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The Craig – Sneyd Analytic Solutions to the Parker Problem

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Abstract This paper follows up on the conclusion by Craig and Sneyd (2005) that the solutions to a linearized magnetostatic problem are counterexamples to the magnetostatic model of Parker (1972), demonstrating a general absence of continuous equilibrium for a magnetic field with an arbitrarily prescribed topology. The analysis presented here shows that Craig and Sneyd had incorrectly rejected an important subset of those solutions in a misunderstanding of the Parker model. The complete set of solutions when correctly interpreted is, in fact, physically consistent with the Parker model. A general discussion of the Parker theory of spontaneous current sheets is given.

Keywords MHD · Sun: corona · Sun: magnetic fields

1. Introduction

In an interesting study, hereafter referred to as C&S05, Craig and Sneyd (2005) presented particular analytic solutions to a linearized magnetostatic problem as counterexamples to the study of spontaneous current sheets described by Parker (1972, hereafter referred to as Parker72). Here we analyze the complete set of solutions to that problem to show that they are all quite consistent with Parker72, contrary to the interpretation of C&S05.

It is widely agreed that the Parker theory, if correct, is fundamental to our understanding of the heating of the solar corona, showing how electric current dissipation may readily occur in spite of the high coronal conductivity (Aschwanden, 2004; Priest, 1982; Parker 1979, 1994; Shibata, 1999; Low, 2001; Zhang and Low, 2005). The physical statement of this theory was developed further by Parker (1986a, 1986b, 1989a, 1989b, 1989c, 1990a, 1990b, 1990c, 1990d, 1991) from general considerations, synthesized in Parker (1994). There are solid physical and mathematical grounds for accepting this theory while truly interesting questions have remained, as explained by Janse, Low, and Parker (2010). We hope that the clarification here of the physical misunderstanding in C&S05 will better motivate future

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attempts at demonstration of the theory, or, construction of possible counterexamples to it by those holding the opposite point of view. In Section 2, the magnetostatic problem of C& S05 is given a brief but complete treatment. The results obtained are related to Parker72 in the general discussion in Section 3, centered on the methods of series solutions previously discussed by Rosner and Knobloch (1982). The paper ends with a brief summary and the Appendix containing a comment on the nonlinear numerical computations in C&S05 in light of our clarification of the physics here.

2. The Craig – Sneyd Linear Problem

Consider the magnetostatic equilibrium equations

$$\frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B} - \nabla P = 0, \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{2}$$

where $\mathbf{B}$ and P respectively denote the magnetic field and pressure. This equation implies

$$\mathbf{B} \cdot \nabla P = 0, \tag{3}$$

showing the absence of the pressure force along the field because the Lorentz force is always perpendicular to the field.

2.1. The Linearized Parker Problem

The following is a self-contained reformulation of the linear problem treated in C&S05. The same notations are used except for obvious modifications that better suit our purpose and also avoid a minor confusion of notations in C&S05.

We construct the entire set of magnetostatic equilibria, each of which is a linear departure from a uniform field $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$ in a uniform plasma with pressure P_0 and density ρ_0, brought about by an infinitesimal static plasma Cartesian displacement

$$\boldsymbol{\xi} = (f, g, h). \tag{4}$$

We assume perfect electrical conductivity and the polytropic law

$$P = \frac{c^2}{\gamma}\rho, \tag{5}$$

γ being the constant polytropic index, ρ the plasma density, and $c(\rho)$ the sound speed. Using the representations

$$\mathbf{B} = B_0\big(\hat{\mathbf{z}} + \mathbf{b}(x, y, z)\big), \tag{6}$$

$$P = P_0\big(1 + p(x, y, z)\big), \tag{7}$$

the linear departures in field and pressure from the initial uniform state are related to the plasma displacement by

$$\mathbf{b} = \nabla \times (\boldsymbol{\xi} \times \hat{\mathbf{z}}) = (f_z, g_z, -f_x - g_y), \tag{8}$$

$$\begin{aligned} p &= -\gamma(\nabla\cdot\boldsymbol{\xi}) \\ &= -\gamma(f_x + g_y + h_z), \end{aligned} \tag{9}$$

where we have used $\rho_0 c_0^2(\rho_0) = \gamma P_0$ and the subscripts to denote the usual partial differentiations.

The linearized equilibrium equation takes the form

$$(\nabla\times\mathbf{b})\times\hat{\mathbf{z}} - \frac{1}{2}\beta\nabla p = 0, \tag{10}$$

to be solved for displacement $\boldsymbol{\xi}(x, y, z)$ in a fixed domain, with $\beta = 8\pi P_0/B_0^2 = 8\pi\rho_0 c_0^2/\gamma B_0^2$. The magnetostatic problem in Parker72, called the Parker problem in C&S05, is posed for the infinite domain sandwiched between two rigid plates $z = \pm 1$, in some unit of length, perpendicular to the uniform field $\mathbf{B}_0$. So the solution is subject to the boundary conditions

$$\begin{aligned} z &= -1, \qquad \boldsymbol{\xi} = \big(f_{\mathrm{B}}(x, y), g_{\mathrm{B}}(x, y), 0\big), \\ z &= +1, \qquad \boldsymbol{\xi} = 0, \end{aligned} \tag{11}$$

where $(f_{\mathrm{B}}, g_{\mathrm{B}})$ are arbitrarily prescribed, combined with $|\boldsymbol{\xi}| \to 0$ at $\sqrt{x^2 + y^2} \to \infty$. Throughout this paper, this condition at infinity is assumed. A converse set of boundary conditions with the tangential boundary displacements, $\boldsymbol{\xi}(x, y, -1) \equiv 0$ and $\boldsymbol{\xi}(x, y, 1) \neq 0$, produces a complementary set of solutions. Using the two sets, linear superposition accounts for all the equilibria produced by the arbitrary plasma displacements one may prescribe on $z = \pm 1$.

In terms of the fluid displacement, Equation (10) gives

$$f_{zz} + f_{xx} + g_{xy} = \frac{1}{2}\beta p_x, \tag{12}$$

$$g_{zz} + f_{yx} + g_{yy} = \frac{1}{2}\beta p_y, \tag{13}$$

$$p_z = 0. \tag{14}$$

Equation (14) is the linear version of Equation (3), and we should rewrite Equation (9) as a restriction on the spatial variation of $\boldsymbol{\xi}$:

$$\gamma(f_x + g_y + h_z) = -p(x, y). \tag{15}$$

These partial differential equations (PDEs) have an unusual feature. The pressure $p(x, y)$ as an unknown in the equations can be eliminated from Equations (12) and (13) to obtain the final set

$$f_{zz} + f_{xx} + g_{xy} = -\frac{1}{2}\beta\gamma(f_{xx} + g_{xy} + h_{xz}), \tag{16}$$

$$g_{zz} + f_{yx} + g_{yy} = -\frac{1}{2}\beta\gamma(f_{yx} + g_{yy} + h_{yz}). \tag{17}$$

This is an under-determined set of two PDEs in three unknowns. The important condition is not explicit that the three unknowns (f, g, h) must vary in 3-D space such that the left-hand side of Equation (15) does not depend on z. It is thus necessary to retain the pressure $p(x, y)$ explicitly by using Equations (12), (13), and (15), subject to the boundary conditions (11).

2.2. Classification of the Solutions

The following solution classification is useful for our purpose. Define the compression and rotation variables as

$$C = f_x + g_y, \tag{18}$$

$$R = g_x - f_y, \tag{19}$$

respectively, as an alternative to (f, g). Note from Equation (8) that $b_z = -C$. If $C(x, y, z)$ and $R(x, y, z)$ are known, the corresponding f and g are uniquely defined by

$$\nabla_\perp^2 f = C_x - R_y, \tag{20}$$

$$\nabla_\perp^2 g = R_x + C_y, \tag{21}$$

on each constant-z plane, under the boundary condition on the vanishing of the plasma displacement at $\sqrt{x^2 + y^2} \to \infty$; the subscript $\perp$ denoting the Laplacian in x and y. The prescribed boundary displacement is purely compressive or rotational, when either R or C respectively vanishes at $z = \pm 1$. All the possible boundary displacements are spanned linearly by these two pure modes.

Let us rewrite Equations (12) and (13) as

$$f_{zz} + C_x = \frac{1}{2}\beta p_x, \tag{22}$$

$$g_{zz} + C_y = \frac{1}{2}\beta p_y, \tag{23}$$

which are readily transformed into

$$\nabla^2 C = \frac{1}{2}\beta \nabla_\perp^2 p, \tag{24}$$

$$\frac{\partial^2 R}{\partial z^2} = 0. \tag{25}$$

Their solutions are

$$C = \frac{1}{2}\beta p(x, y) + q(x, y, z), \tag{26}$$

$$R = a_0(x, y) + a_1(x, y)z, \tag{27}$$

where q is the unique solution of the boundary value problem

$$\nabla^2 q = 0, \tag{28}$$

$$q(x, y, \pm 1) = -\frac{1}{2}\beta p(x, y) + C(x, y, \pm 1), \tag{29}$$

and, $a_0(x, y)$ and $a_1(x, y)$ are arbitrary. It can be shown that $2q$ is just the total pressure to first order. Note that the pressure $p(x, y)$ is to be determined as a part of the problem using Equation (15). The mathematical concern is whether the equations admit a solution p that does not vary with z.

Prescribing f and g at $z=\pm 1$ defines C and R on $z=\pm 1$, so that for the problem in terms of C and R, we have the alternative boundary conditions

$$\begin{aligned} C(x,y,-1) &= C_{\mathrm{B}}(x,y), \qquad R(x,y,-1)=R_{\mathrm{B}}(x,y),\\ C(x,y,1) &= R(x,y,1)=0,\\ h(x,y,\pm 1) &= 0, \end{aligned} \tag{30}$$

where $C_{\mathrm{B}}(x,y)$ and $R_{\mathrm{B}}(x,y)$ are prescribed. These boundary conditions determine $R = a_0(x,y)+a_1(x,y)z$ uniquely, *i.e.*, $a_0=-a_1=\frac{1}{2}R_{\mathrm{B}}(x,y)$ and

$$R=\frac{1}{2}R_{\mathrm{B}}(x,y)(1-z), \tag{31}$$

whereas a separate boundary value problem for q determines p and C.

The orthogonal base functions

$$(\phi_1,\phi_2,\phi_3,\phi_4)\equiv(\sin k_1x\sin k_2y, \sin k_1x\cos k_2y, \cos k_1x\sin k_2y, \cos k_1x\cos k_2y) \tag{32}$$

generated by wavenumbers k_1 and k_2 span the space of continuous functions of x and y. So it suffices to treat the generic case of the prescribed boundary displacement $C_{\mathrm{B}}(x,y)$ being a certain given mode with wavenumbers (k_1,k_2),

$$\begin{aligned} C_{\mathrm{B}}(x,y) &= C_1\sum_{i=1}^{4}\nu_i\phi_i(k_1,k_2,x,y)\\ &= C_1\phi(k_1,k_2,x,y), \end{aligned} \tag{33}$$

where ν_i are prescribed constants to fix the dependence of C_{B} on (x,y), expressed by ϕ, and C_1 is just a free constant to parametrize the amplitude of C_{B}. Then, the linearity of the problem implies that the solution has the form

$$C(x,y,z)=\sigma(z)\phi(k_1,k_2,x,y), \tag{34}$$

$$q(x,y,z)=Q(z)\phi(k_1,k_2,x,y), \tag{35}$$

$$p(x,y)=p_0\phi(k_1,k_2,x,y), \tag{36}$$

where $\sigma(z)$ and $q(z)$ are unknown functions to be determined by Equations (26), (28), and (29), and p_0 is an unknown constant defining the pressure.

Direct solution of these equations gives

$$C(x,y,z)=\left(\frac{1}{2}\beta p_0\left[1-\frac{\cosh kz}{\cosh k}\right]-C_1\frac{\sinh k(z-1)}{\sinh 2k}\right)\phi(x,y,k_1,k_2), \tag{37}$$

where the constant p_0 is to be determined via an application of the boundary conditions on the vertical displacement h in Equation (15). Rewriting this equation,

$$p(x,y)=-\gamma(C+h_z). \tag{38}$$

Substituting for p and C and integrating once with respect to z, we obtain

$$h=\left(h_0-p_0\left[\left(\frac{1}{2}\beta+\frac{1}{\gamma}\right)z-\frac{\sinh kz}{k\cosh k}\right]+C_1\frac{\cosh k(z-1)}{k\sinh 2k}\right)\phi(x,y,k_1,k_2), \tag{39}$$

where h_0 is an integration constant. The boundary conditions $h(x, y, \pm 1) = 0$ then give

$$h_0 = -C_1 \frac{1 + \cosh 2k}{2k \sinh 2k}, \tag{40}$$

$$p_0 = C_1 \frac{1 - \cosh 2k}{2 \sinh 2k\big(\big[\frac{1}{2}\beta + \frac{1}{\gamma}\big]k - \tanh k\big)}, \tag{41}$$

to determine p and h given by Equations (36) and (39).

Thus, every compressive ($C_{\rm B}(x, y) \neq 0$) boundary displacement has a unique equilibrium solution described by a z-varying compression variable $C(x, y, z) \neq 0$. This is the principal result established by C&S05 with a particular analytical solution. This result was previously established by Zweibel and Li (1987). There is the second distinct class of equilibria that escaped the attention of C&S05, namely, those generated by noncompressive rotational boundary displacements with $C_{\rm B}(x, y) \equiv 0$ and $R_{\rm B}(x, y) \neq 0$ which we now examine.

2.3. The $C_{\rm B}(x, y) \equiv 0$, $R_{\rm B}(x, y) \neq 0$ Equilibria

If $C_1 = 0$, the single-mode displacement at the boundary represents a general case of $C_{\rm B}(x, y) \equiv 0$. Then $p_0 = h_0 = 0$, so that we have $C(x, y, z) \equiv 0$ throughout the domain $|z| < 1$. Prescribing $C_1 = 0$ means only that the boundary tangential displacement is not compressive, without implying that the plasma in the domain $|z| < 1$ is incompressible. Despite the plasma's compressibility, it responds to the noncompressive boundary displacement by attaining a nearby equilibrium via an interior displacement that has turned out to be also noncompressive. To be seen in Section 3, compressibility is necessarily involved, but, in this case, that is a higher-order effect. With $C \equiv 0$, the displacement components f and g, given by Equations (20) and (21), are defined by just the rotation variable R. In this case, the pressure and B_z remain uniform, *i.e.*, $p = 0$ and $b_z = -C = 0$. Moreover, R as well as f and g are linear in z which, together with $b_z = 0$, means that the perturbation $\mathbf{b}$, given by Equation (8), is translationally invariant in the z direction, *i.e.*,

$$\frac{\partial}{\partial z}\mathbf{b} = 0. \tag{42}$$

The result (42) is remarkable in that if the deformed field is to be in equilibrium, it cannot vary with z for *all* the possible noncompressive, rotational footpoint displacements.

The C&S05 study incorrectly rejected the subset of $C_1 = 0$ solutions, and thus did not encounter the invariance condition (42). Instead, the study claimed that this condition required the assumption of incompressibility throughout $|z| < 1$, that is, $C \equiv 0$ was an extraneous assumption to be made for Equation (42) to be valid; see the paragraph on page 47, following their Equation (20), and the Appendix on page 61 of Craig and Sneyd (2005). We see here that this claim is not valid. The correct physics is the opposite of their claim. The noncompressibility of the plasma displacement obtaining in $|z| < 1$ is implied by the governing equations for a compressible plasma, not by an extraneous assumption.

In contrast, the z-dependence of a $C_1 \neq 0$ field $\mathbf{b}$ is expected *a priori*. The prescribed boundary displacement compresses and expands both field and plasma at the boundary from an initially uniform state. Magnetic and plasma pressures go up or down together. The field compression is held in place by the rigid anchoring at the boundaries, with the boundary flux $B_z(x, y, \pm 1) = B_0 + b_z(x, y, \pm 1)$ no longer uniform. Although the plasma may relieve a local pressure increase by flowing away from the boundary, the frozen-in condition bottles the plasma in each flux tube so that some residual pressure compression remains in the equilibrium state. Just the fact that magnetic and plasma pressures go up together in a locality on the boundary means that the flux tubes with footprints in that locality must expand with height in the equilibrium state. Even if the plasma is cold, *i.e.*, $P \equiv 0$, variation with height is inevitable when the boundary normal field component $B_z(x, y, -1)$ is not uniform. This variation extends into the domain to a characteristic height comparable to the horizontal scales of the imposed footpoint displacement, as first illustrated by Zweibel and Li (1987) and also by C&S05.

Of particular interest is the case of these horizontal scales being very small compared to the separation between $z = \pm 1$. If C and R are both nonzero, we will see two features. The boundary effects are then confined to a thin layer due to the anchoring of the field at the boundary. In this boundary layer the field and plasma are distributed for the equilibration of the plasma pressure and magnetic twist along the main part of the field. The solutions for $k \gg 1$ show that this equilibration takes the general form of an approximately z-invariant distribution of the plasma pressure and magnetic twist throughout the main part of the domain. In other words, the field in the main part of the domain also tends to the invariance condition (42) as found in the $C_1 = 0$ equilibria. The physically significant point is not about z-dependence in the case of $C_1 \neq 0$, which is expected a priori, but about the translational invariance in z as a common feature of the equilibria for all boundary displacements of (horizontal) scales small compared to the plate separation, whether or not $C_1 = 0$.

It is simple to construct an infinite set of analytical z-dependent equilibria, each deformable from an initial uniform field under the frozen-in condition, even without being limited to infinitesimal displacements. Consider the case of the cold gas so that Equation (9) can be ignored. Linearly superpose the initial field $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$ with an arbitrary potential field $\mathbf{B}^{\text{pot}} = \nabla\Phi(x, y, z)$ such that $B_z^{\text{pot}}(x, y, 1) = 0$ and $|\mathbf{B}^{\text{pot}}| < B_0$. The latter requirement ensures that all the field lines of $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}^{\text{pot}}$ thread from one plate boundary to the other as is the case of $\mathbf{B}_0$. Then it is straightforward, although tedious, to work out the plasma displacement $\boldsymbol{\xi} = (f, g, h)$ that takes $\mathbf{B}_0$ into $\mathbf{B}$ under the frozen-in condition, such that the displacement vanishes at $z = 1$ and is tangential on $z = -1$ where it has continuously changed the normal flux from $B_z = B_0$ to $B_z = B_0 + B_z^{\text{pot}}(x, y, -1)$. In this construction, $\boldsymbol{\xi}$ is infinitesimal if $|\mathbf{B}^{\text{pot}}| \ll B_0$. Otherwise, $\boldsymbol{\xi}$ is of finite amplitude, requiring, in place of Equation (8), the Lundquist integral to relate $\boldsymbol{\xi}$, $\mathbf{B}$ and $\mathbf{B}_0$ (Parker, 1979; Priest, 1982). In either case the boundary displacement is purely compressive with no rotation so that the field in its force-free equilibrium is potential. The z-variation is due to field expansion from the nonuniform flux distribution on $z = -1$ created by the boundary displacement; similar to the case of the $C_1 \neq 0$ solution of C&S05. By choosing the potential field $\mathbf{B}^{\text{pot}}$ such that $B_z^{\text{pot}}(x, y, -1)$ varies horizontally across the boundary $z = -1$ on scales that are small compared to the plate separation, the superposition $\mathbf{B}$ will naturally show a thin boundary layer at $z = -1$ above which the uniform field $\mathbf{B}_0$ dominate. Any general claim, completely independent of context, that there can be no equilibrium varying with z between the two plates can thus be cleanly dismissed.

Once we understand the above magnetostatic properties, their consistency with Parker72 should be easy to see. This is the principal point of the present paper that we now address.

3. The Theory of Spontaneous Current Sheets

We first relate our findings to Parker72 and then follow up on the insightful discussion of Rosner and Knobloch (1982) to lead to more general physical concerns.

3.1. The Parker Series Expansion

The magnetostatic problem in Parker72 is nonlinear, treated by the use of expansions in infinite series. Boundary displacements are introduced at one plate, say, $z = -1$, on some characteristic scale l small compared to the separation between $z = \pm 1$. Despite the assumption of $l \ll 1$, thin flux tubes of the initially uniform field $\mathbf{B}_0$ are imagined to be twisted individually and around each other many times, by boundary displacements under the frozen-in condition. For cell-like chaotic boundary displacements, there is also the characteristic correlational length $\epsilon > l$ that may be used as an expansion parameter. The assumption is made in Parker72 that for a small ϵ an equilibrium with the displaced magnetic footpoints anchored rigidly at the boundary exists and can be described by the analytic infinite series:

$$\mathbf{B} = \mathbf{B}_0 + \sum_{n=1}^{\infty} \mathbf{b}_n(x, y, z)\epsilon^n, \tag{43}$$

$$P = P_0 + \sum_{n=1}^{\infty} p_n(x, y, z)\epsilon^n. \tag{44}$$

It is pointed out in Parker72 at the outset that, in general, the build up of stress at the boundary means that this solution can vary with z; see pages 500–501 of Parker (1972). Such variations are argued to be confined to a boundary layer on each of the two boundary plates, as thin as the scale ϵ is small. Hence, to dispense with the complications arising from boundary conditions on these plates, Parker72 analyzed in the limit of $\epsilon \to 0$. Then, neglecting the z-variation in the boundary layers, the analysis in Parker72 arrived at the necessary conditions for this infinite series to exist,

$$\frac{\partial}{\partial z}[\mathbf{b}_n, p_n] = 0, \tag{45}$$

for all integer n. A basic mathematical property used is that a harmonic function in unbounded spaces must be a constant (Courant and Hilbert, 1962).

This result is derived strictly from the equilibrium equation. Suppose the footpoint displacement on $z = -1$ launches and locks a sequence of uncorrelated twists into the flux tubes to be preserved topologically under the frozen condition. Such a topology necessarily varies along z and is thus incompatible with the invariance conditions (45) imposed by the equilibrium conditions. It is then argued in Parker72 that the field with the imposed field topology cannot find an equilibrium. This merely means mathematically that no continuous equilibrium exists for the field. The preservation of field topology is the fundamental property of the perfectly conducting plasma that cannot be compromised. So the equilibrium of this field must contain current sheets; not any current sheets, but those that are magnetic tangential discontinuities with the sum of magnetic and plasma pressures continuous across them. Such a discontinuous equilibrium is governed by two complementary sets of equations (Janse, Low, and Parker, 2010). The magnetostatic equations (1) and (2) describe the continuous part of the solution with the integrated forms of these equations describing the current sheets, an example of the so-called weak solutions of classical analysis (Courant and Hilbert, 1962). That an equilibrium should contain magnetic tangential discontinuities is physically not so surprising, because infinitesimally thin current sheets are admissible in the physics of the perfectly conducting plasma.

The demonstration of the existence of z-dependent solutions in C&S05 seems to be motivated by a misinterpretation that equates the result of Parker72 with an absolute claim of impossibility for any z-dependent equilibrium to exist between the two plates, taken to be true independent of context. On the contrary, the Parker72 formulation of the problem recognizes that z-variation is expected simply from boundary conditions alone. By focusing attention on small scale boundary displacements, this formulation proposes that the z-variation in boundary layers may be neglected. To begin with, the above absolute claim is readily demonstrated to be false, using potential theory without any need for complicated mathematical solutions, and, when taken out of context, this claim has nothing to do with Parker72.

A similar misinterpretation seems to have motivated the criticism by Bogoyavlenskij (2000a, 2000b) of the Parker theory; see the reply to this criticism by Parker (2000). C&S05 cited this criticism to be supporting their erroneous result. In addition to imputing Parker with a context-independent claim he did not make, the magnetostatic solutions of Bogoyavlenskij are topologically not relevant to Parker's two-plate problem. This problem treats the 3-D equilibrium states deformed continuously from an initially uniform field perpendicular to the two plates, created by prescribed, continuous, footpoint displacements on those plates. This means that one plate is uniformly of positive magnetic polarity and the other uniformly of the opposite polarity. Consequently, as demanded by the physics of the investigation, no separatrix flux surfaces and neutral points, of the kind made well known in 2D fields, are allowed in the deformed field. Moreover, the deformed field cannot contain any interior flux-ropes fully detached from the boundary plates. The Bogoyavlenskij solutions have neither of these two defining topological features of the two-plate problem. Field topology is not a constraint in the construction of these solutions. By assuming axisymmetry and a constant "alpha" for the equilibrium field-aligned current, in order to achieve tractability, the solutions Bogoyavlenskij found have a B_z reversing sign across successive concentric cylinders. Such multi-polar fields contain separatrices as well as interior flux-ropes, both explicitly excluded under the topological consideration of the Parker problem. Well known from 2D studies (Low and Wolfson, 1988), the Bogoyavlenskij solutions actually have a ready propensity for current-sheet forming at its separatrices, a point not mentioned by this author.

The following analysis brings into focus the topological constraint in the Parker problem. The $C_{\mathrm{B}} = 0$ noncompressive, rotational, footpoint displacements were employed by Parker (1979, 1989a, 1989b, 1989c). Prescribe one of these displacements at the boundary and continue it into a continuous interior plasma displacement. The interior plasma is fully compressible. The rotation in the $C_{\mathrm{B}} = 0$ boundary displacement fixes the field topology whereas this displacement can be continued into a diversity of interior displacements. Thus a continuum of continuous, deformed fields is created, having exactly the same topology in the sense that any one field can be further deformed continuously into another while holding the footpoints fixed at the boundary. The fundamental point of the Parker problem is that this continuum of deformed fields does not necessarily include one that is in equilibrium. Suppose it does. Then any one of the deformed fields may evolve under the frozen-in condition into this continuous equilibrium state. Our derivation in Section 2 shows that, at the level of linear perturbations, this equilibrium state is accessible from the uniform state by an incompressible, continuous interior displacement matching the prescribed footpoint. To emphasize the point, that the displacement turned out to be incompressible comes not from an assumption, but from solving the compressive magnetostatic equations.

The Parker theory goes a step further, stating that, for most prescribed footpoint displacements, the associated continuum of continuous, deformed fields contains no equilibrium field. As long as we impose the frozen-in condition, the field cannot change its given topology. In this case, any of the deformed fields can attain equilibrium only by evolving to one containing current sheets. The compressive, irrotational footpoint displacements, with $C_B \neq 0$, $R_B = 0$, exhibit this property in an interesting way, discovered after the submission of the present paper for publication (Low, 2010).

Parker72 did not specifically identify the noncompressive, rotational footpoint displacements ($C_1 = 0$) to make its case. At the level of linear perturbations, a general footpoint displacement has both compressive and rotational components, *i.e.*, $C_B \neq 0$ and $R_B \neq 0$. The $C_B \neq 0$ contribution gives rise to variations with z. Parker72 pointed out that those variations are confined in a thin boundary layer in the limit of $\epsilon \ll 1$, so that even for these $C_B \neq 0$ modes, the necessary conditions (45) is required in the main part of the field if the equilibrium is continuous. Both points are verified by the solutions of the linear magnetostatic problem for a large horizontal wave number $k = \sqrt{k_1^2 + k_2^2}$. So, the $C_1 \neq 0$ solutions in that limit are also physically consistent with Parker72.

3.2. The Nature of Expansion Methods

The use of infinite series in treating the nonlinear magnetostatic equations has its limitations as pointed out by Rosner and Knobloch (1982). In astrophysical studies of this kind, as opposed to rigorous mathematics, we typically tacitly assume rather than prove that a series in ϵ exists with some finite radius of convergence. To establish this radius of convergence, it may be necessary to explicitly treat the boundary conditions directly. Complications can arise when a family of magnetostatic equilibria require multi-parameter infinite series to describe them. If a single-parameter series serves the purpose, there is the possible complication that more than one expansion may be needed for different spatial regions of the domain. It is not difficult to construct exact potential examples showing that the assumption of a single universal expansion is not always justified, especially for domains that are unbounded.

The relationship between the small plasma displacement $\boldsymbol{\xi}$ used in C&S05 and the small expansion parameter ϵ in Parker72 is not simple. The magnetostatic problem of C&S05 is posed under the constraint of the frozen-in condition, analyzed only to first order in a one-parameter expansion, this parameter taken to be the amplitude of $\boldsymbol{\xi}$. The higher-order terms are more complicated, given, in the case of the magnetic field, by the Lundquist integral to the induction equation expressed as an expansion in terms of $\boldsymbol{\xi}$.

In contrast, the series in ϵ generate magnetostatic solutions without a specific reference to the frozen-in condition. For $\epsilon \ll 1$, these solutions are in the infinitesimally small neighborhood Σ of the uniform field $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$ in a phase space of *continuous* equilibria. An infinite subset σ of these continuous equilibria are accessible under the frozen-in condition to the uniform field by continuous boundary footpoint displacements. These equilibria are recoverable up to first order by the linearized magnetostatic problem of C&S05. Not all of the equilibria in Σ are accessible to the uniform field under the frozen-in condition. These are the equilibria that can evolve one into another by resistive effects in the domain while the boundaries $z = \pm 1$ remain rigid and perfectly conducting, for example.

The fundamental physical point made by the Parker72 calculation is that the subset σ is topologically restricted in the following sense. The term topology refers to magnetic footpoint connectivity and flux-tube twist and interweaving. As a field anchored rigidly at the boundary evolves continuously under the frozen-in condition, its geometry changes but not these topological properties (Berger and Field, 1984; Berger, 1999; Longcope and Malanushenko, 2008; Low, 2006a). The claim in Parker72 is that the topologies to be found in the subset σ are only a subset of the larger set of topologies that the uniform field can acquire by continuous footpoint displacements. In creating this larger set, the fields deformed from the uniform field are not required to be in equilibrium. It is important to review how well has this physical point been made in Parker72 by the use of infinite series. To avoid confusion, we must begin with the recognition of the above structure of the infinitesimal neighborhood Σ in phase space, understand what Parker72 is saying physically, and see clearly that the solutions of the linearized problem of C&S05 are, in fact, consistent with Parker72.

Assuming that the ϵ expansion used in Parker72 has a finite radius of convergence, it is still not necessarily a unique representation of the equilibrium solutions. Thus, the failure to find a continuous solution with the imposed complex topology in one representation does not rule out the possibility of such a solution recoverable by the use of another representation. This possibility was demonstrated by Van Ballegooijen (1985) using a different nonlinear expansion to treat the two-plate problem. This is a true counterexample to the Parker72 expansion. This important development led to an alternative theory of field evolution based on the idea of a cascade of energy towards large wavenumbers, as a field evolves through a continuous sequence of equilibria of increasingly greater complexities in response to footpoint motions, without losing analyticity (Van Ballegooijen, 1985, 1986).

To relate van Ballegooijen's results to the Parker theory, we need to separate what the mathematics of the ϵ-series is teaching us from the larger physical theory it represents. This theory is the above point stated more generally as follows, that the infinite set of field topologies to be found in the continuous equilibria admissible in a specific hydromagnetic system is generally a true subset of the infinite set of topologies an arbitrary, continuous field, not in equilibrium, may possess in that system. Thus, a nonequilibrium field with a topology not found in any continuous equilibrium must develop current sheets to arrive at a state of equilibrium. The theory of spontaneous current sheets has been developed from this more general physical (topological) consideration by Parker (1986a, 1986b, 1989a, 1989b, 1989c, 1990a, 1990b, 1990c, 1990d, 1991) and synthesized in Parker (1994). Progress is hampered by two challenges, finding the mathematical tools to *i*) describe field topology directly, and *ii*) treat nonperturbative problems. Some recent progress has been made in these directions (Berger and Field, 1984; Low, 2006a, 2006b, 2007; Longcope and Malanushenko, 2008; Janse and Low, 2009, 2010; Low and Janse, 2009; Huang, Bhattacharjee, and Zweibel, 2009). This topological point of view is the essence of the theory, with solid grounds for accepting this theory but also with many interesting open problems, as described by Janse, Low, and Parker (2010).

This topological point of view relates the two expansions of Parker72 and van Ballegooijen in the following way. In Parker72, the necessary condition of z-independence; for equilibrium in the main part of the field, implies that the solutions cannot span the infinite set of topologies realizable in nonequilibrium fields, assuming that the boundary layers may be neglected. The z-dependent van Ballegooijen solutions make the important correction to the theory that the Parker72 solutions are not the only continuous equilibria available. Each of these z-dependent solutions is constructed not by solving a boundary value problem but by a progressive (rigorous to the lowest order) analytical continuation of the equilibrium field through successive planes of constant z from one plate boundary to the other. The footpoint locations on the second boundary of the field lines of the solution are not prescribed but are self-consistently determined by that continuation; see Aly and Amari (2007) and Low and Flyer (2007) for general discussions. From this it can be argued that the van Ballegooijen solutions also span only a subset of the topologies of nonequilibrium fields, and are topologically no more general that the z-independent solutions of Parker72 (Parker, 1986a; Low, 1990). So the central physical question is clear, as we move away from the particularity of these two expansions, whether the nonequilibrium fields, in general, have topologies of a greater variety and diversity than the equilibrium fields for a given physical system.

3.3. Magnetic Singularity at the Boundary

In another important study, van Ballegooijen (1988) identified a property of the current sheets that challenges the validity of the Parker theory, which we should also briefly discuss. The magnetic tangential discontinuities associated with the current sheets in an equilibrium field lie on magnetic flux surfaces. In a field anchored to the boundary, such discontinuities extend along their respective flux surfaces to intersect the boundary. On the two sides of such a discontinuity at a boundary intersection, the field lines from a common footpoint diverge as they enter the domain. Since the normal field component is prescribed to be continuous everywhere on the boundary, the fields on the two sides of the discontinuity on the boundary have the same component normal to the boundary. The transverse components of these fields are discontinuous, but such that the total pressure remains continuous across the discontinuity. This gives rise to the divergence of the above two field lines as they leave their common footpoint.

Suppose the current-sheet flux surface is a simple geometric surface extending from one plate boundary to the other, that is, with no bifurcations connecting this surface to a complexity of surfaces like, for example, the surfaces of tightly packed honeycomb tubes (Parker, 1994). Now, the above pair of field lines diverging from a common footpoint at one boundary must meet again at a common footpoint on the intersection of the same flux surface made with the opposite plate boundary. For such a simple geometric situation, a contradiction was derived by van Ballegooijen between the topological connectivity of the two field lines and the requirement of the force balance of the current-sheet flux surface as a macroscopic structure, thus proving that no such current sheet may form.

This proof is significant, but it only rules out that for unidirectional fields passing from one plate boundary to the other, if current sheets do form, they do not form in simple surfaces but in some complexity of flux surfaces. This reasoning is motivated by more than just to go around the van Ballegooijen proof. In the absence of neutral points in the field, the slippage of two flux tubes in fully 3-D geometry is to be expected in a perfectly conducting plasma. The van Ballegooijen proof may be interpreted to imply that in the relaxation into an equilibrium state, if current sheets are inevitable, they form in complexities of multiple surfaces. Physical and mathematical analyses of such a situation in equilibrium fields suggest that complex formation of sheet surfaces is a natural development; see Parker (1994), including the possibility of dense formation of sheets (Low, 2007; Janse and Low, 2009). In fact, where a current sheet intersects a rigid boundary a unique variety of singularity is found, with an integrable point infinity in field intensity that is readily demonstrated even with a 2D field (Low and Wolfson, 1988; Low, 1990). Such an inevitable singularity was not considered in the van Ballegooijen's (1988) proof. It is clear that the construction of equilibrium fields by mathematically continuing a set of magnetic conditions, given at one plate boundary, out to the other boundary must begin with conditions that are not analytic to begin with, if the equilibria with inevitable current sheets are to be so constructed. Perhaps hitherto undiscovered important physics is to be found in the boundary layers, a point that is suggestive from the calculations by Zweibel and Boozer (1985), Zweibel and Li (1987), and C&S05. These physical issues merit further study.

This last remark serves to illustrate two points with which we conclude this paper. The first is that the Parker theory involves hydromagnetic properties that are still being discovered and investigated; see the recent studies (Janse and Low, 2009; Low and Janse, 2009; Huang, Bhattacharjee, and Zweibel, 2009; Janse, Low, and Parker, 2010). The other point is that numerical methods are useful tools for exploring these properties, but a clear analytical understanding of the Parker theory is crucial for designing these tools to meet the special challenges of this theory. We have concentrated on the analytical part of C&S05. The Appendix makes a comment on the numerical part of that study, kept separate in order not to interrupt our discussion.

4. Summary

The solutions to the linearized magnetostatic problem in C&S05 are physically consistent with the theory of spontaneous current sheets described in Parker72, contrary to the opposite claim in C&S05. We have shown this erroneous claim to be based on an incomplete set of solutions and misinterpretations of both these solutions and the theory in Parker72. Our discussion on the topological aspect of that theory relates the magnetostatic solutions as series expansions to the larger physical question of the general incompatibility between two demands imposed on a magnetic field, to be in an everywhere-continuous equilibrium and to possess an arbitrarily prescribed topology (Parker, 1994; Janse, Low, and Parker, 2010).

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Appendix

The numerical part of C&S05 treats the frictional relaxation, in a domain V with rigid boundary S, of a magnetic field $\mathbf{B}$ from some arbitrary, initially nonequilibrium field $\mathbf{H}(x, y, z)$ described by the equations

$$\mathbf{v} = \mu(\nabla \times \mathbf{B}) \times \mathbf{B}, \tag{46}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{v} \times \mathbf{B}) = 0, \tag{47}$$

where μ is a constant coefficient of friction and $\mathbf{v}$ the plasma velocity. The boundary conditions are

$$\mathbf{v}|_S = 0, \tag{48}$$

$$\mathbf{B} \cdot \hat{\mathbf{n}}|_S = \mathbf{H} \cdot \hat{\mathbf{n}}|_S, \tag{49}$$

stipulating a boundary flux distribution fixed in time by the initial field, where $\hat{\mathbf{n}}$ is the outward normal on S. For simplicity of discussion we neglect the effect of plasma pressure. The vanishing of $\mathbf{v}$ follows from the continuity of the tangential electric field across S with $\mathbf{B} \cdot \hat{\mathbf{n}}|_S \neq 0$ (Roberts, 1967). Since the flow is in the direction of the Lorentz force, the magnetic field is always doing positive work and must lose energy monotonically to settle into a minimum energy force-free field, of the same topology, satisfying

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = 0, \tag{50}$$

with $\mathbf{v}$ vanishing everywhere.

Quite independent of the specific numerical code used to implement this model, the above evolution equations are not compatible with the physical boundary conditions posed, a point not discussed in C&S05 and other applications of this model (see, *e.g.*, Chodura and Schluter, 1981; Craig and Sneyd, 1986; Yang, Sturrock, and Antiochos, 1986). Initially and in subsequent evolution at any one time when the field is not yet force-free, **v** is algebraically determined instantly at each point in space. The value of **v** at any point is completely determined independently from that at any other point, including the points on the boundary. During the evolution, the Lorentz force is generally nonzero all the way to the boundary. Equation (46) therefore determines a velocity that does not vanish at S in the limit of approaching S from within V, contradicting the boundary condition (48). In this exact system, the velocity is necessarily discontinuous at S if we insist on applying this boundary condition. When this model is treated numerically, this discontinuity is artificially smeared into a smooth distribution by the so-called numerical diffusion, severely compromising the induction equation (47). Thus, the formation of current sheets which is an effect of the frozen-in condition cannot be treated without modifying this model. Moreover, we expect current sheets forming in an anchored field to intersect the boundary where a severe form of singularity develops (Low and Wolfson, 1988), exacerbating this numerical limitation of the model. An obvious alternative to the algebraic equation (46), without the serious problems described is

$$\nabla^2 \mathbf{v} + (\nabla \times \mathbf{B}) \times \mathbf{B} = 0, \tag{51}$$

a static PDE for **v** that admits boundary conditions (48) meaningfully. In the recent numerical study of magnetic frictional relaxation by Bhattacharyya, Low, and Smolarkiewicz (2010), the Navier – Stokes equation is used to describe the velocity.

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