 0 \in \text{cl}(\text{Dom}(f_j^*)) \\ (H''_3) \quad & \exists \gamma_j > 0: \phi_j(y) \geq \gamma_j \|y\| \quad \forall y \in \mathcal{K} \\ (H'_8) \quad & \exists C > 0: \Delta^* \cap (-\mathcal{K}^*) \subset B(0, C) \\ (H'_{10}) \quad & \partial f_j(0) = \emptyset, \end{aligned}$$

there exist $(x, (\pi_j), (\alpha_j)) \in \mathcal{K} \times (\mathcal{K}^*)^m \times \text{Int}(\mathbb{R}_+^m)$, $\|x\| = 1$, such that (76) holds.

Remark 4 In Theorem 6, if (H'_9) does not hold for some $j \in \{1, \dots, m\}$ then there might happen that $\phi_j(x) = -\langle x, \pi_j \rangle = 0$ and $\alpha_j = +\infty$ (cf. [34]). This is a warning signal for j 's consumer that the bankruptcy may occur.

8 Constrained Multiobjective Optimization Problem

In this section some existence results for a class of multiobjective optimization problems with “budget-like” constraints will be established. For any $\beta = (\beta_j) \in \text{Int } \mathbb{R}^m$ define

$$f_j^\beta(y) = f_j\left(\frac{1}{\beta_j}y\right), \quad y \in \mathcal{K}, \quad j = 1, \dots, m.$$

It is easy to see that

$$\pi \in \partial f_j^\beta(x) \Leftrightarrow \beta_j \pi \in \partial f_j\left(\frac{1}{\beta_j}x\right).$$

One can check immediately that if f_0, f_j, ϕ_j fulfill the assumptions of Corollary 3 then for the data system (f_0, f_j^β, ϕ_j) all these assumptions are fulfilled as well. Accordingly, the result follows.

Proposition 11 *Assume that $\mathcal{K}^*$ is normal. Moreover, suppose that for each $j = 1, \dots, m$, the hypotheses below hold:*

$$\begin{aligned} (H'_1) \quad & 0 \in \text{cl}(\text{Dom}(f_j^*)) \\ (H'_3) \quad & \exists \gamma_j > 0: \phi_j(y) \geq \gamma_j \|y\| \quad \forall y \in \mathcal{K} \\ (H'_8) \quad & \exists C > 0: \Delta^* \cap (-\mathcal{K}^*) \subset B(0, C) \\ (H'_{10}) \quad & \partial f_j(0) = \emptyset. \end{aligned}$$

Then for each $\beta \in (\text{Int } \mathbb{R}_+)^m$ there exists $(x^\beta, (\pi_j^\beta), (\alpha_j^\beta)) \in (\mathcal{K} \setminus \{0\}) \times (\mathcal{K}^)^m \times (\mathbb{R}_+)^m$ such that*

$$\left. \begin{aligned} \pi_j^\beta &\in \frac{1}{\beta_j} \partial f_j\left(\frac{\alpha_j^\beta}{\beta_j} x^\beta\right) \\ -\sum_{j=1}^m \pi_j^\beta &\in \partial f_0(x^\beta) \\ -\langle x^\beta, \pi_j^\beta \rangle - \phi_j(x^\beta) &= 0 \end{aligned} \right\} \quad j = 1, \dots, m. \quad (77)$$

Corollary 4 *Under the assumptions of Proposition 11 it follows*

$$\left. \begin{aligned} \sum_{j=1}^m \frac{1}{\alpha_j^\beta} f_j\left(\frac{\alpha_j^\beta}{\beta_j} x^\beta\right) + f_0(x^\beta) + \sum_{j=1}^m \frac{1}{\alpha_j^\beta} f_j^*(\beta_j \pi_j^\beta) &= 0 \\ \sum_{j=1}^m \frac{1}{\alpha_j^\beta} f_j\left(\frac{\alpha_j^\beta}{\beta_j} x^\beta\right) + f_0(x^\beta) &\leq \sum_{j=1}^m \frac{1}{\alpha_j^\beta} f_j\left(\frac{\alpha_j^\beta}{\beta_j} y\right) + f_0(y) \quad \forall y \in \mathcal{K} \\ \sum_{j=1}^m \frac{1}{\alpha_j^\beta} f_j^*(\beta_j \pi_j^\beta) &\leq \sum_{j=1}^m \frac{1}{\alpha_j^\beta} f_j^*(\beta_j \tau_j), \quad \forall (\tau_j) \in (\mathcal{K}^*)^m, -\sum_{j=1}^m \tau_j \in \Delta^*. \end{aligned} \right\} \quad (78)$$

Define a mapping $\Gamma : \text{Int}(\mathbb{R}_+^m) \rightarrow 2^{\mathbb{R}^m}$ by assigning to each $\beta \in \text{Int}(\mathbb{R}_+^m)$ the set of all corresponding Lagrange multipliers, i.e.

$$\begin{aligned} \Gamma(\beta) &:= \{(\alpha_j^\beta) \in \mathbb{R}_+^m : ((x^\beta, (\pi_j^\beta), (\alpha_j^\beta)) \text{ is a solution of (77)} \\ &\quad \text{for some } (x^\beta, (\pi_j^\beta)) \in (\mathcal{K} \times (\mathcal{K}^*)^m) \text{ with } f_0(x^\beta) = c\}. \end{aligned} \quad (79)$$

Here c is a positive constant which can be arbitrarily specified thanks to the positive homogeneity of ϕ_j and $f_0 = \sum_{j=1}^m \phi_j$. Note that $\Gamma(\cdot)$ has nonempty, compact values, but it is not convex valued.

Our aim now is to examine the conditions under which Γ has at least one fixed point. Notice, that if Γ happens to have a fixed point, say β , the corresponding x^β is a solution of the minimizing problem

$$\sum_{j=1}^m \frac{1}{\beta_j} f_j(x^\beta) + f_0(x^\beta) \leq \sum_{j=1}^m \frac{1}{\beta_j} f_j(y) + f_0(y) \quad \forall y \in \mathcal{K} \quad (80)$$

with the “budget-like” constraints

$$-(x^\beta, \pi_j^\beta) - \phi_j(x^\beta) = 0, \quad j = 1, \dots, m. \quad (81)$$

It is easy to observe that in such a case x^β is a Pareto optimal solution of a multiobjective optimization problem related to the vector objective $(f_0, f_1, \dots, f_m)$. Moreover, the validity of “budget-like” constraints (81) is ensured.

Proposition 12 *Under the assumptions of Proposition 11 and*

$$(H_{11}) \quad \partial f_j^*(0) = \emptyset,$$

there exist $\sigma, S \in \text{Int } \mathbb{R}_+^m$ with $\sigma_j < S_j$, $j = 1, \dots, m$, such that

$$\beta = (\beta_j) \in [\sigma, S] \implies (\alpha_j^\beta) \in [\sigma, S], \quad (82)$$

where $[\sigma, S] = \{(t_j) : \sigma_j \leq t_j \leq S_j, j = 1, \dots, m\}$, i.e. $\Gamma([\sigma, S]) \subset [\sigma, S]$.

Proof Let $\beta \in \text{Int } \mathbb{R}_+^m$ and

$$(x^\beta, (\pi_j^\beta), (\alpha_j^\beta)), \quad x^\beta \neq 0,$$

be a solution of (77). Since f_0, ϕ_j are positive homogeneous, one can suppose that

$$\|x^\beta\| = c > 0$$

for some fixed $c > 0$ which will be specified later.

Since f_j^* is convex, proper and lower semicontinuous, there exist $a_j, b_j \geq 0$ with

$$f_j^*(\tau) \geq -a_j - b_j \|\tau\|, \quad \forall \tau \in \text{Dom}(f_j^*). \quad (83)$$

In view of (H'_8) and (42), from (77)₂ it follows

$$\|\pi_j^\beta\| \leq \frac{C}{\mu^{m-1}}. \quad (84)$$

Now we claim that for each $S = (S_j) \in \text{Int } \mathbb{R}_+^m$ there exists $\sigma_S = (\sigma_{Sj})$, $\sigma_{Sj} > 0$, such that

$$\beta = (\beta_j) \in [\sigma_S, S] \implies (\alpha_j^\beta) \in [\sigma_S, \infty). \quad (85)$$

On the contrary, suppose that for some $j \in \{1, \dots, m\}$ there exists $\sigma_j^n \rightarrow 0$ as $n \rightarrow \infty$, such that $\sigma_j^n \leq \beta_j^n \leq S_j$ and $\alpha_j^{\beta^n} < \sigma_j^n$. Thus $\alpha_j^{\beta^n} \rightarrow 0$ as $n \rightarrow \infty$ and

$$\frac{\alpha_j^{\beta^n}}{\beta_j^n} \leq \frac{\alpha_j^{\beta^n}}{\sigma_j^n} < 1.$$

One can suppose that

$$\frac{\alpha_j^{\beta^n}}{\beta_j^n} \rightarrow r, \quad \text{as } n \rightarrow \infty, \quad r \in [0, 1].$$

Now consider two possible cases: 1) $r = 0$ and 2) $r \in (0, 1]$.

1) If $r = 0$, then

$$\frac{\alpha_j^{\beta^n}}{\beta_j^n} x^{\beta^n} \rightarrow 0, \quad \text{strongly in } X.$$

By the boundedness of (β_j^n) we are allowed to suppose that $\beta_j^n \pi_j^n \rightharpoonup \hat{\pi}_j$ weakly in X^* for some $\hat{\pi}_j \in X^*$. Since $\beta_j^n \pi_j^{\beta^n} \in \partial f_j(\frac{\alpha_j^{\beta^n}}{\beta_j^n} x^{\beta^n})$, thanks to the maximal monotonicity of ∂f_j it follows that $\hat{\pi}_j \in \partial f_j(0)$, in contrary to (H'_{10}) .

2) If $r \in (0, 1]$, then $\beta_j^n \rightarrow 0$. By the boundedness of the corresponding x^{β^n} and $\pi_j^{\beta^n}$ it follows that $\beta_j^n \pi_j^{\beta^n} \rightarrow 0$ strongly in X^* and $\frac{\alpha_j^{\beta^n}}{\beta_j^n} x^{\beta^n} \rightharpoonup rx$ weakly in X for some $x \in X$ (passing to a subsequence, if necessary). Since $\frac{\alpha_j^{\beta^n}}{\beta_j^n} x^{\beta^n} \in \partial f_j^*(\beta_j^n \pi_j^{\beta^n})$, thanks to the maximal monotonicity of ∂f_j^* it follows that $rx \in \partial f_j^*(0)$, in contrary to (H_{11}) . The claim has been established.

From now on, fix $S_j = S$, $j = 1, \dots, m$, with $S \geq 2$. For simplicity of notations we use the same symbol S for the vector $(S, \dots, S)$. As already established, there exists σ_S corresponding to S such that

$$\beta_j \in [\sigma_{Sj}, S] \implies \alpha_j^\beta \in [\sigma_{Sj}, \infty), \quad j = 1, \dots, m. \quad (86)$$

Using (77)₁ we get

$$\begin{aligned} \alpha_j^\beta \hat{\gamma} c &\leq \alpha_j^\beta \phi_j(x^\beta) = -\alpha_j^\beta \langle x^\beta, \pi_j^\beta \rangle \leq -\alpha_j^\beta \langle x^\beta, \tau \rangle + f_j^*(\beta_j \tau) - f_j^*(\beta_j \pi_j^\beta) \\ &\leq \alpha_j^\beta \|x^\beta\| \|\tau\| + f_j^*(\beta_j \tau) + a_j + b_j \beta_j \|\pi_j^\beta\| \\ &\leq \alpha_j^\beta c \|\tau\| + f_j^*(\beta_j \tau) + a_j + b_j \beta_j \frac{C}{\mu^{m-1}}, \end{aligned}$$

$\hat{\gamma} = \min_{j=1, \dots, m} \{\gamma_j\}$, which is valid for any $\tau \in \mathcal{K}^*$ and $\beta \in \text{Int } \mathbb{R}_+^m$. Substituting $\tau \in \mathcal{K}^*$ with $\|\tau\| \leq \frac{\hat{\gamma}}{2}$ yields

$$\alpha_j^\beta \frac{\hat{\gamma}}{2} c \leq f_j^*(\beta_j \tau) + a_j + b_j \beta_j \frac{C}{\mu^{m-1}},$$

Now define

$$d_{Sj} := \max \left\{ 0, \inf \{ f_j^*(t_j \tau) : \|\tau\| \leq \frac{\hat{\gamma}}{2}, \sigma_{Sj} \leq t_j \} \right\}.$$

Then the following estimate follows

$$\alpha_j^\beta \frac{\hat{\gamma}}{2} c \leq d_{Sj} + a_j + b_j \beta_j \frac{C}{\mu^{m-1}}, \quad \forall \beta_j \in [\sigma_{Sj}, S] \quad (87)$$

from which we obtain

$$\begin{aligned} \sum_{j=1}^m \alpha_j^\beta &\leq \frac{2}{c\hat{\gamma}} \sum_{j=1}^m \left(d_{Sj} + a_j + b_j \beta_j \frac{C}{\mu^{m-1}} \right) \\ &\leq \frac{2}{c\hat{\gamma}} \sum_{j=1}^m (d_{Sj} + a_j) + S \frac{2C}{c\hat{\gamma}\mu^{m-1}} \sum_{j=1}^m b_j \quad \forall \beta \in [\sigma_S, S]. \end{aligned}$$

Now we are ready to choose $c > 0$ in such a way to fulfill the desired condition

$$\beta \in [\sigma_S, S] \implies \alpha^\beta \in [\sigma_S, S]. \quad (88)$$

Namely, if

$$\max \left\{ \frac{2}{\hat{\gamma}} \sum_{j=1}^m (d_{Sj} + a_j), \frac{4C}{\hat{\gamma}\mu^{m-1}} \sum_{j=1}^m b_j \right\} \leq c$$

then

$$\sum_{j=1}^m \alpha_j^\beta \leq 1 + \frac{1}{2} S \leq S, \quad (89)$$

because $S \geq 2$. The proof of Proposition 12 is complete. $\square$

Proposition 13 *Under the assumptions of Proposition 12 the mapping Γ has at least one fixed point.*

Proof By (77) and (78)₁, and the boundedness of $(x^\beta, (\pi_j^\beta))$ in $X \times (X^*)^m$ we conclude that Γ has compact, nonempty values and it is closed in the sense that whenever $\beta^n \rightarrow \beta$ as $n \rightarrow \infty$ then each cluster point of $\{\alpha^{\beta^n}\} \subset \Lambda(\beta^n)$ lies in $\Lambda(\beta)$. Therefore, a function $\Psi : [\sigma, S] \times [\sigma, S] \rightarrow \mathbb{R}$ given by

$$\Psi(\beta, \alpha) = \min_{\tilde{\alpha} \in \Lambda(\beta)} \langle \beta - \tilde{\alpha}, \beta - \alpha \rangle_{\mathbb{R}^m}, \quad (\beta, \alpha) \in [\sigma, S] \times [\sigma, S], \quad (90)$$

where $\langle \cdot, \cdot \rangle_{\mathbb{R}^m}$ stands for the inner product in $\mathbb{R}^m$, is concave with respect to α (fixed β) and lower semicontinuous as a function of β (fixed α). By the Ky Fan inequality [8] there exists $\hat{\beta} \in [\sigma, S]$ with the property that

$$\Psi(\hat{\beta}, \alpha) \leq \Psi(\alpha, \alpha) = 0, \quad \forall \alpha \in [\sigma, S], \quad (91)$$

This implies the existence of $\alpha^{\hat{\beta}} \in \Lambda(\hat{\beta})$ such that $\alpha^{\hat{\beta}} = \hat{\beta}$, i.e. $\hat{\beta} \in [\sigma_S, S]$ is a fixed point of Λ . The proof is complete. $\square$

Proposition 14 *Under the assumptions of Proposition 12 there exists $(x^\beta, (\pi_j^\beta), (\beta_j))$ such that*

$$\left. \begin{aligned} \pi_j^\beta &\in \frac{1}{\beta_j} \partial f_j(x^\beta) \\ -\sum_{j=1}^m \pi_j^\beta &\in \partial f_0(x^\beta) \\ -\langle x^\beta, \pi_j^\beta \rangle - \phi_j(x^\beta) &= 0 \end{aligned} \right\} \quad j = 1, \dots, m. \quad (92)$$

Moreover,

$$\left. \begin{aligned} \sum_{j=1}^m \frac{1}{\beta_j} f_j(x^\beta) + f_0(x^\beta) + \sum_{j=1}^m \frac{1}{\beta_j} f_j^*(\beta_j \pi_j^\beta) &= 0 \\ \sum_{j=1}^m \frac{1}{\beta_j} f_j(x^\beta) + f_0(x^\beta) &\leq \sum_{j=1}^m \frac{1}{\beta_j} f_j(y) + f_0(y) \quad \forall y \in \mathcal{K} \\ \sum_{j=1}^m \frac{1}{\beta_j} f_j^*(\beta_j \pi_j^\beta) &\leq \sum_{j=1}^m \frac{1}{\beta_j} f_j^*(\beta_j \tau_j), \quad \forall (\tau_j) \in (\mathcal{K}^*)^m, -\sum_{j=1}^m \tau_j \in \Delta^*. \end{aligned} \right\} \quad (93)$$

In other words, a multiobjective optimization problem with “budget-like” constraints (related to the vector objective $(f_0, f_1, \dots, f_m)$ and the “budget functions” $(\phi_1, \dots, \phi_m)$) admits at least one Pareto optimal solution.

Now we are ready to formulate the main result of this chapter.

Theorem 7 *Assume that $\mathcal{K}^*$ is normal and $f_0(y) = \sum_{j=1}^m \phi_j(y)$, $y \in \mathcal{K}$, where $\phi_j(\cdot)$ are convex, lower semicontinuous, positive homogeneous, nonnegative valued. Suppose that for each $j = 1, \dots, m$ the hypotheses below hold:*

- $(H'_1) \quad 0 \in \text{cl}(\text{Dom}(f_j^*))$
- $(H'_3) \quad \exists \gamma_j > 0: \phi_j(y) \geq \gamma_j \|y\| \quad \forall y \in \mathcal{K}$
- $(H'_8) \quad \exists C > 0: \Delta^* \cap (-\mathcal{K}^*) \subset B(0, C)$
- $(H_{13}) \quad \exists G > 0: \sum_{j=1}^m \phi_j(y) \leq G \|y\| \quad \forall y \in \mathcal{K}.$

Moreover, let one of the two assumptions below hold:

- $(H''_{10}) \quad \partial f_j(0) = \partial f_j^*(0) = \emptyset, \quad j = 1, \dots, m,$

or

- $(H_{12}) \quad \limsup_{\|y\| \rightarrow \infty} \frac{\langle y, \partial f_j(y) \rangle}{\phi_j(y)} < 0, \quad j=1, \dots, m.$

Then there exist $(x^\beta, (\pi_j^\beta), (\beta_j))$ such that (92) and (93) hold.

Proof The case (H''_{10}) has been already considered in Proposition 14.

Suppose that (H_{12}) is satisfied. Fix any $\varepsilon \in (0, 1)$ and introduce the data system on $(X \times \mathbb{R}) \times (X^* \times \mathbb{R})$ by

$$\left. \begin{aligned} \tilde{f}_j^\varepsilon(\tau, q) &= f_j^*(\tau) + \varphi^\varepsilon(q), \quad \tau \in \mathcal{K}^*, \quad q \in \mathbb{R}_- \\ \tilde{f}_0^\varepsilon(y, z) &= f_0(y) + z, \quad y \in \mathcal{K}, \quad z \in \mathbb{R}_+ \\ \tilde{\phi}_j^\varepsilon(y, z) &= \phi_j(y) + s_j z, \quad y \in \mathcal{K}, \quad z \in \mathbb{R}_+, \end{aligned} \right\} \quad (94)$$

where $\varphi^\varepsilon(q) = -\varepsilon(-q)^{1-\varepsilon}$ and $s_j > 0$ are a positive constants with $\sum_{j=1}^m s_j = 1$. Note that $\partial \tilde{f}_j^{\star^\varepsilon}(0, 0) = \emptyset$ and $\partial(\tilde{f}_j^{\star^\varepsilon})^\star(0, 0) = \emptyset$. The remaining assumptions of Proposition 14 hold as well. Therefore, for sufficiently large c there exists $(x^s, z^s, (\pi_j^s), (q_j^s), (\beta_j^s))$ with $\|x^s\| + z^s = c^s$, such that

$$\begin{aligned} \pi_j^s &\in \frac{1}{\beta_j^s} \partial f_j(x^s), \quad z^s = \varepsilon(1 - \varepsilon) \left(\sum_{j=1}^m \frac{1}{\beta_j^s} \right)^\varepsilon, \quad -q_j^s = \frac{1}{\beta_j^s} \left(\sum_{k=1}^m \frac{1}{\beta_k^s} \right)^{-1} \\ &\quad - \sum_{j=1}^m \pi_j^s \in \partial f_0(x^s), \quad - \sum_{j=1}^m q_j^s = 1 \\ &\quad - \langle x^s, \pi_j^s \rangle - \phi_j(x^s) - z^s(s_j + q_j^s) = 0. \end{aligned} \quad (95)$$

From (95)₃ we get

$$-\frac{z^s}{\phi_j(x^s)}(s_j + q_j^s) \in 1 + \frac{1}{\beta_j^s} \frac{\langle x^s, \partial f_j(x^s) \rangle}{\phi_j(x^s)}. \quad (96)$$

Taking into account (H_{12}) we can assume that

$$\frac{\langle x^s, \partial f_j(x^s) \rangle}{\phi_j(x^s)} \leq -\rho, \quad j = 1, \dots, m$$

for some $\rho > 0$. Thus

$$-\frac{z^s}{\phi_j(x^s)}(s_j + q_j^s) \leq 1 - \frac{\rho}{\beta_j^s}$$

from which it follows

$$-z^s(s_j + q_j^s) \leq \phi_j(x^s) - \frac{\rho}{\beta_j^s} \phi_j(x^s) \leq \phi_j(x^s) - \min_{j=1, \dots, m} \{\gamma_j\} \|x^s\| \frac{\rho}{\beta_j^s}.$$

Adding by sides these inequalities (recall that $\sum_{j=1}^m s_j = -\sum_{j=1}^m q_j = 1$) yields

$$0 \leq f_0(x^s) - \rho \|x^s\| \min_{j=1, \dots, m} \{\gamma_j\} \sum_{j=1}^m \frac{1}{\beta_j^s} \leq G \|x^s\| - \rho \|x^s\| \min_{j=1, \dots, m} \{\gamma_j\} \sum_{j=1}^m \frac{1}{\beta_j^s}.$$

Hence

$$\sum_{j=1}^m \frac{1}{\beta_j^s} \leq \frac{G}{\rho \min_{j=1, \dots, m} \{\gamma_j\}}$$

and, consequently,

$$-q_j^s = \frac{1}{\beta_j^s} \left(\sum_{k=1}^m \frac{1}{\beta_k^s} \right)^{-1} \geq \frac{\rho \min_{j=1, \dots, m} \{\gamma_j\}}{SG}, \quad j = 1, \dots, m. \quad (97)$$

Accordingly, it has been shown that for a convex, closed set Q given by

$$Q = \left\{ (s_j) : s_j \geq \frac{\rho \min_{j=1, \dots, m} \{\nu_j\}}{SG}, \sum_{j=1}^m s_j = 1 \right\}$$

the implication holds

$$s = (s_j) \in Q \implies -q^s = (-q_j^s) \in Q. \quad (98)$$

Following the lines of the proof of Proposition 13 we assert that the mapping S which assigns to any $s \in Q$ the set of all corresponding q^s 's, i.e. $S(s) = \{q^s\}$, is closed and compact valued. Moreover, if $Q \ni s^n \rightarrow s \in Q$ as $n \rightarrow \infty$, then each cluster point of $\{q^{s^n}\}$ lies in $S(s)$. Again, by the Ky Fan inequality the existence of a fixed point of S is ensured. By (95)₃ we arrive at the desired conclusion. The proof is complete. $\square$

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References

1. Aliprantis, C.D., Brown, D.J., Burkinshaw, O.: Existence and Optimality of Competitive Equilibria. Springer, Berlin (1989)
2. Aliprantis, C.D., Florenzano, M., Tourky, R.: Linear and non-linear price decentralization. *J. Econ. Theory* **121**, 51–74 (2005)
3. Aliprantis, C.D., Montero, P.K., Tourky, R.: Non-marketed options, non-existence of equilibria, and nonlinear prices. *J. Econ. Theory* **114**, 345–357 (2004)
4. Aliprantis, C.D., Tourky, R., Yannelis, N.C.: A theory of value with nonlinear prices. *J. Econ. Theory* **100**, 22–72 (2001)
5. Arrow, K., Debreu, G.: Existence of an equilibrium for a competitive economy. *Econometrica* **22**, 264–290 (1954)
6. Arrow, K.J., Intriligator, M.D.: Handbook of Mathematical Economics, vol. II. North-Holland, Amsterdam (1982)
7. Attouch, H., Brézis, H.: Duality for the sum of convex functions. In: Barroso, J.A. (ed.) *Aspects of Mathematics and its Applications*. Elsevier, Amsterdam (1986)
8. Aubin, J.P.: Optima and Equilibria. Springer, Berlin (1993)
9. Baiocchi, C., Buttafazzo, G., Gastaldi, F., Tomarelli, F.: General existence theorems for unilateral problems in continuum mechanics. *Arch. Ration. Mech. Anal.* **100**, 149–188 (1988)
10. Bao, T.Q., Gupta, P., Mordukhovich, B.S.: Necessary conditions in multiobjective optimization with equilibrium constraints. *J. Optim. Theory Appl.* **134**, 179–203 (2007)
11. Breckner, W.W., Kolmán, I.: Dualität bei Optimierungsaufgaben in topologischen vektorräumen. *Mathematica* **10**(33), 229–244 (1968)
12. Browder, F.E., Hess, P.: Nonlinear mappings of monotone type in Banach spaces. *J. Funct. Anal.* **11**, 251–294 (1972)
13. Chen, G.Y., Huang, X.X., Yang, X.Q.: Vector optimization: set-valued and variational analysis. In: *Lect. Notes Econ. Math. Syst.*, vol. 541. Springer (2005)
14. Chichilnisky, G.: A unified perspective on resource allocation: limited arbitrage is necessary and sufficient for the existence of a competitive equilibrium, the core and social choice. In: Chichilnisky, G. (ed.) *Topology and Markets*. American Mathematical Society and the Fields Institute for Research in Mathematical Sciences, Toronto (1999)

15. Chichilnisky, G., Heal, G.: A unified treatment of finite and infinite economies: limited arbitrage is necessary and sufficient for the existence of equilibrium and the core. *Econ. Theory* **12**, 163–176 (1998)
16. Eaves, C.: Homotopies for computation of fixed points. *Math. Program.* **3**, 1–22 (1972)
17. Ekeland, I., Temam, R.: *Convex Analysis and Variational Problems*. North-Holland, Amsterdam (1976)
18. Elster, R., Gerth, C., Göpfert, A.: Duality in geometric vector optimization. *Optimization* **20**, 457–476 (1989)
19. Gale, D., Mas-Colell, A.: An equilibrium existence theorem for a general model without ordered preferences. *J. Math. Econ.* **2**, 9–15 (1975)
20. Geoffrion, A.M.: Proper efficiency and the theory of vector maximization. *J. Math. Anal. Appl.* **22**, 618–630 (1968)
21. Göpfert, A., Riahi, H., Tammer, C., Zalinescu, C.: *Variational methods in partially ordered spaces*. In: CMS Books in Mathematics, vol. 17. Springer (2003)
22. Hirsh, M., Smale, S.: On algorithms for solving $f(x) = 0$. *Commun. Pure Appl. Math.* **32**, 281–312 (1979)
23. Jahn, J.: *Vector Optimization: Theory, Applications and Extensions*. Series Oper. Res. Springer (2004)
24. Jofré, A., Rockafellar, R.T., Wets, R.J-B.: Variational inequalities and economic equilibrium. *Math. Oper. Res.* **32**, 32–50 (2007)
25. Luc, D.T.: Theory of vector optimization. In: *Lecture Notes in Economics and Mathematical Systems*, vol. 139. Springer (1989)
26. Mas-Colell, A.: The price equilibrium existence problem in topological vector lattices. *Econometrica* **54**, 1039–1053 (1986)
27. Mas-Colell, A., Richard, S.F.: A new approach to the existence of equilibria in vector lattices. *J. Econ. Theory* **53**, 1–11 (1991)
28. Mas-Colell, A., Whinston, M.D., Green, J.R.: *Microeconomic Theory*. Oxford University Press, New York (1995)
29. McKenzie, L.: On the existence of general equilibrium for a competitive market. *Econometrica* **27**, 54–71 (1959)
30. Miettinen, K.M.: *Nonlinear Multiobjective Optimization*. Kluwer, Boston (1998)
31. Mordukhovich, B.S.: Variational analysis and generalized differentiation. I: Basic theory. In: *Grundlehren Series (Fundamental Principles of Mathematical Sciences)*, vol. 330. Springer, Berlin (2006)
32. Mordukhovich, B.S.: Variational analysis and generalized differentiation. II: Applications. In: *Grundlehren Series (Fundamental Principles of Mathematical Sciences)*, vol. 331. Springer, Berlin (2006)
33. Nagurney, A., Siokos, S.: *Financial Networks: Statics and Dynamics*. Springer, Berlin (1997)
34. Naniewicz, Z.: Pseudo-monotonicity and economic equilibrium problem in reflexive Banach space. *Math. Oper. Res.* **32**, 436–466 (2007)
35. Naniewicz, Z.: Pareto optimality and Walrasian equilibria. *J. Math. Anal. Appl.* **341**, 1365–1381 (2008)
36. Nash, J.F.: Equilibrium points in n -person games. *Proc. Natl. Acad. Sci. U. S. A.* **36**, 48–49 (1950)
37. Negishi, T.: Welfare economics and existence of an equilibrium for a competitive economy. *Metroeconomica* **12**, 92–97 (1960)
38. Neumann, V.J.: A model of general economic equilibrium. *Rev. Econ. Stud.* **XIII**, 1–9 (1945–1946). G. Morgenstern, transl.
39. Nockowska-Rosiak, M.: Economic equilibrium through variational inequalities. *Appl. Math.* **36**, 43–58 (2009)
40. Pallaschke, D., Rolewicz, S.: *Foundation of Mathematical Optimization*. Kluwer, Boston (1997)
41. Peressini, A.L.: *Ordered topological vector spaces*. Harper & Row, New York (1967)
42. Rockafellar, R.T.: Extension of Fenchel's duality theorem for convex functions. *Duke Math. J.* **33**, 81–90 (1966)
43. Rockafellar, R.T.: *Convex Analysis*. Princeton University Press, Princeton (1970)
44. Sawaragi, Y., Nakayama, H., Tanino, T.: *Theory of Multiobjective Optimization*. Academic, New York (1985)
45. Scarf, H.E.: *The Computation of Economic Equilibria*. Yale University Press, New Haven (1973)
46. Smale, S.: Exchange processes with price adjustment. *J. Math. Econ.* **3**, 211–226 (1976)
47. Tanino, T.: Conjugate duality in vector optimization. *J. Math. Anal. Appl.* **167**, 84–97 (1992)

48. Wanka, G., Boř, R.-I.: Multiobjective duality for convex-linear problems II. *Math. Methods Oper. Res.* **53**, 419–433 (2001)
49. Wanka, G., Boř, R.-I.: A new duality approach for multiobjective convex optimization problems. *J. Nonlinear Convex Anal.* **3**, 41–57 (2002)
50. Wanka, G., Boř, R.-I.: Duality for multiobjective optimization problems with convex objective functions and D.C. constraints. *J. Math. Anal. Appl.* **315**, 526–543 (2006)