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Estimation of parameters for trend-renewal processes

Alicja Jokiel-Rokita · Ryszard Magiera

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Abstract Methods of estimating unknown parameters of a trend function for trend-renewal processes are investigated in the case when the renewal distribution function is unknown. If the renewal distribution is unknown, then the likelihood function of the trend-renewal process is unknown and consequently the maximum likelihood method cannot be used. In such a situation we propose three other methods of estimating the trend parameters. The methods proposed can also be used to predict future occurrence times. The performance of the estimators based on these methods is illustrated numerically for some trend-renewal processes for which the statistical inference is analytically intractable.

Keywords Parameter estimation · Trend-renewal process · Power-law process · Weibull-power-law trend-renewal process

1 Introduction

We consider a class of estimation problems for the stochastic model determined by the trend-renewal process (TRP) which is defined to be a time-transformed renewal process (time-transformed RP), where the time transformation is given by a trend function $\lambda(\cdot)$. It contains the nonhomogeneous Poisson process (NHPP) and RP as special cases and serves as a useful reliability model for repairable systems.

The TRP was introduced and investigated first by Lindqvist (1993) and by Lindqvist et al. (1994) (see also Lindqvist and Doksum (2003), and Lindqvist (2006)). Parametric inference on the parameters of the TRP was considered in the paper of Lindqvist et al. (2003), where the authors also proposed corresponding models, called heterogeneous trend-renewal processes, that extend the TRP to cases involving unobserved heterogeneity. Nonparametric maximum likelihood (ML) estimation of the trend function of a TRP under the often natural condition that $\lambda(\cdot)$ is monotone was considered by Heggland and Lindqvist (2007).

Peña and Hollander (2004) presented a general class of models that allows the researcher to incorporate the effect of interventions performed on a unit after each event occurrence, the impact of accumulating events on a unit, the effect of unobservable random effects of frailties, and the effect of covariates that could be time-dependent. The ML estimators of this general models parameters were presented, and their finite and asymptotic properties were ascertained by Stocker and Peña (2007).

In the present paper we take up the problem of estimating unknown trend parameters of a TRP in the case when its renewal distribution is unknown. If the renewal distribution is unknown, then the likelihood function of the TRP is unknown and consequently the ML method cannot be used. In this case we propose three possible methods of estimating the trend parameters. The problem of estimating trend parameters of the TRP with unknown renewal distribution may be of interest in the situation when we observe several systems, of the same kind, working in different environments and we are interesting in examining and comparing their trend functions, whatsoever their renewal distribution is.

The article is organized as follows. In Sect. 2 we recall the definition of the TRP and the form of its likelihood function. A special case of the TRP with a Weibull renewal distribution and power-law type trend function is considered in

A. Jokiel-Rokita · R. Magiera (✉)
Institute of Mathematics and Computer Science, Wrocław University of Technology, 50-370 Wrocław, Poland
e-mail: Ryszard.Magiera@pwr.wroc.pl

Sect. 3. The likelihood function and the likelihood equations for estimating the parameters of that process are given. The likelihood equations are also presented in the form which is used in simulation study to obtain the ML estimators of the TRP parameters. In Sect. 4 we propose three methods of estimating the trend parameters in the case when the ML methods can not be used. The estimation problem of the trend parameters in some special case of the TRP is considered in Sect. 5. In Sect. 6 the estimators proposed are examined and compared with the ML estimators (obtained under the additional assumption that the renewal distribution has a known parametric form) through a computer simulation study. Some real data are examined in Sect. 6.3. Section 7 contains conclusions and some prospects.

2 Definitions and preliminaries

Let $N(t)$ denote the number of jumps (failures) in the time interval $(0,t]$ and let T_i be the time of the ith failure. Define $T_0 = 0$ and denote $X_i = T_i - T_{i-1}$—the time between failure number $i-1$ and failure number i. In the context of failure-repair models it is assumed here that all repair times are equal to 0. In practice this corresponds to the situation, when repair actions are conducted immediately or the repair times can be neglected in comparison to the times X_i between failures (the so called working times or waiting times). The observed sequence $\{T_i, i = 1,2,\dots\}$ of occurrence times $T_1, T_2, \dots$ (failure times) forms a point process, and $\{N(t), t \geq 0\}$ is the corresponding counting process.

Let $\lambda(t)$ be a nonnegative function defined for $t \geq 0$, and let $\Lambda(t) = \int_0^t \lambda(u)du$. The process $\{N(t), t \geq 0\}$ is called a trend renewal process TRP$(F, \lambda(\cdot))$ if the time-transformed process $\Lambda(T_1), \Lambda(T_2), \dots$ is a renewal process RP(F), i.e. if the random variables $\Lambda(T_i) - \Lambda(T_{i-1}), i = 1,2,\dots$, are i.i.d. with cdf F.

The cdf F is meant as the renewal distribution function, and $\lambda(t)$ is called the trend function. If for instance, $F(t) = 1 - \exp(-t)$, then the TRP$(1-\exp(-t), \lambda(\cdot))$ becomes the non-homogeneous Poisson process NHPP$(\lambda(\cdot))$. Let us also remark that in particular, the TRP$(F, 1)$ is the RP(F).

Equivalently, the corresponding counting process $\{N(t), t \geq 0\}$ can be considered, where $N(t) = \widetilde{N}(\Lambda(t))$ and $\{\widetilde{N}(t), t \geq 0\}$ represents a RP.

Note that the representation TRP$(F, \lambda(\cdot))$ is not unique. For uniqueness we assume that the expected value of the renewal distribution defined by F equals 1.

The conditional intensity function of a point process is defined by

$$\begin{aligned}\gamma(t) &= \lambda(t|\mathcal{F}_{t-})\\ &= \lim_{\Delta t \to 0} \frac{P(\text{failure in point process in } (t, t+\Delta t)|\mathcal{F}_{t-})}{\Delta t},\end{aligned}$$

where $\mathcal{F}_{t-} = \sigma\{N(u), u < t\}$. For a TRP$(F, \lambda(\cdot))$ we have

$$\begin{aligned}\gamma(t) &= \lim_{\Delta t \to 0} \frac{P(\text{failure in TRP in } (t, t+\Delta t)|\mathcal{F}_{t-})}{\Delta t}\\ &= \lim_{\Delta t \to 0} \frac{P(\text{failure in RP}(F) \text{ in } (\Lambda(t), \Lambda(t+\Delta t))|\mathcal{F}_{t-})}{\Delta t}\\ &= \lim_{\Delta t \to 0} \frac{P(\text{failure in RP}(F) \text{ in } (\Lambda(t), \Lambda(t+\Delta t))|\mathcal{F}_{t-})}{\Delta \Lambda(t)}\\ &\quad \times \frac{\Delta\Lambda(t)}{\Delta t}.\end{aligned}$$

In the case of RP(F) the conditional intensity is given by $\gamma(t) = z(t - T_{N(t-)})$, where $z(t)$ is the hazard rate corresponding to F: $z(t) = \frac{f(t)}{1-F(t)}$, where $f(t) = \frac{d}{dt}F(t)$. Thus, for a TRP$(F, \lambda(\cdot))$,

$$\begin{aligned}\gamma(t) &= z(\Lambda(t) - \Lambda(T_{N(t-)})) \lim_{\Delta t \to 0} \frac{\Lambda(t+\Delta t) - \Lambda(t)}{\Delta t}\\ &= z(\Lambda(t) - \Lambda(T_{N(t-)}))\lambda(t). \qquad (1)\end{aligned}$$

For a point process $N(t)$ observed in the interval time $[0, \sigma]$ with the realizations $t_1, t_2, \dots, t_{N(\sigma)}$ of the jump (failure) times $T_1, T_2, \dots, T_{N(\sigma)}$ and conditional intensity function $\gamma(t)$, the likelihood function is of the form

$$L(\sigma) = \left[\prod_{i=1}^{N(\sigma)} \gamma(t_i)\right] \exp\left(-\int_0^\sigma \gamma(u)du\right)$$

(Andersen et al. 1993). For a random stopping time σ with respect to the filtration $\mathcal{F}_t = \sigma\{N(u) : u \leq t\}$, this formula follows from the fundamental identity of sequential analysis as a consequence of the optional stopping theorem.

Taking into account formula (1), the likelihood function of a TRP$(F, \lambda(\cdot))$ takes the form

$$\begin{aligned}L(\sigma) = &\left[\prod_{i=1}^{N(\sigma)} z(\Lambda(t_i) - \Lambda(t_{i-1}))\lambda(t_i)\right.\\ &\left.\times \exp\left(-\int_{t_{i-1}}^{t_i} z(\Lambda(u) - \Lambda(t_{i-1}))\lambda(u)du\right)\right]\\ &\times \exp\left(-\int_{t_{N(\sigma)}}^{\sigma} z(\Lambda(u) - \Lambda(t_{N(\sigma)}))\lambda(u)du\right).\end{aligned}$$

Consequently, for a TRP$(F, \lambda(\cdot))$ observed in the time interval $[0, \sigma]$, by applying the substitution $v = \Lambda(u) - \Lambda(T_{i-1})$, the likelihood function takes the form

$$\begin{aligned}L(\sigma) = &\left[\prod_{i=1}^{N(\sigma)} z\big(\Lambda(t_i) - \Lambda(t_{i-1})\big)\lambda(t_i)\right.\\ &\left.\times \exp\left(-\int_0^{\Lambda(t_i)-\Lambda(t_{i-1})} z(v)dv\right)\right]\end{aligned}$$

$$\times \exp\left(-\int_0^{\Lambda(\sigma)-\Lambda(t_{N(\sigma)})} z(v)dv\right) \tag{2}$$

(Lindqvist et al. (2003, formula (2), Chap. 2)) and the log-likelihood function is defined by

$$\begin{aligned}\ell(\sigma) :=& \log L(\sigma)\\ =& \sum_{i=1}^{N(\sigma)}\Big[\log\big(z(\Lambda(t_i)-\Lambda(t_{i-1}))\big)\\ &+\log\big(\lambda(t_i)\big)-\int_0^{\Lambda(t_i)-\Lambda(t_{i-1})} z(v)dv\Big]\\ &-\int_0^{\Lambda(\sigma)-\Lambda(t_{N(\sigma)})} z(v)dv. \end{aligned}\tag{3}$$

3 An example—the Weibull-Power-Law TRP

3.1 The likelihood function

Let us consider the TRP$(F,\lambda(\cdot))$ with

$$\lambda(t;\alpha,\beta)=\alpha\beta t^{\beta-1}, \quad \alpha>0, \beta>0,$$
$$\Lambda(t;\alpha,\beta)=\alpha t^{\beta}$$

and

$$\begin{aligned}F(x)&=F(x;\gamma)\\ &=1-\exp\big[-\big(\Gamma(1+1/\gamma)x\big)^{\gamma}\big] \quad (\gamma>0),\end{aligned}$$

studied by Lindqvist et al. (2003). The renewal distribution function F corresponds to the Weibull distribution $\mathcal{W}e(\gamma, 1/\Gamma(1+1/\gamma))$ with the parametrization resulting in the expectation 1. The hazard function corresponding to F is $z(x)=(\Gamma(1+\frac{1}{\gamma}))^{\gamma}\gamma x^{\gamma-1}$. This TRP$(F,\lambda(\cdot))$ will be called the Weibull-Power-Law TRP and will be denoted shortly by WPLP(α,β,γ).

For the WPLP(α,β,γ) the likelihood function defined by (2) takes the form

$$\begin{aligned}L(\sigma)&=L(\sigma;\vartheta)\\ &=\prod_{i=1}^{N(\sigma)}\varphi\beta\gamma t_i^{\beta-1}(t_i^{\beta}-t_{i-1}^{\beta})^{\gamma-1}\\ &\quad\times\exp\Big[-\sum_{i=1}^{N(\sigma)}\varphi(t_i^{\beta}-t_{i-1}^{\beta})^{\gamma}-\varphi(\sigma^{\beta}-t_{N(\sigma)}^{\beta})^{\gamma}\Big],\end{aligned}$$

where $\vartheta=(\varphi,\beta,\gamma)$ and

$$\varphi=\varphi(\alpha,\gamma)=[\alpha\Gamma(1+1/\gamma)]^{\gamma}.$$

In the case $\gamma=1$ the renewal distribution function F corresponds to the exponential distribution $\mathcal{E}(1)$ and the WPLP$(\alpha,\beta,1)$ becomes NHPP$(\lambda(t))$ with $\lambda(t)=\alpha\beta t^{\beta-1}$, i.e., the so called Power-Law Process. We denote this process by PLP(α,β). Note that in this case $\varphi=\alpha$.

If $\gamma=1$ and $\beta=1$, then the WPLP$(\alpha,1,1)$ is the TRP$(1-\exp(-t),\alpha)$, i.e., it is the HPP(α).

The log-likelihood function for the WPLP(α,β,γ) is

$$\begin{aligned}\ell(\sigma;\vartheta):=&\log L(\sigma;\vartheta)\\ =&N(\sigma)(\ln\varphi+\ln\beta+\ln\gamma)\\ &+(\beta-1)\sum_{i=1}^{N(\sigma)}\ln t_i+(\gamma-1)\sum_{i=1}^{N(\sigma)}\ln(t_i^{\beta}-t_{i-1}^{\beta})\\ &-\varphi\Big[\sum_{i=1}^{N(\sigma)}(t_i^{\beta}-t_{i-1}^{\beta})^{\gamma}+(\sigma^{\beta}-t_{N(\sigma)}^{\beta})^{\gamma}\Big].\end{aligned}$$

In the case when the observation is finished at the n-th failure time point, i.e., $\sigma=t_{N(\sigma)}$ and $N(\sigma)=n$, the likelihood function is given by

$$\begin{aligned}\widetilde{L}(n;\vartheta)=&(\varphi\beta\gamma)^n\prod_{i=1}^{n}t_i^{\beta-1}[t_i^{\beta}-t_{i-1}^{\beta}]^{\gamma-1}\\ &\times\exp\Big\{-\varphi\sum_{i=1}^{n}[t_i^{\beta}-t_{i-1}^{\beta}]^{\gamma}\Big\},\end{aligned}$$

and the log-likelihood function is

$$\begin{aligned}\widetilde{\ell}(n;\vartheta)=&n(\ln\varphi+\ln\beta+\ln\gamma)\\ &+\sum_{i=1}^{n}\big[(\beta-1)\ln t_i+(\gamma-1)\ln(t_i^{\beta}-t_{i-1}^{\beta})\\ &-\varphi[t_i^{\beta}-t_{i-1}^{\beta}]^{\gamma}\big].\end{aligned}$$

In other words, the observation time is the random stopping time $\sigma=\inf\{t\ge 0; N(t)=n\}$ determining the so called inverse estimation plan.

3.2 The ML estimators

The solution to the equation $\partial\ell/\partial\varphi=0$ with respect to φ is

$$\widetilde{\varphi}=\widetilde{\varphi}(\beta,\gamma)=\frac{N(\sigma)}{\sum_{i=1}^{N(\sigma)}(t_i^{\beta}-t_{i-1}^{\beta})^{\gamma}+(\sigma^{\beta}-t_{N(\sigma)}^{\beta})^{\gamma}}. \tag{4}$$

Performing the likelihood equations for the parameters β and γ we have the following fact.

Fact 1 *The ML estimators $\widehat{\varphi}_{ML}$, $\widehat{\beta}_{ML}$ and $\widehat{\gamma}_{ML}$ of the parameters φ, β and γ, based on the observation up to any stopping time σ, are determined as follows:*

$$\widehat{\varphi}_{ML}=\frac{N(\sigma)}{\sum_{i=1}^{N(\sigma)}(t_i^{\widehat{\beta}_{ML}}-t_{i-1}^{\widehat{\beta}_{ML}})^{\widehat{\gamma}_{ML}}+(\sigma^{\widehat{\beta}_{ML}}-t_{N(\sigma)}^{\widehat{\beta}_{ML}})^{\widehat{\gamma}_{ML}}},$$

where $\widehat{\beta}_{ML}$ and $\widehat{\gamma}_{ML}$ are the solutions of the following system of likelihood equations

$$\frac{N(\sigma)}{\beta}+\sum_{i=1}^{N(\sigma)}\Big\{(t_i^{\beta}\ln t_i-t_{i-1}^{\beta}\ln t_{i-1})\times\Big[\frac{\gamma-1}{t_i^{\beta}-t_{i-1}^{\beta}}-\widetilde{\varphi}\gamma(t_i^{\beta}-t_{i-1}^{\beta})^{\gamma-1}\Big]+\ln t_i\Big\}-\widetilde{\varphi}\gamma\big(\sigma^{\beta}-t_{N(\sigma)}^{\beta}\big)^{\gamma-1}\big(\sigma^{\beta}\ln\sigma-t_{N(\sigma)}^{\beta}\ln t_{N(\sigma)}\big)=0,$$

$$\frac{N(\sigma)}{\gamma}+\sum_{i=1}^{N(\sigma)}\ln(t_i^{\beta}-t_{i-1}^{\beta})\big[1-\widetilde{\varphi}(t_i^{\beta}-t_{i-1}^{\beta})^{\gamma}\big]-\widetilde{\varphi}(\sigma^{\beta}-t_{N(\sigma)}^{\beta})^{\gamma}\ln(\sigma^{\beta}-t_{N(\sigma)}^{\beta})=0,$$

where $\widetilde{\varphi}=\widetilde{\varphi}(\beta,\gamma)$ is defined by (4).

In the inverse sequential estimation plan, the solution to the equation $\partial\widetilde{\ell}/\partial\varphi=0$ with respect to φ is

$$\widetilde{\varphi}=\widetilde{\varphi}(\beta,\gamma)=\frac{n}{\sum_{i=1}^{n}(t_i^{\beta}-t_{i-1}^{\beta})^{\gamma}},\tag{5}$$

and we have the following special case of Fact 1.

Fact 2 *The ML estimators $\widehat{\varphi}_{ML}$, $\widehat{\beta}_{ML}$ and $\widehat{\gamma}_{ML}$ of the parameters φ,β and γ in the inverse estimation plan are determined as follows*:

$$\widehat{\varphi}_{ML}=\frac{n}{\sum_{i=1}^{n}[t_i^{\widehat{\beta}_{ML}}-t_{i-1}^{\widehat{\beta}_{ML}}]^{\widehat{\gamma}_{ML}}},\tag{6}$$

where $\widehat{\beta}_{ML}$ and $\widehat{\gamma}_{ML}$ are the solutions of the following system of likelihood equations

$$\frac{n}{\beta}+\sum_{i=1}^{n}\Big\{[t_i^{\beta}\ln t_i-t_{i-1}^{\beta}\ln t_{i-1}]\times\Big[\frac{\gamma-1}{t_i^{\beta}-t_{i-1}^{\beta}}-\widetilde{\varphi}\gamma(t_i^{\beta}-t_{i-1}^{\beta})^{\gamma-1}\Big]+\ln t_i\Big\}=0,\tag{7}$$

$$\frac{n}{\gamma}+\sum_{i=1}^{n}\ln(t_i^{\beta}-t_{i-1}^{\beta})\big[1-\widetilde{\varphi}(t_i^{\beta}-t_{i-1}^{\beta})^{\gamma}\big]=0,$$

where $\widetilde{\varphi}=\widetilde{\varphi}(\beta,\gamma)$ is defined by (5).

The estimator $\widehat{\alpha}$ of α is evaluated according to the formula

$$\widehat{\alpha}=\frac{\widehat{\varphi}^{1/\widehat{\gamma}}}{\Gamma(1+1/\widehat{\gamma})},\tag{8}$$

where $\widehat{\varphi}$ and $\widehat{\gamma}$ are estimators of φ and γ.

Regarding that $t_0=0$, to avoid indeterminate expressions $0\cdot(-\infty)$ in the numerical evaluations we express the formula for the log-likelihood function in the following form

$$\widetilde{\ell}(n;\vartheta)=n(\ln\varphi+\ln\beta+\ln\gamma)+(\beta-1)\ln t_1+(\gamma-1)\ln t_1^{\beta}-\varphi t_1^{\beta\gamma}+\sum_{i=2}^{n}\big[(\beta-1)\ln t_i+(\gamma-1)\ln(t_i^{\beta}-t_{i-1}^{\beta})-\varphi(t_i^{\beta}-t_{i-1}^{\beta})^{\gamma}\big].\tag{9}$$

The derivative $\partial\widetilde{\ell}/\partial\beta$ is

$$\frac{\partial\widetilde{\ell}}{\partial\beta}=\frac{n}{\beta}+\gamma(1-\varphi t_1^{\beta\gamma})\ln t_1+\sum_{i=2}^{n}\Big\{[t_i^{\beta}\ln t_i-t_{i-1}^{\beta}\ln t_{i-1}]\Big[\frac{\gamma-1}{t_i^{\beta}-t_{i-1}^{\beta}}-\varphi\gamma(t_i^{\beta}-t_{i-1}^{\beta})^{\gamma-1}\Big]+\ln t_i\Big\}$$

and in the numerical computation we use the likelihood equation

$$\frac{n}{\beta}+\gamma\Big(1-\varphi(\beta,\gamma)t_1^{\beta\gamma}\Big)\ln t_1+\sum_{i=2}^{n}\Big\{[t_i^{\beta}\ln t_i-t_{i-1}^{\beta}\ln t_{i-1}]\Big[\frac{\gamma-1}{t_i^{\beta}-t_{i-1}^{\beta}}-\varphi(\beta,\gamma)\gamma(t_i^{\beta}-t_{i-1}^{\beta})^{\gamma-1}\Big]+\ln t_i\Big\}=0$$

instead of (7).

In particular, for the WPLP$(\alpha,\beta,1)$, i.e. for the PLP(α,β), the ML estimators of α and β can be explicitly determined (see e.g. Rigdon and Basu (2000, pp. 136–137)).

Fact 3 *For the PLP(α,β) the ML estimators $\widehat{\alpha}_{ML}$ and $\widehat{\beta}_{ML}$ of α and β, based on the observation up to any stopping time σ, are of the form*

$$\widehat{\alpha}_{ML}=\frac{N(\sigma)}{\sigma^{\widehat{\beta}_{ML}}}\tag{10}$$

and

$$\widehat{\beta}_{ML}=N(\sigma)\bigg(\ln\frac{\sigma^{N(\sigma)}}{\prod_{i=1}^{N(\sigma)}t_i}\bigg)^{-1}.\tag{11}$$

4 The alternative methods of estimating trend parameters of a TRP

In the case when both the form of the renewal distribution function F and the form of the trend function $\lambda(\cdot)$ of the

TRP($F, \lambda(\cdot)$) are known one can estimate unknown parameters of this process using the maximum likelihood (ML) method. The problem is to find the estimators of unknown parameters of F and λ for which the likelihood function defined by (2) or the log-likelihood defined by (3) takes its maximum.

In the case where the form of F is unknown, we propose in Sects. 4.1, 4.2 and 4.3 three methods for estimating unknown parameters of the trend function of a TRP($F, \lambda(\cdot)$), where $\lambda(\cdot) = \lambda(t; \vartheta)$ and ϑ is a vector of unknown parameters. The problem of estimating trend parameters of the TRP with unknown renewal distribution may be of interest in the situation when we observe several systems, of the same kind, working in different environments and we are interesting in examining and comparing their trend functions, whatsoever their renewal distribution is. Moreover, the following limit results for the TRP hold

$$\frac{V(t)}{\Lambda(t)} \to 1, \quad a.s.,$$
$$\frac{N(t)}{\Lambda(t)} \to 1 \quad \text{as } t \to \infty,$$

where $V(t) = E(N(t))$ (see Lindqvist et al. (2003)). For NHPP($\lambda(t)$) the equality $V(t) = \Lambda(t)$ holds for every t. Thus we may, at least asymptotically, think of $\Lambda(t)$ as the expected number of failures until time t. Therefore, we can use $\widehat{\Lambda}(t_0) = \Lambda(t_0; \widehat{\vartheta})$ as an estimator of $V(t_0)$, for some t_0 large enough, whatsoever renewal distribution of the TRP is.

4.1 The least squares method

The least squares (LS) method consists of determining the value of $\hat{\vartheta}_{LS}$ that minimizes the quantity

$$S^2_{LS}(\vartheta) = \sum_{i=1}^{N(\sigma)} [\Lambda(t_i; \vartheta) - \Lambda(t_{i-1}; \vartheta) - 1]^2, \tag{12}$$

where t_i are the realizations of random variables T_i, $i = 1, \dots, N(\sigma)$, and $\Lambda(t_0) := 0$. Let us note that $W_i = \Lambda(T_i) - \Lambda(T_{i-1})$ are the observations from the distribution with the expected value 1 (this is assumed for the uniqueness of the representation of a TRP). Thus the LS method consists of deriving such estimate of the unknown parameter ϑ (of the trend function) which minimizes the sum of squares of deviations of the random variables W_i from the expected value 1 (i.e. minimizes the sample variance).

4.2 The constrained least squares method

The constrained least squares (CLS) method consists of determining the value of ϑ that minimizes the quantity $S^2_{LS}(\vartheta)$ defined by (12) subject to the constraint

$$\frac{1}{N(\sigma)} \sum_{i=1}^{N(\sigma)} [\Lambda(t_i; \vartheta) - \Lambda(t_{i-1}; \vartheta)] = 1,$$

i.e., under the condition

$$\Lambda(t_{N(\sigma)}; \vartheta) = N(\sigma). \tag{13}$$

Thus in the CLS method we assume additionally that the sample mean

$$\overline{W} = \frac{1}{N(\sigma)} \sum_{i=1}^{N(\sigma)} W_i$$

is equal to the theoretical expected value 1 of the distribution defined by F.

4.3 The method of moments

If the value of the variance of the renewal distribution F is known, say s, then we can state the following condition on the sample variance:

$$\frac{1}{N(\sigma) - 1} \sum_{i=1}^{N(\sigma)} [\Lambda(t_i; \vartheta) - \Lambda(t_{i-1}; \vartheta) - 1]^2 = s.$$

Taking into account (13) we have the following first two sample moment conditions:

$$\begin{cases} \Lambda(t_{N(\sigma)}; \vartheta) = N(\sigma), \\ \sum_{i=1}^{N(\sigma)} [\Lambda(t_i; \vartheta) - \Lambda(t_{i-1}; \vartheta)]^2 = (s+1)N(\sigma) - s. \end{cases} \tag{14}$$

If $\vartheta = (\vartheta_1, \vartheta_2)$, then the method of moments (M method) consists of determining any solution $\hat{\vartheta}_M$ to the system of (14).

4.4 Some remarks

Remark 1 The LS, CLS and M methods can be useful when we do not know the form of the cumulative distribution function F (the renewal distribution), and consequently, when we do not know the likelihood function of the TRP($F, \lambda(\cdot)$).

Remark 2 The LS, CLS and M methods can be used to predict the next failure time. For example, we have $\widehat{T}_{N(\sigma)+1} = \widehat{\Lambda}^{-1}[\widehat{\Lambda}(T_{N(\sigma)}) + 1]$, where $\widehat{\Lambda}(t) = \Lambda(t; \widehat{\vartheta})$.

Remark 3 The LS, CLS and M methods can be the alternative methods of obtaining the estimators of an unknown parameter ϑ in the case when the maximum likelihood estimator does not exist. For example, in the case of k-stage

Erlangian NHPP, which was first mentioned in Khoshgoftaar (1988), the maximum likelihood estimator exists and is unique if and only if some condition concerning the realizations of the process is satisfied (see Zhao and Xie (1996, Theorem 2.1(ii)).

Remark 4 In the NHPP models for which the number of failures is bounded there are no consistent estimators of the unknown parameters (see Nayak et al. (2008, Theorem 1)). Thus, in these cases of the TRP, the estimators obtained by the ML, LS, CLS or M method are not consistent.

5 Estimation of trend parameters in special models of the TRP

Consider a TRP$(F,\lambda(\cdot))$, where $\lambda(t)=\alpha\beta t^{\beta-1}$, $\alpha>0$, $\beta>0$. If the renewal distribution function F is not specified, we will call this process the Power TRP$(F,\lambda(\cdot))$ and denote it by PTRP(α,β).

5.1 The LS method

Using the LS method we denote

$$S_{LS}^2(\alpha,\beta)=\sum_{i=1}^{N(\sigma)}[\Lambda(t_i;\alpha,\beta)-\Lambda(t_{i-1};\alpha,\beta)-1]^2,$$

and the optimization problem considered is to find

$$(\widehat{\alpha}_{LS},\widehat{\beta}_{LS})=\arg\min_{(\alpha,\beta)\in\mathbf{R}_+\times\mathbf{R}_+} S_{LS}^2(\alpha,\beta).$$

For the PTRP(α,β) considered, the equality $\sum_{i=1}^{N(\sigma)}(t_i^\beta-t_{i-1}^\beta)=t_{N(\sigma)}^\beta$ holds, and consequently

$$S_{LS}^2(\alpha,\beta)=\alpha^2\sum_{i=1}^{N(\sigma)}(t_i^\beta-t_{i-1}^\beta)^2-2\alpha t_{N(\sigma)}^\beta+N(\sigma). \tag{15}$$

Substituting the value

$$\alpha=\alpha_{LS}(\beta)=\frac{t_{N(\sigma)}^\beta}{\sum_{i=1}^{N(\sigma)}(t_i^\beta-t_{i-1}^\beta)^2}, \tag{16}$$

which minimizes the trinomial $S_{LS}^2(\alpha,\beta)$, into formula (15) we have

$$S_{LS}^2(\alpha_{LS}(\beta),\beta)=N(\sigma)-\frac{t_{N(\sigma)}^{2\beta}}{\sum_{i=1}^{N(\sigma)}(t_i^\beta-t_{i-1}^\beta)^2},$$

and the optimization problem reduces to the problem of finding

$$\widehat{\beta}_{LS}=\arg\min_{\beta\in\mathbf{R}_+}\widetilde{S}_{LS}^2(\beta), \tag{17}$$

where

$$\widetilde{S}_{LS}^2(\beta)=-\frac{t_{N(\sigma)}^{2\beta}}{\sum_{i=1}^{N(\sigma)}(t_i^\beta-t_{i-1}^\beta)^2}.$$

For numerical reasons (to avoid $\ln 0$ in evaluating the estimator $\widehat{\beta}_{LS}$), formula (15) is expressed in the form

$$S_{LS}^2(\alpha,\beta)=\alpha^2 t_1^{2\beta}+\alpha^2\sum_{i=2}^{N(\sigma)}(t_i^\beta-t_{i-1}^\beta)^2 -2\alpha t_{N(\sigma)}^\beta+N(\sigma).$$

The condition $\frac{\partial S_{LS}^2(\alpha,\beta)}{\partial\beta}=0$ leads to the equation

$$2\alpha\Bigg[\alpha t_1^{2\beta}\ln t_1-t_{N(\sigma)}^\beta\ln t_{N(\sigma)} +\alpha\sum_{i=2}^{N(\sigma)}(t_i^\beta-t_{i-1}^\beta)(t_i^\beta\ln t_i-t_{i-1}^\beta\ln t_{i-1})\Bigg]=0.$$

Taking into account formula (16) gives

$$t_1^{2\beta}\ln t_1-\ln t_{N(\sigma)}\sum_{i=1}^{N(\sigma)}(t_i^\beta-t_{t-i}^\beta)^2 +\sum_{i=2}^{N(\sigma)}(t_i^\beta-t_{i-1}^\beta)(t_i^\beta\ln t_i-t_{i-1}^\beta\ln t_{i-1})=0,$$

which can be rewritten in the form

$$t_1^{2\beta}\ln\frac{t_1}{t_{N(\sigma)}}+\sum_{i=2}^{N(\sigma)}(t_i^\beta-t_{i-1}^\beta) \times\left(t_i^\beta\ln\frac{t_i}{t_{N(\sigma)}}-t_{i-1}^\beta\ln\frac{t_{i-1}}{t_{N(\sigma)}}\right)=0. \tag{18}$$

Consequently, we have

Proposition 1 *The LS estimators $\widehat{\alpha}_{LS}$ and $\widehat{\beta}_{LS}$ of α and β are determined by*

$$\widehat{\alpha}_{LS}=\frac{t_{N(\sigma)}^{\widehat{\beta}_{LS}}}{\sum_{i=1}^{N(\sigma)}(t_i^{\widehat{\beta}_{LS}}-t_{i-1}^{\widehat{\beta}_{LS}})^2} \tag{19}$$

and the $\widehat{\beta}_{LS}$ which is the solution to (18).

5.2 The CLS method

Using the CLS method we denote

$$C(\sigma)=\Big\{(\alpha,\beta):\Lambda(t_{N(\sigma)};\alpha,\beta)=N(\sigma)\Big\}, \tag{20}$$

and the optimization problem considered is to find

$$(\widehat{\alpha}_{CLS}, \widehat{\beta}_{CLS}) = \arg \min_{(\alpha,\beta)\in C(\sigma)} S^2_{LS}(\alpha, \beta).$$

For the PTRP(α, β) considered the restriction set defined by (20) takes the form

$$C(\sigma) = \left\{(\alpha, \beta) : \alpha t^{\beta}_{N(\sigma)} = N(\sigma)\right\}.$$

Denote

$$\alpha_{CLS} = \alpha_{CLS}(\beta) = \frac{N(\sigma)}{t^{\beta}_{N(\sigma)}}$$

and

$$S^2_{CLS}(\beta) = S^2_{LS}(\alpha_{CLS}(\beta), \beta).$$

Thus, under the CLS criterion the optimization problem reduces to the problem of finding

$$\widehat{\beta}_{CLS} = \arg \min_{\beta\in\mathbf{R}_+} S^2_{CLS}(\beta),$$

where

$$S^2_{CLS}(\beta) = \frac{N^2(\sigma)}{t^{2\beta}_{N(\sigma)}} \sum_{i=1}^{N(\sigma)} (t_i^{\beta} - t_{i-1}^{\beta})^2 - 2\frac{N(\sigma)}{t^{\beta}_{N(\sigma)}} t^{\beta}_{N(\sigma)} + N(\sigma)$$
$$= N(\sigma)\left(\frac{N(\sigma)}{t^{2\beta}_{N(\sigma)}} \sum_{i=1}^{N(\sigma)} (t_i^{\beta} - t_{i-1}^{\beta})^2 - 1\right).$$

Hence, the problem of finding the estimator $\widehat{\beta}_{CLS}$ is equivalent to the problem of finding

$$\widehat{\beta}_{CLS} = \arg \min_{\beta\in\mathbf{R}_+} \widetilde{S}^2_{CLS}(\beta), \tag{21}$$

where

$$\widetilde{S}^2_{CLS}(\beta) = \frac{1}{t^{2\beta}_{N(\sigma)}} \sum_{i=1}^{N(\sigma)} (t_i^{\beta} - t_{i-1}^{\beta})^2.$$

Observe that

$$\widetilde{S}^2_{CLS}(\beta) = -\left[\widetilde{S}^2_{LS}(\beta)\right]^{-1}$$

and the extrema appear at the same points as in the LS method, so $\widehat{\beta}_{CLS} = \widehat{\beta}_{LS}$.

The condition $\frac{\partial \widetilde{S}^2_{CLS}(\beta)}{\partial\beta} = 0$ leads to the equation

$$t_1^{2\beta} \ln \frac{t_1}{t_{N(\sigma)}} + \sum_{i=2}^{N(\sigma)} (t_i^{\beta} - t_{i-1}^{\beta}) \times \left(t_i^{\beta} \ln \frac{t_i}{t_{N(\sigma)}} - t_{i-1}^{\beta} \ln \frac{t_{i-1}}{t_{N(\sigma)}}\right) = 0, \tag{22}$$

which has the same form as that one defined by (18) for deriving $\widehat{\beta}_{LS}$ in the LS method.

Proposition 2 *The CLS estimators $\widehat{\alpha}_{CLS}$ and $\widehat{\beta}_{CLS}$ of α and β are determined by*

$$\widehat{\alpha}_{CLS} = \frac{N(\sigma)}{t^{\widehat{\beta}_{CLS}}_{N(\sigma)}} \tag{23}$$

and the $\widehat{\beta}_{CLS}$ which is the solution to (22).

5.3 The M method

For the PTRP(α, β) considered, the system of equations of (14) takes the form

$$\begin{cases} \alpha t^{\beta}_{N(\sigma)} = N(\sigma), \\ \alpha^2 \sum_{i=1}^{N(\sigma)} (t_i^{\beta} - t_{i-1}^{\beta})^2 = (s+1)N(\sigma) - s. \end{cases}$$

Thus we have the following

Proposition 3 *The M estimators $\widehat{\alpha}_M$ and $\widehat{\beta}_M$ of α and β are determined by*

$$\widehat{\alpha}_M = \frac{N(\sigma)}{t^{\widehat{\beta}_M}_{N(\sigma)}} \tag{24}$$

and the $\widehat{\beta}_M$ which is the solution to the equation

$$\frac{N^2(\sigma)}{t^{2\beta}_{N(\sigma)}} \sum_{i=1}^{N(\sigma)} (t_i^{\beta} - t_{i-1}^{\beta})^2 - (s+1)N(\sigma) + s = 0. \tag{25}$$

For numerical computation reasons the following equivalent form of (25)

$$N^2(\sigma)\left\{\left(\frac{t_1}{t_{N(\sigma)}}\right)^{2\beta} + \sum_{i=2}^{N(\sigma)} \left[\left(\frac{t_i}{t_{N(\sigma)}}\right)^{\beta} - \left(\frac{t_{i-1}}{t_{N(\sigma)}}\right)^{\beta}\right]^2\right\} - (s+1)N(\sigma) + s = 0 \tag{26}$$

is more useful.

Note that for the WPLP(α, β, γ) the variance s of the renewal distribution defined by F is given by

$$s = \frac{\Gamma(1 + 2/\gamma)}{\Gamma^2(1 + 1/\gamma)} - 1. \tag{27}$$

In particular, $s = 1$ for the PLP(α, β).

6 Numerical results

In this section we present some numerical results illustrating the accuracy of the LS, CLS and M estimators proposed in the PTRP(α, β) model (with F unspecified) and in the WPLP(α,β,γ). The samples of the PLP(α,β) and the WPLP(α,β,γ) were generated up to a fixed number n of jumps is reached and for $k=500$ samples for each chosen combination of the parameters α, β and γ. The estimates of the unknown parameters α, β and γ are evaluated as the means of the estimates derived on the basis of individual realizations of the process considered. The variability of an estimator $\widehat{\eta}$ of an unknown parameter η was measured by the root mean squared error (RMSE) which is expressed by $\mathrm{RMSE}(\widehat{\eta})=\sqrt{(\mathrm{sd}(\widehat{\eta}))^2+(\mathrm{mean}(\widehat{\eta})-\eta)^2}$, where sd stands for the standard deviation. In the tables the abbreviation $\mathrm{se}(\widehat{\eta})$ is used for this error.

In constructing the executable computer programs, procedures of the package Mathematica 8.0 were used.

6.1 The estimates in the PLP

The values of the estimators of α and β were evaluated numerically using two numerical methods: constrained local optimization through solving equations (CLOSE method) and constrained global optimization (CGO method).

The CLOSE method in obtaining ML estimators relies on using the explicit formulae for $\widehat{\alpha}_{ML}$ and $\widehat{\beta}_{ML}$ given by (10) and (11), respectively, in Fact 3. The CLOSE method in evaluating the LS estimators relies on Proposition 1, i.e. on solving numerically equation (18) with respect to $\beta>0$ and then substituting the solution for the estimator $\widehat{\beta}_{LS}$ into formula (19) for the estimator $\widehat{\alpha}_{LS}$. The CLOSE method in evaluating the CLS estimators relies on Proposition 2, i.e. on solving numerically equation (22) with respect to $\beta>0$ and then substituting the solution for the estimator $\widehat{\beta}_{CLS}$ into formula (23) for the estimator $\widehat{\alpha}_{CLS}$. The M estimators were obtained by Proposition 3, i.e. by solving numerically equation (26) (for $s=1$) with respect to $\beta>0$ and then substituting the solution for the estimator $\widehat{\beta}_M$ into formula (24) for the estimator $\widehat{\alpha}_M$.

To investigate the numerical results for those processes for which the optimization problems can not be even partially solved explicitly (in contrast to the PLP(α,β)), we conducted analogous numerical investigation by using CGO method. The CGO method in evaluating ML estimators relies on solving the problem $(\widehat{\alpha}_{ML},\widehat{\beta}_{ML})=\arg\max_{(\alpha,\beta)\in\mathbf{R}_+\times\mathbf{R}_+} L(\sigma;\alpha,\beta)$ or equivalently $(\widehat{\alpha}_{ML},\widehat{\beta}_{ML})=\arg\max_{(\alpha,\beta)\in\mathbf{R}_+\times\mathbf{R}_+} \ell(\sigma;\alpha,\beta)$ by using a constrained global optimization procedure with respect to both variables α and β. The CGO method in evaluating LS and CLS estimators relies on solving the problems defined by (17) and (21), respectively, by using constrained global optimization procedures with respect to the variable β, and then substituting the solutions into formulas (19) and (23), respectively. The results carried out by the CGO numerical method have had the same accuracy as those carried out by the CLOSE numerical method, and the latter are not presented in the paper.

The estimates $\widehat{\alpha}_{LS}$, $\widehat{\alpha}_{CLS}$, $\widehat{\beta}_{(C)LS}$, $\widehat{\alpha}_M$ and $\widehat{\beta}_M$ proposed in the PTRP(α,β) are evaluated on the basis of the realizations (samples) of the generated PLP(α,β) and compared with the ML estimates $\widehat{\alpha}_{ML}$ and $\widehat{\beta}_{ML}$ for the latter model. The values of the estimators and their measures of variability are contained in Tables 1–4 for $n=50$, $n=100$ and for $k=500$ samples for each pair (α,β).

6.2 The estimates in the WPLP

The WPLP(α,β,γ) process is generated according to the following formula for the jump times:

$$T_i=\left[T_{i-1}^{\beta}+\frac{1}{\alpha\Gamma(1+1/\gamma)}\left(\ln\frac{1}{1-U_i}\right)^{1/\gamma}\right]^{1/\beta},\quad i=1,2,\ldots,\tag{28}$$

$T_0=0$, where U_i are random numbers from uniform distribution $\mathcal{U}(0,1)$. The generating formula is equivalent to

$$T_i=\left[T_{i-1}^{\beta}-\frac{1}{\alpha\Gamma(1+1/\gamma)}(\ln U_i)^{1/\gamma}\right]^{1/\beta},\quad i=1,2,\ldots,$$

but for numerical computation reasons formula (28) is more useful.

Table 1 The ML estimates of α and β in the PLP(α,β) and the LS, CLS and M estimates of α and β in the PTRP(α,β). The number of jumps $n=50$

No.	α	β	$\overline{T}_n$	$\widehat{\alpha}_{ML}$	$\widehat{\beta}_{ML}$	$\widehat{\alpha}_{LS}$	$\widehat{\alpha}_{CLS}$	$\widehat{\beta}_{(C)LS}$	$\widehat{\alpha}_M$	$\widehat{\beta}_M$
1	20	0.8	3.20	19.7126	0.8305	11.2212	21.0444	0.7766	20.6414	0.8309
2	15	1	3.31	14.6992	1.0545	8.6936	16.3002	0.9714	15.8796	1.0459
3	5	2	3.15	4.7882	2.1119	3.0020	5.6651	1.9511	6.1284	2.0992
4	1	3	3.68	0.9896	3.1501	0.6486	1.2292	2.9107	1.9286	3.0972
5	0.5	4	3.15	0.5461	4.1386	0.3245	0.6139	3.9135	1.1481	4.0914
6	0.2	5	3.02	0.2144	5.2257	0.1276	0.2429	4.8894	0.5600	5.2572
7	5	1	9.98	4.8411	1.0507	3.1241	5.8862	0.9779	6.4392	1.0340
8	1	2	7.04	1.0137	2.0918	0.6966	1.3172	1.9453	1.8174	2.0593
9	0.5	3	4.64	0.4993	3.1460	0.3283	0.6205	2.9369	1.0099	3.1879
10	0.2	4	3.98	0.2279	4.1454	0.1310	0.2504	3.9148	0.5561	4.1892

Table 2 The measures of variability of the ML, LS, CLS and M estimates of α and β. The number of jumps $n = 50$

No.	α	β	$se(\widehat{\alpha}_{ML})$	$se(\widehat{\beta}_{ML})$	$se(\widehat{\alpha}_{LS})$	$se(\widehat{\alpha}_{CLS})$	$se(\widehat{\beta}_{(C)LS})$	$se(\widehat{\alpha}_{M})$	$se(\widehat{\beta}_{M})$
1	20	0.8	4.0017	0.1252	9.2031	4.6789	0.1547	6.7959	0.3138
2	15	1	3.3799	0.1700	6.6997	4.2260	0.1902	6.3619	0.3737
3	5	2	1.8023	0.3391	2.2577	1.9989	0.2823	4.2556	0.7833
4	1	3	0.6163	0.4766	0.4315	0.5171	0.3212	2.1844	1.2008
5	0.5	4	0.3527	0.6276	0.2203	0.2621	0.3520	1.4657	1.5090
6	0.2	5	0.1785	0.8114	0.0867	0.0975	0.3565	0.8084	2.0052
7	5	1	1.7517	0.1666	2.3094	2.6501	0.1955	4.5391	0.4048
8	1	2	0.6019	0.3200	0.4780	0.7487	0.2928	2.0740	0.7657
9	0.5	3	0.3317	0.4674	0.2288	0.3041	0.3247	1.3625	1.1364
10	0.2	4	0.1889	0.6537	0.0901	0.1183	0.3500	0.8735	1.5046

Table 3 The ML estimates of α and β in the PLP(α, β) and the LS, CLS and M estimates of α and β in the PTRP(α, β). The number of jumps $n = 100$

No.	α	β	$\overline{T}_n$	$\widehat{\alpha}_{ML}$	$\widehat{\beta}_{ML}$	$\widehat{\alpha}_{LS}$	$\widehat{\alpha}_{CLS}$	$\widehat{\beta}_{(C)LS}$	$\widehat{\alpha}_{M}$	$\widehat{\beta}_{M}$
1	20	0.8	7.57	19.6564	0.8168	10.8982	21.0258	0.7890	21.4116	0.8138
2	15	1	6.64	14.8118	1.0243	8.4449	16.3125	0.9797	16.6752	1.0127
3	5	2	4.46	4.8790	2.0593	2.9003	5.6046	1.9758	6.4385	2.0319
4	1	3	4.63	0.9943	3.0943	0.6258	1.2199	2.9398	1.7474	3.0440
5	0.5	4	3.75	0.5424	4.0651	0.3144	0.6066	3.9344	1.1275	3.9990
6	0.2	5	3.47	0.2049	5.1247	0.1228	0.2384	4.9218	0.5083	5.0422
7	5	1	20.00	4.9431	1.0223	2.9833	5.7906	0.9820	6.5638	1.0126
8	1	2	9.93	1.0050	2.0583	0.6507	1.2635	1.9767	1.7931	2.0102
9	0.5	3	5.86	0.5024	3.0737	0.3193	0.6181	2.9465	0.9616	3.0427
10	0.2	4	4.73	0.2142	4.0854	0.1252	0.2430	3.9479	0.5168	4.0244

Table 4 The measures of variability of the ML, LS, CLS and M estimates of α and β. The number of jumps $n = 100$

No.	α	β	$se(\widehat{\alpha}_{ML})$	$se(\widehat{\beta}_{ML})$	$se(\widehat{\alpha}_{LS})$	$se(\widehat{\alpha}_{CLS})$	$se(\widehat{\beta}_{(C)LS})$	$se(\widehat{\alpha}_{M})$	$se(\widehat{\beta}_{M})$
1	20	0.8	3.9049	0.0862	9.5037	5.0847	0.1182	8.9502	0.2311
2	15	1	3.2128	0.1090	6.9290	4.4487	0.1401	7.6363	0.2744
3	5	2	1.6100	0.2225	2.3639	2.1384	0.2453	4.5891	0.5753
4	1	3	0.5071	0.3460	0.4576	0.5521	0.2964	1.8783	0.8647
5	0.5	4	0.2913	0.4330	0.2266	0.2662	0.3281	1.4226	1.1547
6	0.2	5	0.1251	0.5209	0.0904	0.0980	0.3408	0.6981	1.4549
7	5	1	1.5318	0.1046	2.3453	2.4555	0.1438	4.8208	0.2934
8	1	2	0.4922	0.2230	0.5025	0.7488	0.2518	1.9405	0.5515
9	0.5	3	0.2680	0.3088	0.2341	0.3074	0.2909	1.1825	0.8043
10	0.2	4	0.1360	0.4353	0.0945	0.1174	0.3188	0.7747	1.1057

The ML estimates $\widehat{\beta}_{ML}$ and $\widehat{\gamma}_{ML}$ of β and γ are found by maximizing the log-likelihood function in solving the optimization problem

$$(\widehat{\beta}_{ML}, \widehat{\gamma}_{ML}) = \arg\max_{(\beta,\gamma)} \widetilde{\ell}(n; (\varphi, \beta, \gamma)),$$

by using a constrained global optimization (CGO) procedure, where $\widetilde{\ell}(n; (\varphi, \beta, \gamma))$ is given by (9) with $\varphi = \widetilde{\varphi}(\beta, \gamma)$ defined by (5). The ML estimate $\widehat{\alpha}_{ML}$ of α is evaluated using formula (8) with $\widehat{\varphi}$ defined by (6). In the optimization problem the procedure NMaximize of Mathematica package is used.

The tables provide the numerical results for the WPLP(α, β, γ) in comparison to a PTRP(α, β) (the TRP with unknown F). The CLS estimates $\widehat{\alpha}_{CLS}$ and $\widehat{\beta}_{CLS}$ of the parameters α and β are evaluated on the basis of the realizations of the generated WPLP(α, β, γ) supposing that we do know nothing about the renewal distribution function F, i.e., that we observe the PTRP(α, β). The estimates $\widehat{\alpha}_{CLS}$ and $\widehat{\beta}_{CLS}$ are evaluated using Proposition 2 and the CGO

Table 5 The ML estimates of α, β and γ in the WPLP(α, β, γ). The number of jumps $n = 50$

No.	α	β	γ	$\overline{T}_n$	$\widehat{\alpha}_{ML}$	$\widehat{\beta}_{ML}$	$\widehat{\gamma}_{ML}$
1	15	1	1	3.32962	15.1619	1.0368	1.0885
2	5	2	1	3.17269	5.3976	1.9964	1.0771
3	1	3	1	3.66570	1.3147	2.9155	1.0592
4	0.5	4	1	3.15200	0.7100	3.8705	1.0442
5	15	1	2	3.33842	15.0668	0.9969	2.2657
6	5	2	2	3.16118	5.2531	1.9725	2.1992
7	1	3	2	3.68059	1.1986	2.9013	2.1112
8	0.5	4	2	3.15758	0.6307	3.8560	2.0721
9	15	1	4	3.33383	14.8703	0.9926	4.8756
10	5	2	4	3.15973	5.1408	1.9724	4.5728
11	1	3	4	3.68595	1.0875	2.9454	4.1946
12	0.5	4	4	3.16234	0.5643	3.9147	4.0732

Table 6 The LS, CLS and M estimates of α and β in the PTRP(α, β). The number of jumps $n = 50$

No.	α	β	γ	$\widehat{\alpha}_{LS}$	$\widehat{\alpha}_{CLS}$	$\widehat{\beta}_{(C)LS}$	$\widehat{\alpha}_{M}$	$\widehat{\beta}_{M}$
1	15	1	1	8.4899	15.5415	1.0099	15.9159	1.0466
2	5	2	1	2.9514	5.4646	1.9763	6.6879	1.9894
3	1	3	1	0.6510	1.2038	2.9419	1.8761	3.0830
4	0.5	4	1	0.3055	0.5710	3.9710	1.1167	4.1872
5	15	1	2	12.0570	14.9190	1.0128	15.1474	1.0136
6	5	2	2	4.0270	4.9722	2.0237	5.2634	2.0276
7	1	3	2	0.8286	1.0247	3.0201	1.2190	3.0156
8	0.5	4	2	0.4107	0.5075	4.0335	0.6213	4.0406
9	15	1	4	13.9616	14.7163	1.0182	15.0521	1.0038
10	5	2	4	4.5967	4.8461	2.0332	5.0424	2.0149
11	1	3	4	0.8883	0.9368	3.0594	1.0349	3.0255
12	0.5	4	4	0.4392	0.4625	4.0821	0.5365	4.0183

Table 7 The measures of variability of the ML estimates of α, β and γ in the WPLP(α, β, γ). The number of jumps $n = 50$

No.	α	β	γ	se($\widehat{\alpha}_{ML}$)	se($\widehat{\beta}_{ML}$)	se($\widehat{\gamma}_{ML}$)
1	15	1	1	3.33457	0.15557	0.16199
2	5	2	1	1.78142	0.27868	0.93242
3	1	3	1	0.73484	0.39724	1.94492
4	0.5	4	1	0.45766	0.53774	2.95805
5	15	1	2	1.80166	0.07315	1.30066
6	5	2	2	0.97649	0.15192	0.34360
7	1	3	2	0.39435	0.23876	0.92567
8	0.5	4	2	0.23856	0.30699	1.94106
9	15	1	4	0.94914	0.04010	3.93850
10	5	2	4	0.50364	0.07720	2.63491
11	1	3	4	0.18483	0.12200	1.29288
12	0.5	4	4	0.11374	0.16595	0.46185

Table 8 The measures of variability of the LS, CLS and M estimates of α and β in the PTRP(α, β). The number of jumps $n = 50$

No.	α	β	γ	se($\widehat{\alpha}_{LS}$)	se($\widehat{\alpha}_{CLS}$)	se($\widehat{\beta}_{(C)LS}$)	se($\widehat{\alpha}_{M}$)	se($\widehat{\beta}_{M}$)
1	15	1	1	6.94033	4.13278	0.19882	6.72476	0.39719
2	5	2	1	2.30323	1.96049	0.29032	4.59692	0.73614
3	1	3	1	0.44004	0.52805	0.32762	2.22384	1.12291
4	0.5	4	1	0.22956	0.23550	0.34672	1.45936	1.57426
5	15	1	2	3.33391	1.96074	0.08493	3.31304	0.17297
6	5	2	2	1.25120	0.98412	0.16400	2.06921	0.35251
7	1	3	2	0.29908	0.31353	0.22921	0.78828	0.53535
8	0.5	4	2	0.15346	0.15749	0.25820	0.45190	0.66774
9	15	1	4	1.36704	1.02954	0.04660	1.85616	0.09534
10	5	2	4	0.60101	0.51363	0.08390	1.07829	0.18743
11	1	3	4	0.18418	0.17018	0.13490	0.38390	0.28681
12	0.5	4	4	0.09863	0.09214	0.17502	0.22913	0.37755

method. The estimates $\widehat{\alpha}_M$ and $\widehat{\beta}_M$ were obtained by Proposition 3 for s evaluated according to formula (27).

The values of the estimators and their measures of variability are contained in Tables 5–12. We assumed $n = 50$ and $n = 100$, and used $k = 500$ simulated realizations for every combination of the three parameters α, β and γ.

6.3 Some real data

Let us take into account some real data of failure times, namely the data set contained in the paper of Lindqvist et al. (2003), given in Table 13. These data contain 41 failure times of a gas compressor with time censoring at time 7571 (days).

Supposing that the set of failure times of Table 13 forms a TRP belonging to the class of WPLP(α, β, γ), the ML estimates $\widehat{\alpha}_{ML}$, $\widehat{\beta}_{ML}$ and $\widehat{\gamma}_{ML}$ of α, β and γ have been evaluated and presented in Table 14. On the other hand, if no assumptions are made on the renewal distribution function F, the estimates $\widehat{\alpha}_{CLS}$ and $\widehat{\beta}_{CLS}$ of α and β are given as the parameters of the PTRP(α, β).

As the results of Table 14 show, the real data of failure times considered can be recognized as the WPLP(0.048, 0.763, 0.842) or the PTRP(0.028, 0.823). In both cases, the estimates of β are almost the same. In Table 14 the relative errors re($\widehat{\alpha}_{CLS}$) $= |\widehat{\alpha}_{CLS} - \widehat{\alpha}_{ML}|/\widehat{\alpha}_{ML}$ and re($\widehat{\beta}_{CLS}$) $= |\widehat{\beta}_{CLS} - \widehat{\beta}_{ML}|/\widehat{\beta}_{ML}$ are given too. For comparison, in Table 14 there are also given the sum of squares $SS_{CLS} := S^2_{LS}(\widehat{\alpha}_{CLS}, \widehat{\beta}_{CLS})$ and $SS_{ML} := S^2_{LS}(\widehat{\alpha}_{ML}, \widehat{\beta}_{ML})$,

Table 9 The ML estimates of α, β and γ in the WPLP(α, β, γ). The number of jumps $n = 100$

No.	α	β	γ	$\overline{T}_n$	$\widehat{\alpha}_{ML}$	$\widehat{\beta}_{ML}$	$\widehat{\gamma}_{ML}$
1	15	1	1	6.65959	15.2537	1.0106	1.0432
2	5	2	1	4.49654	5.3817	1.9808	1.0361
3	1	3	1	4.62962	1.2442	2.9300	1.0246
4	0.5	4	1	3.75459	0.6478	3.9004	1.0193
5	15	1	2	6.67911	15.1817	0.9957	2.1258
6	5	2	2	4.47066	5.2200	1.9805	2.0860
7	1	3	2	4.64077	1.1413	2.9350	2.0476
8	0.5	4	2	3.75763	0.5914	3.9068	2.0315
9	15	1	4	6.66444	15.0487	0.9947	4.3797
10	5	2	4	4.47187	5.1345	1.9817	4.2772
11	1	3	4	4.64240	1.0568	2.9697	4.0841
12	0.5	4	4	3.76311	0.5447	3.9437	4.0555

Table 10 The LS, CLS and M estimates of α and β in the PTRP(α, β). The number of jumps $n = 100$

No.	α	β	γ	$\widehat{\alpha}_{LS}$	$\widehat{\alpha}_{CLS}$	$\widehat{\beta}_{(C)LS}$	$\widehat{\alpha}_M$	$\widehat{\beta}_M$
1	15	1	1	8.2955	15.8372	0.9962	16.6449	1.0165
2	5	2	1	2.8901	5.5234	1.9765	6.5350	2.0163
3	1	3	1	0.6340	1.2202	2.9356	1.7638	3.0461
4	0.5	4	1	0.3055	0.5885	3.9480	1.0603	4.0136
5	15	1	2	11.9492	14.9728	1.0050	15.5671	0.9975
6	5	2	2	3.9958	5.0149	2.0086	5.3982	2.0018
7	1	3	2	0.8237	1.0336	3.0002	1.2009	2.9886
8	0.5	4	2	0.4120	0.5169	4.0073	0.5902	4.0298
9	15	1	4	13.8816	14.8064	1.0082	15.0318	1.0047
10	5	2	4	4.6145	4.9207	2.0135	5.0642	2.0089
11	1	3	4	0.9025	0.9628	3.0300	1.0275	3.0194
12	0.5	4	4	0.4549	0.4850	4.0290	0.5416	4.0005

where $S^2_{LS}(\vartheta)$ is defined by (12). Note that the sum of squares SS_{CLS} is somewhat smaller then SS_{ML}.

Let us denote by $\text{ENF}_{ML}(t) = \Lambda(t; \widehat{\alpha}_{ML}, \widehat{\beta}_{ML}) = \widehat{\alpha}_{ML} t^{\widehat{\beta}_{ML}}$ the estimated number of failures up to time t evaluated on the basis of the ML estimators, and analogously by $\text{ENF}_{CLS}(t) = \Lambda(t; \widehat{\alpha}_{CLS}, \widehat{\beta}_{CLS}) = \widehat{\alpha}_{CLS} t^{\widehat{\beta}_{CLS}}$ the estimated number of failures up to time t evaluated on the basis of the CLS estimators. In Table 15 we compare the estimated numbers of failures with the observed number of failures ONF(t) for some chosen values of t. The CLS method provides satisfactory estimates of the number of failures.

Table 11 The measures of variability of the ML estimates of α, β and γ in the WPLP(α, β, γ). The number of jumps $n = 100$

No.	α	β	γ	se($\widehat{\alpha}_{ML}$)	se($\widehat{\beta}_{ML}$)	se($\widehat{\gamma}_{ML}$)
1	15	1	1	3.14694	0.09949	0.09575
2	5	2	1	1.62619	0.19181	0.96731
3	1	3	1	0.59630	0.28980	1.97709
4	0.5	4	1	0.33198	0.37474	2.98187
5	15	1	2	1.77660	0.05361	1.14043
6	5	2	2	0.84431	0.10308	0.19245
7	1	3	2	0.29881	0.16427	0.96563
8	0.5	4	2	0.18406	0.22068	1.97425
9	15	1	4	0.90038	0.02865	3.40073
10	5	2	4	0.45525	0.05646	2.30391
11	1	3	4	0.14197	0.08341	1.13197
12	0.5	4	4	0.08742	0.11630	0.33783

Table 12 The measures of variability of the LS, CLS and M estimates of α and β in the PTRP(α, β). The number of jumps $n = 100$

No.	α	β	γ	se($\widehat{\alpha}_{LS}$)	se($\widehat{\alpha}_{CLS}$)	se($\widehat{\beta}_{(C)LS}$)	se($\widehat{\alpha}_M$)	se($\widehat{\beta}_M$)
1	15	1	1	7.11759	4.51018	0.14632	7.98056	0.28053
2	5	2	1	2.37319	2.12398	0.24919	4.68815	0.58681
3	1	3	1	0.44665	0.53891	0.28675	1.99083	0.85777
4	0.5	4	1	0.22869	0.24739	0.31908	1.25473	1.13285
5	15	1	2	3.40104	1.90465	0.05921	3.88282	0.13443
6	5	2	2	1.20606	0.84533	0.11109	2.09510	0.26953
7	1	3	2	0.26997	0.26255	0.16401	0.69686	0.38839
8	0.5	4	2	0.14504	0.14744	0.21073	0.37910	0.50564
9	15	1	4	1.40465	0.94442	0.03051	2.17527	0.07461
10	5	2	4	0.55396	0.44468	0.05755	1.13060	0.15219
11	1	3	4	0.15325	0.13267	0.08847	0.34732	0.22260
12	0.5	4	4	0.07923	0.07255	0.11161	0.22158	0.31295

7 Concluding remarks

We observe a good performance of the estimators $\widehat{\alpha}_{CLS}$ and $\widehat{\beta}_{CLS}$ obtained by the CLS method. This method leads to satisfactory accuracy of these estimators in the TRP($F, \lambda(\cdot)$) model considered with unspecified F in comparison to the ML estimators $\widehat{\alpha}_{ML}$ and $\widehat{\beta}_{ML}$ for this model with specified F.

The CLS method leads in average to more accurate estimators than the LS and M methods.

The LS method considerably underestimates the parameter α. In most cases considered we have RMSE($\widehat{\alpha}_{CLS}$) < RMSE($\widehat{\alpha}_{LS}$), and in all cases, RMSE($\widehat{\alpha}_{CLS}$) < RMSE($\widehat{\alpha}_M$).

The RMSE($\widehat{\alpha}_M$) is about two, or even more, times greater than the RMSE($\widehat{\alpha}_{CLS}$). A similar remark concerns the RMSE($\widehat{\beta}_M$) and RMSE($\widehat{\beta}_{CLS}$).

Table 13 The real data

1	4	305	330	651	856	996	1016	1155	1520	1597	1729
1758	1852	2070	2073	2093	2213	3197	3555	3558	3724	3768	4103
4124	4170	4270	4336	4416	4492	4534	4578	4762	5474	5573	5577
5715	6424	6692	6830	6999							

Table 14 The ML and CLS estimates applied to the real data of Table 13

$\widehat{\alpha}_{ML}$	$\widehat{\beta}_{ML}$	$\widehat{\gamma}_{ML}$	$\widehat{\alpha}_{CLS}$	$\widehat{\beta}_{CLS}$	re($\widehat{\alpha}_{CLS}$)	re($\widehat{\beta}_{CLS}$)	SS_{ML}	SS_{CLS}
0.047985	0.763104	0.842064	0.027980	0.823383	0.41690	0.078993	58.08	56.8

In some cases the RMSE($\widehat{\alpha}_{CLS}$) is even less than the RMSE($\widehat{\alpha}_{ML}$), and the RMSE($\widehat{\beta}_{CLS}$) is less than the RMSE($\widehat{\beta}_{ML}$).

For a given number n of failures, the RMSE's of all the estimators in the WPLP(α, β, γ) become significantly smaller as the parameter γ increases. Remark that, according to formula (27), the variance s of the renewal distribution F decreases evidently as γ increases. For example, $s = 1$ for $\gamma = 1$, $s = 0.2732$ for $\gamma = 2$, $s = 0.0787$ for $\gamma = 4$. A smaller value of γ (a larger value of the variance s) causes larger variability of the estimators (recall that the RMSE determines the mean squared deviation of the estimate from the true value of the parameter—the risk). In the WPLP($\alpha, \beta, 1$), i.e. in the PLP(α, β), the variance of F is equal to 1 and constitutes a great value in reference to the same value of the expectation of the renewal distribution as well as in reference to the assumed value 1 of the sample mean of the transformed working times $W_i = \Lambda(T_i) - \Lambda(T_{i-1}), i = 1, \ldots, N(\sigma)$.

A great value, such as 1, of the variance of the renewal distribution causes larger variability and instability of the RMSE's of the LS, CLS and M estimators in the case of relatively small sample sizes n. It may then happen that in some cases for $\gamma = 1$ the RMSE($\widehat{\alpha}_{LS}$), RMSE($\widehat{\alpha}_{CLS}$) and/or RMSE($\widehat{\alpha}_{M}$) increase as n increases. Further simulation study for $\gamma = 1$ and much greater than $n = 100$ numbers of failures shows that these RMSE's decrease as n increases. They decrease much more slowly than the RMSE's of the parameter β.

For $\gamma \geq 2$, the RMSE($\widehat{\alpha}_{CLS}$) decreases as n increases.

If the number of jumps n increases then all the RMSE's of the parameter β, i.e. the RMSE($\widehat{\beta}_{ML}$), RMSE($\widehat{\beta}_{CLS}$) and RMSE($\widehat{\beta}_{M}$) decrease.

If the renewal distribution function is unknown we recommend using the CLS method to obtain the estimators of the unknown parameters of the trend function or the expected number of failures.

The CLS method can also be used to predict the next failure time. Examination of asymptotic properties of the CLS estimators would be desirable, among others, for constructing the confidence intervals for unknown parameters or for the expected number of failures.

Table 15 Comparisons of estimated numbers of failures with the observed number of failures for the real data

t	ENF$_{ML}(t)$	ENF$_{CLS}(t)$	ONF(t)
1000	9.341	8.260	7
2000	15.854	14.617	14
3000	21.603	20.410	18
4000	26.906	25.866	23
5000	31.901	31.083	33
6000	36.663	36.117	37
7000	41.240	41.005	41

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