Prevailing-frequency approximation of the coupling ray theory along the SH and SV reference rays in a heterogeneous generally anisotropic medium which is approximately uniaxial

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ABSTRACT

The behaviour of the actual polarization of an electromagnetic wave or elastic S-wave is described by the coupling ray theory, which represents the generalization of both the zero-order isotropic and anisotropic ray theories and provides continuous transition between them. The coupling ray theory is usually applied to anisotropic common reference rays, but it is more accurate if it is applied to reference rays which are closer to the actual wave paths. In a generally anisotropic or bianisotropic medium, the actual wave paths may be approximated by the anisotropic-ray-theory rays if these rays behave reasonably. In an approximately uniaxial (approximately transversely isotropic) anisotropic medium, we can define and trace the SH (ordinary) and SV (extraordinary) reference rays, and use them as reference rays for the prevailing-frequency approximation of the coupling ray theory. In both cases, i.e. for the anisotropic-ray-theory rays or the SH and SV reference rays, we have two sets of reference rays. We thus obtain two arrivals along each reference ray of the first set and have to select the correct one. Analogously, we obtain two arrivals along each reference ray of the second set and have to select the correct one. In this paper, we suggest the way of selecting the correct arrivals. We then demonstrate the accuracy of the resulting prevailing-frequency approximation of the coupling ray theory using elastic S waves along the SH and SV reference rays in four different approximately uniaxial (approximately transversely isotropic) velocity models.

Keywords: wave propagation, elastic anisotropy, electromagnetic bianisotropy, heterogeneous media, wave coupling, approximately uniaxial (approximately transversely isotropic) anisotropic medium, anisotropic-ray-theory rays, SH (ordinary) and SV (extraordinary) reference rays

1. INTRODUCTION

There are two different high-frequency asymptotic ray theories for two polarizations of electromagnetic waves or two elastic S waves with frequency-independent amplitudes: the *isotropic ray theory* based on the assumption of equal velocities of both electromagnetic waves or S waves, and the *anisotropic ray theory* assuming both electromagnetic waves or S waves strictly decoupled. In the isotropic ray theory, the elastic S-wave or electromagnetic polarization vectors do not rotate about the ray, whereas in the anisotropic ray theory they coincide with the eigenvectors of the Kelvin–Christoffel matrix which may rotate rapidly about the ray. In “weakly anisotropic” velocity models, at moderate frequencies, the actual wave polarization tends to remain unrotated round the ray, but is partly attracted by the rotation of the eigenvectors of the Kelvin–Christoffel matrix. The intensity of the attraction increases with frequency. This behaviour of the actual wave polarization is described by the *coupling ray theory*. The coupling ray theory was proposed, e.g., by *Kravtsov (1968)*, *Naida (1977, 1979)* or *Fuki et al. (1998)* for electromagnetic waves or by *Coates and Chapman (1990)* for elastic S waves. The frequency-dependent coupling ray theory is the generalization of both the zero-order isotropic and anisotropic ray theories and provides continuous transition between them. The coupling ray theory is applicable to electromagnetic waves or elastic S waves at all degrees of anisotropy, from isotropic to considerably anisotropic velocity models. The numerical algorithm for calculating the frequency-dependent coupling-ray-theory tensor Green function has been designed by *Bulant and Klimeš (2002)*. The accuracy of the coupling ray theory depends on the reference rays (*Bulant and Klimeš, 2008*).

On the one hand, the coupling ray theory is usually applied to anisotropic common reference rays (*Bakker, 2002; Klimeš and Bulant, 2004, 2006; Klimeš, 2006; Bulant and Klimeš, 2008*). On the other hand, the coupling ray theory is more accurate if it is applied to reference rays which are closer to the actual wave paths (*Klimeš and Bulant, 2014a, 2015*).

In a generally anisotropic or bianisotropic medium, the actual wave paths may be approximated by the anisotropic-ray-theory rays if these rays behave reasonably, i.e. outside the conical refraction (*Chen, 1983, Sec. 5.7*) from the point singularities. In an approximately uniaxial (approximately transversely isotropic) anisotropic medium, we can define and trace the SH (ordinary) and SV (extraordinary) reference rays, and use them as reference rays for the prevailing-frequency approximation of the coupling ray theory. In both cases, i.e. for the anisotropic-ray-theory rays or the SH and SV reference rays, we have two sets of reference rays. The coupling ray theory tensor Green function along the anisotropic-ray-theory reference rays, along the SH reference rays or along the SV reference rays can be calculated using the algorithm by *Bulant and Klimeš (2002)*. The decomposition of the coupling ray theory tensor Green function into two arrivals can be determined using the algorithm by *Klimeš and Bulant (2012, 2016)*.

In this way, we obtain two arrivals along each reference ray of the first set and have to select the correct one. Analogously, we obtain two arrivals along each reference ray of the second set and have to select the correct one. The main topic of this paper consists in the selection of the arrivals which is proposed in Section 3.

1.1. Introduction for electromagnetic waves
and for elastic waves in media
which are not approximately uniaxial
(anisotropic-ray-theory reference rays)

For electromagnetic waves in a generally anisotropic medium, the slowness sheets of the slowness surface usually intersect at four point singularities, with neither intersection (line) singularity nor split intersection singularity. We thus have to separate the slowness surface into the faster slowness sheet and the slower slowness sheet. In this case, we can use the anisotropic-ray-theory rays as the reference rays for the prevailing-frequency approximation of the coupling ray theory in regions where these rays behave reasonably, i.e. outside the conical refraction (*Chen, 1983, Sec. 5.7*) from the point singularities. For the singularities of the slowness surface of electromagnetic waves in a generally bianisotropic medium refer to *Favaro and Hehl (2016)*. For the intersection (line) singularities and the split intersection singularities of the slowness surface of electromagnetic waves in an uniaxial bianisotropic medium refer to *Klimeš (2017a)*.

In a generally anisotropic elastic medium, the S-wave slowness sheets of the slowness surface are usually mostly separated and intersect in as many as 32 point S-wave singularities (*Vavryčuk, 2005a,b*). In this case, outside the point singularities, the anisotropic-ray-theory rays stay at the faster or slower S-wave slowness sheet, respectively. In approaching the point singularities, the limiting case again corresponds to staying at the faster or slower S-wave slowness sheet, respectively. In a generally anisotropic medium, we thus have to separate the slowness surface into the P-wave slowness sheet, the faster S-wave slowness sheet and the slower S-wave slowness sheet. If a generally anisotropic elastic medium is not approximately uniaxial, i.e. if it contains neither intersection (line) singularity nor split intersection singularity, we can use the anisotropic-ray-theory rays as the reference rays for the prevailing-frequency approximation of the coupling ray theory at least in regions not influenced by point singularities.

1.2. Introduction for elastic waves
in approximately uniaxial media
(SH and SV reference rays)

In an uniaxial (transversely isotropic) elastic medium, the S-wave slowness sheets may intersect along line intersection singularities (*Vavryčuk, 2001, 2003*), see Fig. 1. In this special case, we can separate the slowness surface into the P-wave, SH-wave and SV-wave slowness sheets. The SH-wave slowness sheet and the SV-wave slowness sheet form an intersection singularity, see Fig. 2. Unfortunately, this intersection singularity is unstable, and any deviation from an uniaxial medium causes splitting of this intersection singularity (*Crampin, 1981*). The slower S-wave slowness sheet then separates from the faster S-wave slowness sheet, forming a split intersection singularity with smooth but strongly curved edges on both sheets (*Bulant and Klimeš, 2014; Klimeš and Bulant, 2014b*), see Fig. 3.

Fig. 1. Example of singularities at the slowness surface. The red rectangle marks the detail zoomed in Figs 2–5. The slowness vector smoothly moves along the slowness surface according to Hamilton’s equations of rays.

When the slowness vector of a ray smoothly passes through a split intersection singularity, the ray–velocity vector rapidly changes its direction and creates a sharp bend of the ray (Bulant and Klimeš, 2014; Klimeš and Bulant, 2014b). This sharp bend is connected with a rapid rotation of the eigenvectors of the Kelvin–Christoffel matrix, see Fig. 4. The sharply bent rays thus cannot describe the correct wave propagation.

Since any rounding error can perturb an uniaxial medium to a generally anisotropic medium and split the intersection singularity, we cannot determine numerically whether a medium is uniaxial (transversely isotropic), but we can determine numerically whether a medium is approximately uniaxial (approximately transversely isotropic) (Klimeš, 2015, 2016a).

The actual S waves do not propagate along the sharply bent rays, but smoothly tunnel through a split intersection singularity (Klimeš and Bulant, 2012, 2016). If a medium is close to uniaxially anisotropic, the actual wave paths resemble the SH (ordinary) and SV (extraordinary) rays. Fortunately, in an approximately uniaxial anisotropic medium, we can define and trace the SH (ordinary) and SV (extraordinary) reference rays according to Klimeš and Bulant (2015, 2017), see Fig. 5. These SH and SV reference rays represent a generalization of the SH (ordinary) and SV (extraordinary) rays from an uniaxial medium towards a generally anisotropic medium.

For elastic waves in an approximately uniaxial medium, we thus should use the SH (ordinary) and SV (extraordinary) reference rays as the reference rays for the prevailing–frequency approximation of the coupling ray theory.

In Section 4, we demonstrate numerically the improvement of the coupling–ray–theory seismograms calculated along the SH and SV reference rays proposed

Fig. 2. Intersection singularity at the slowness surface.

Fig. 3. Split intersection singularity at the slowness surface.

Fig. 4. Anisotropic-ray-theory ray passing through a split intersection singularity at the slowness surface.

Fig. 5. SH reference ray passing through a split intersection singularity at the slowness surface.

by *Klimeš and Bulant (2015, 2017)*, compared to the coupling-ray-theory seismograms calculated along the anisotropic common reference rays. The comparison is performed in approximately uniaxial elastic velocity models QI2 and QI4 without a split intersection singularity, and in approximately uniaxial elastic velocity models SC1-I and SC1-II with a split intersection singularity (*Pšenčík et al., 2012*). We compare the prevailing-frequency approximation along the SH and SV reference rays with the prevailing-frequency approximation along the anisotropic common

reference rays and with the Fourier pseudospectral method (Pšenčík et al., 2012) which is considered as a nearly exact reference.

In velocity model SC1_II, Klimeš and Bulant (2012, Fig. 7; 2016, Fig. 7) observed the greatest inaccuracy of the coupling-ray-theory seismograms calculated along the anisotropic common reference rays. Bulant and Klimeš (2014) and Klimeš and Bulant (2014b) then demonstrated that the anisotropic-ray-theory reference rays are not applicable in velocity model SC1_II, which lead to the proposal of tracing the SH and SV reference rays in generally anisotropic velocity models which are approximately uniaxial.

2. PREVAILING-FREQUENCY APPROXIMATION OF THE COUPLING RAY THEORY

We trace two systems of reference rays, which may be represented by two systems of the anisotropic-ray-theory rays, or by the SH and SV reference rays. The SH and SV reference rays can be traced according to Klimeš and Bulant (2015, 2017). The algorithm for calculating the prevailing-frequency approximation of the coupling ray theory tensor Green function is the same along the rays of the first and of the second system of reference rays.

Along each reference ray, we denote g_{iA} , $A = 1, 2$, two elastic S-wave or electromagnetic eigenvectors of the Kelvin-Christoffel matrix (Červený, 2001, Eq. 2.2.19; Klimeš, 2016b, Eq. 41; 2016c, Eq. 31). Hereinafter, the lower-case Roman indices take values 1, 2 and 3, and the upper-case Roman indices take values 1 and 2, unless otherwise stated.

We approximate the anisotropic-ray-theory elastic S-wave or electromagnetic travel times τ_1 and τ_2 by linear perturbation expansions. Note that we may also improve the accuracy of linear perturbation expansions by including the second-order perturbations of travel time according to Klimeš (2002), Klimeš and Bulant (2006) and Bulant and Klimeš (2008).

We denote by $\mathbf{\Pi} = \mathbf{\Pi}(\omega)$ the 2×2 frequency-domain propagator matrix of the coupling ray theory defined by the coupling equations (Bulant and Klimeš, 2002, Eq. 11; Klimeš and Bulant, 2012, Eq. 1; 2016, Eq. 3). We denote by

$$\mathbf{D}(\omega) = \frac{\partial \mathbf{\Pi}}{\partial \omega}(\omega) \quad (1)$$

its derivative with respect to circular frequency ω . For the algorithm of calculating matrices $\mathbf{\Pi}$ and $\mathbf{D}$ refer to Klimeš and Bulant (2012, Sec. 4; 2016, Sec. 4).

For a detailed description of the prevailing-frequency approximation of the coupling ray theory refer to Klimeš and Bulant (2012, Sec. 3; 2016, Sec. 3). Here we only recall the definitions of the terms useful for the selection of arrivals in Section 3.

Along each reference ray, the prevailing-frequency approximation of the frequency-domain propagator matrix $\mathbf{\Pi} = \mathbf{\Pi}(\omega)$ of the coupling ray theory reads

$$\mathbf{\Pi}(\omega) \approx \mathbf{\Pi}^{(1)}(\omega_0) \exp[-i(\omega - \omega_0) D(\omega_0)] + \mathbf{\Pi}^{(2)}(\omega_0) \exp[i(\omega - \omega_0) D(\omega_0)] , \quad (2)$$

where ω_0 is the prevailing circular frequency. This decomposition represents the approximation of propagator matrix $\mathbf{\Pi} = \mathbf{\Pi}(\omega)$ by two S waves with travel times $\bar{\tau} - D$ and $\bar{\tau} + D$.

We calculate difference D of the coupling-ray-theory travel times from the mean travel time

$$\bar{\tau} = \frac{1}{2} (\tau_1 + \tau_2) \quad (3)$$

using relation

$$D = \sqrt{\det(\mathbf{D})} . \quad (4)$$

We then calculate the 2×2 complex-valued singular amplitude matrices

$$\mathbf{\Pi}^{(1)} = \frac{1}{2} \left[\mathbf{\Pi} + i \frac{\mathbf{D}}{D} \right] , \quad (5)$$

and

$$\mathbf{\Pi}^{(2)} = \frac{1}{2} \left[\mathbf{\Pi} - i \frac{\mathbf{D}}{D} \right] . \quad (6)$$

The prevailing-frequency approximation of the complex-valued frequency-domain dyadic Green function is then composed of two elementary dyadic Green functions,

$$G_{ij}(\omega) \approx G_{ij}^{(1)}(\omega) + G_{ij}^{(2)}(\omega) . \quad (7)$$

These elementary dyadic Green functions read

$$G_{ij}^{(J)}(\omega) = A_{ij}^{(J)}(\omega_0) \exp[i\omega T^{(J)}(\omega_0)] . \quad (8)$$

These S waves have coupling-ray-theory travel times

$$T^{(1)}(\omega_0) = \bar{\tau} - D \quad (9)$$

and

$$T^{(2)}(\omega_0) = \bar{\tau} + D . \quad (10)$$

The corresponding complex-valued dyadic amplitudes read

$$A_{ij}^{(J)}(\omega_0) = A g_{iM} \tilde{g}_{jN} \Pi_{MN}^{(J)}(\omega_0) \exp\{i\omega_0 [\bar{\tau} - T^{(J)}(\omega_0)]\} , \quad (11)$$

no summation over $J = 1, 2$. Note that the Einstein summation over repetitive indices is used throughout the paper unless otherwise stated. Here $\tilde{g}_{jN}$ are the elastic S-wave or electromagnetic eigenvectors of the Kelvin-Christoffel matrix at the source, g_{iM} are the elastic S-wave or electromagnetic eigenvectors of the Kelvin-Christoffel matrix at the receiver, and A is the reference amplitude corresponding to the reference rays. Note that the “dyadic tensor” represents the tensor product of two vectors.

Since the 2×2 matrices $\Pi_{MN}^{(J)}(\omega_0)$ are singular, each complex-valued tensor amplitude (11) is a dyadic product

$$g_{iM} \Pi_{MN}^{(J)} \tilde{g}_{jN} = g_i^{(J)} \tilde{g}_j^{(J)} , \quad (12)$$

no summation over $J = 1, 2$, of two unit complex-valued vectors $g_i^{(J)}$ and $\tilde{g}_j^{(J)}$, multiplied by amplitude

$$A^{(J)} = A \exp\{i\omega_0 [\bar{\tau} - T^{(J)}(\omega_0)]\} , \quad (13)$$

$$A_{ij}^{(J)}(\omega_0) = A^{(J)} g_i^{(J)} \tilde{g}_j^{(J)} , \quad (14)$$

no summation over $J = 1, 2$. The unit complex-valued vectors $\tilde{g}_j^{(1)}$ and $\tilde{g}_j^{(2)}$ are mutually perpendicular, and the unit complex-valued vectors $g_i^{(1)}$ and $g_i^{(2)}$ are also mutually perpendicular.

3. SELECTION OF ARRIVALS

We have two systems of reference rays, which may be represented by two systems of the anisotropic-ray-theory rays, or by the SH and SV reference rays. For the sake of simplicity, we shall refer hereinafter to the first system of rays as the “SH reference rays” and to the second system of rays as the “SV reference rays”, although they may represent the anisotropic-ray-theory rays or even other systems of reference rays.

For each SH reference ray, we now have two arrivals with coupling-ray-theory travel times (9) and (10), and the corresponding complex-valued dyadic amplitudes (11). We need to select just one of them. We may try to select the proper arrival according to its polarization or travel time. If one of the arrivals is close to the reference SH wave in its polarization and its travel time, we may assume that it has probably been calculated with good accuracy and we may select it. Analogously for each SV reference ray. Unfortunately, the criteria based on polarization and travel time may be contradictory as we shall observe in numerical examples. This selection thus represents the critical unresolved issue. If this selection fails, the coupling ray theory along the SH and SV reference rays cannot be used, and we should calculate the coupling-ray-theory approximation along the anisotropic common reference rays.

The prevailing-frequency approximation of the complex-valued frequency-domain tensor Green function is then composed of the SH and SV dyadic Green functions,

$$G_{ij}(\omega) \approx G_{ij}^{\text{SH}}(\omega) + G_{ij}^{\text{SV}}(\omega) , \quad (15)$$

where $G_{ij}^{\text{SH}}(\omega)$ corresponds to the selected part of Green function (7) calculated along the SH reference ray and $G_{ij}^{\text{SV}}(\omega)$ corresponds to the selected part of Green function (7) calculated along the SV reference ray.

3.1. Selection of arrivals based on polarization

The unit SV reference polarization vector reads (*Klimeš and Bulant, 2015, Eq. 1; 2017, Eq. 7*)

$$g_{i\text{SV}} = \frac{g_{iA} g_{kA} t_k}{\sqrt{t_r g_{rN} g_{sN} t_s}} , \quad (16)$$

where t_k is the given reference symmetry vector (*Klimeš, 2015, 2016a, 2017b*). Since vectors $g_j^{(1)}$ and $g_j^{(2)}$ defined by (14) are unit and mutually perpendicular linear combinations of vectors g_{j1} and g_{j2} , $g_i^{(A)*} g_k^{(A)}$ is a projection matrix equal to projection matrix $g_{iA} g_{kA}$. Asterisk * denotes the complex conjugate. We can thus express definition (16) as

$$g_{i\text{SV}} = \frac{g_i^{(A)*} g_k^{(A)} t_k}{\sqrt{t_r g_r^{(N)*} g_s^{(N)} t_s}} . \quad (17)$$

Definitions (14) and (17) yield relations

$$|\tilde{g}_i^{(J)} \tilde{g}_{i\text{SV}}|^2 = \frac{A_{ij}^{(J)}(\omega^0)^* t_j A_{im}^{(J)}(\omega^0) t_m}{A_{nr}^{(1)}(\omega^0)^* t_r A_{ns}^{(1)}(\omega^0) t_s + A_{nr}^{(2)}(\omega^0)^* t_r A_{ns}^{(2)}(\omega^0) t_s} , \quad (18)$$

and

$$|g_i^{(J)} g_{iSV}|^2 = \frac{A_{ji}^{(J)}(\omega^0)^* t_j A_{mi}^{(J)}(\omega^0) t_m}{A_{rn}^{(1)}(\omega^0)^* t_r A_{sn}^{(1)}(\omega^0) t_s + A_{rn}^{(2)}(\omega^0)^* t_r A_{sn}^{(2)}(\omega^0) t_s}, \quad (19)$$

both with no summation over $J = 1, 2$.

Along the SH reference ray, we may then select the arrival according to values (18) and (19):

$$|\tilde{g}_i^{(1)} \tilde{g}_{iSV}|^2 \ll \frac{1}{2} \quad \text{and} \quad |g_i^{(1)} g_{iSV}|^2 \ll \frac{1}{2} \quad \implies \quad G_{ij}^{\text{SH}}(\omega) = G_{ij}^{(1)}(\omega) \quad (20)$$

or

$$|\tilde{g}_i^{(1)} \tilde{g}_{iSV}|^2 \gg \frac{1}{2} \quad \text{and} \quad |g_i^{(1)} g_{iSV}|^2 \gg \frac{1}{2} \quad \implies \quad G_{ij}^{\text{SH}}(\omega) = G_{ij}^{(2)}(\omega). \quad (21)$$

Analogously along the SV reference ray:

$$|\tilde{g}_i^{(1)} \tilde{g}_{iSV}|^2 \gg \frac{1}{2} \quad \text{and} \quad |g_i^{(1)} g_{iSV}|^2 \gg \frac{1}{2} \quad \implies \quad G_{ij}^{\text{SV}}(\omega) = G_{ij}^{(1)}(\omega) \quad (22)$$

or

$$|\tilde{g}_i^{(1)} \tilde{g}_{iSV}|^2 \ll \frac{1}{2} \quad \text{and} \quad |g_i^{(1)} g_{iSV}|^2 \ll \frac{1}{2} \quad \implies \quad G_{ij}^{\text{SV}}(\omega) = G_{ij}^{(2)}(\omega). \quad (23)$$

If both conditions (20) and (21) fail, or both conditions (22) and (23) fail, this selection of arrivals based on polarization fails.

Analogous but even simpler selection can be applied to the anisotropic-ray-theory reference rays.

3.2. Selection of arrivals based on travel time

The selection of arrivals along the SH reference ray according to the corresponding reference travel time τ_{SH} is possible only if both $|\tau_{\text{SH}} - \bar{\tau}|$ and D are not negligible with respect to τ_{SH} . Then

$$\tau_{\text{SH}} - \bar{\tau} < 0 \quad \implies \quad G_{ij}^{\text{SH}}(\omega) = G_{ij}^{(1)}(\omega) \quad (24)$$

or

$$\tau_{\text{SH}} - \bar{\tau} > 0 \quad \implies \quad G_{ij}^{\text{SH}}(\omega) = G_{ij}^{(2)}(\omega). \quad (25)$$

Analogously, the selection of arrivals along the SV reference ray according to the corresponding reference travel time τ_{SV} is possible only if both $|\tau_{\text{SV}} - \bar{\tau}|$ and D are not negligible with respect to τ_{SV} . Then

$$\tau_{\text{SV}} - \bar{\tau} < 0 \quad \implies \quad G_{ij}^{\text{SV}}(\omega) = G_{ij}^{(1)}(\omega) \quad (26)$$

or

$$\tau_{\text{SV}} - \bar{\tau} > 0 \quad \implies \quad G_{ij}^{\text{SV}}(\omega) = G_{ij}^{(2)}(\omega). \quad (27)$$

This selection of arrivals based on travel time is recommended especially as the test of the selection of arrivals based on polarization, because it may fail in the vicinity of a split intersection singularity.

This selection can equally well be applied to the anisotropic-ray-theory reference rays.

4. ELASTIC NUMERICAL EXAMPLES

Bulant et al. (2011) and *Klimeš and Bulant (2012, 2016)* compared the coupling ray theory for S waves and its prevailing-frequency approximation with the Fourier pseudospectral method (*Pšenčík et al., 2012*) which was considered as a nearly exact reference. The coupling-ray-theory seismograms were calculated along the anisotropic common reference rays with the second-order perturbation expansion of travel times. The comparison was performed in seven velocity models: six velocity models QI, QI2, QI4, KISS, SC1-I and SC1-II are approximately uniaxial (approximately transversely isotropic), velocity model ORT is orthorhombic. These anisotropic velocity models with related input data can be found in package DATA (*Bucha and Bulant, 2015*).

The coupling-ray-theory seismograms in approximately uniaxial velocity models QI and KISS were nearly equal to the Fourier pseudospectral method (*Klimeš and Bulant, 2012, Figs 3, 8; 2016, Figs 3, 8*). The coupling-ray-theory seismograms in orthorhombic velocity model ORT differed from the Fourier pseudospectral method approximately by the thickness of line (*Klimeš and Bulant, 2012, Fig. 9; 2016, Fig. 9*). We thus compare the prevailing-frequency approximation of the coupling ray theory with the Fourier pseudospectral method in velocity models QI2, QI4, SC1-I and SC1-II.

Approximately uniaxial velocity models QI2 and QI4 contain no split intersection singularity. In this case, the SH and SV reference rays correspond to the faster and slower anisotropic-ray-theory S waves. We thus sort the SH and SV reference rays in velocity models QI2 and QI4 according to the algorithm of Section 3.2. Note that the algorithm of Section 3.1 is not applicable in these velocity models because of too great rotation of the eigenvectors of the Kelvin-Christoffel matrix.

Approximately uniaxial velocity models SC1-I and SC1-II contain a split intersection singularity. We thus cannot sort the SH and SV reference rays according to the algorithm of Section 3.2 in these velocity models, and use the algorithm of Section 3.1.

We consider the prevailing-frequency approximation of the coupling ray theory calculated (a) along the anisotropic common reference rays with the quadratic perturbation expansions of travel times, (b) along the SH and SV reference rays with the linear perturbation expansions of travel times. We compare these coupling ray theory seismograms with the Fourier pseudospectral method by *Pšenčík et al. (2012)*, which is considered as a nearly exact reference.

4.1. Anisotropic velocity models

We consider four approximately uniaxial (approximately transversely isotropic) velocity models QI2, QI4, SC1-I and SC1-II. These velocity models are laterally homogeneous. The density-reduced elastic moduli are linear functions of depth in all these velocity models. The density is constant. For the sections of the S-wave phase-velocity surface in the source-receiver plane refer to *Pšenčík et al. (2012)*.

Vertically heterogeneous 1-D anisotropic velocity models QI2 and QI4 are approximately transversely isotropic in a vertical plane which forms a 45° angle with the vertical source–receiver plane and is situated between the positive radial and positive transverse seismogram components. Velocity models QI2 and QI4 are derived from velocity model QI (*Pšenčík and Dellinger, 2001, model WA rotated by 45°*) and mutually differ in their degrees of anisotropy. The differences of the elastic moduli in velocity models QI, QI2 and QI4 from the elastic moduli in the reference isotropic velocity model are determined by ratio 1:2:4. For the elastic moduli in velocity models QI2 and QI4 refer to *Bulant and Klimeš (2008)*.

At the depth of 0 km, velocity model SC1_I is approximately uniaxial and its reference symmetry axis is horizontal. At this depth, the slowness surface contains a split intersection singularity, whereas velocity models QI2 and QI4 display no exact or split intersection singularity (*Pšenčík et al., 2012*). At the depth of 1.5 km, velocity model SC1_I is very close to isotropic, but is slightly cubic and its symmetry axes coincide with the coordinate axes. This means that, at depths between 0 km and 1.5 km, velocity model SC1_I is very close to uniaxial, but is slightly tetragonal. Whereas the uniaxial medium contains the intersection singularity through which the rays pass without rotation of the eigenvectors of the Kelvin–Christoffel matrix (*Vavryčuk, 2001, Sec. 4.3; Klimeš and Bulant, 2014a*), in the approximately uniaxial medium, the S–wave slowness surface is split at this unstable singularity (*Crampin, 1981*) and the eigenvectors of the Kelvin–Christoffel matrix rapidly rotate by 90° there. This split intersection singularity then acts as an interface and smoothly but very rapidly converts the actual S–wave polarizations from the approximately anisotropic–ray–theory polarization S1 to the approximately anisotropic–ray–theory polarization S2, and vice versa.

Velocity model SC1_II is analogous to SC1_I, but the axis of symmetry of its uniaxial component is tilted, refer to *Pšenčík et al. (2012)*. The symmetry axes of its weak cubic component coincide with the coordinate axes. The split intersection singularity in velocity model SC1_II is thus positioned differently in comparison with velocity model SC1_I. In the source–receiver plane, the split intersection singularity is close to horizontal slowness vectors. The deviation of the anisotropic–ray–theory rays from both the anisotropic common reference rays and the actual wave paths is much greater in velocity model SC1_II than in velocity model SC1_I (*Klimeš and Bulant, 2014b*).

For a more detailed description of velocity models SC1_I and SC1_II and for the comparison with the anisotropic–ray–theory seismograms in these velocity models refer to *Pšenčík et al. (2012)*.

4.2. Measurement configuration

The synthetic seismograms generated by a vertical force are calculated at the receivers located in a vertical well at a distance of 1 km from the source. The source–receiver configuration in velocity models QI2 and QI4 is displayed in Fig. 6. The source–receiver configuration in velocity models SC1_I and SC1_II is displayed in Fig. 7.

Fig. 6. The source–receiver configuration for calculating synthetic seismograms in velocity models QI2 and QI4, with a sketch of the reference rays. The source is located at the Earth’s surface, the receivers are placed in a vertical well.

The source time function is the Gabor signal $\cos(2\pi ft) \exp[-(2\pi ft/4)^2]$ with reference frequency $f = 50$ Hz, bandpass filtered by a cosine filter specified by frequencies 0 Hz, 5 Hz, 60 Hz and 100 Hz. The depths of the receivers are displayed in the plots of seismograms. The receivers record the following 3 components of displacement: radial component (along the line connecting the source and the top of the well, positive away from the source), transverse component (perpendicular to the source–receiver plane), and vertical component (positive downwards). The recording system is right-handed. For the prevailing-frequency approximation of the coupling ray theory, we naturally use prevailing frequency $f_0 = 50$ Hz.

4.3. Comparison of synthetic seismograms

Along the anisotropic common reference rays, we use the quadratic perturbation expansions of travel times because they yield slightly more accurate synthetic seismograms than the linear expansions. Along the SH and SV reference rays, we use the linear perturbation expansions of travel times, and the presented numerical examples suggest that these linear perturbation expansions are sufficiently accurate in velocity models QI2, QI4, SC1.I and SC1.II.

Since approximately uniaxial velocity models QI2 and QI4 display no split intersection singularity, we applied the selection of arrivals based on travel time according to Section 3.2 in these velocity models. In these numerical tests, we selected the arrivals manually.

The synthetic seismograms in velocity models QI2 and QI4 are displayed in Figs 8–11. In both velocity models, we compare the prevailing-frequency approximation of the coupling ray theory along the anisotropic common reference rays with the Fourier pseudospectral method, and the prevailing-frequency approximation of the coupling ray theory along the SH and SV reference rays with the Fourier pseudospectral method. In each figure, the synthetic seismograms calculated by the prevailing-frequency approximation of the coupling ray theory and by the Fourier pseudospectral method are displayed in different colours. In each velocity model, the

Fig. 7. The source–receiver configuration for calculating synthetic seismograms in velocity models SC1_I and SC1_II, with a sketch of the reference rays. The source is located at the Earth’s surface, the receivers are placed in the vertical well.

amplitude scaling is equal for all components and all receivers. Note that the second (transverse) component would vanish in the isotropic–ray–theory seismograms in these velocity models.

In velocity model QI2 (Figs 8 and 9) which displays the development of S–wave splitting, we can observe that the amplitude differences in the third (vertical) component and also in the first (radial) component (first S–wave arrival at the bottom receivers) have been corrected by the prevailing–frequency approximation of the coupling ray theory along the SH and SV reference rays.

Velocity model QI4 (Figs 10 and 11), whose anisotropy is twice stronger than in model QI2, is considerably anisotropic, and the separation of the two S waves is about 0.06 s which corresponds approximately to three S–wave periods. In velocity model QI4, the amplitude differences in the third (vertical) component, and also in the second (transverse) component (first S–wave arrival at deeper receivers) and in the first (radial) component (first S–wave arrival at deeper receivers) have been corrected by the prevailing–frequency approximation of the coupling ray theory along the SH and SV reference rays.

The synthetic seismograms in velocity models SC1_I and SC1_II are displayed in Figs 12–15. Since approximately uniaxial velocity models SC1_I and SC1_II

Fig. 8. Radial, transverse and vertical component of the seismograms calculated in velocity model QI2. The **red** seismograms are calculated using the prevailing-frequency approximation of the coupling ray theory according to *Klimeš and Bulant (2012, 2016)* along the anisotropic common reference rays with the quadratic perturbation expansions of travel times. They are overlaid by the **black** seismograms calculated using the Fourier pseudospectral method by *Pšenčík et al. (2012)* which is considered here as a nearly exact reference.

Fig. 9. Radial, transverse and vertical component of the seismograms calculated in velocity model QI2. The **red** seismograms are calculated using the prevailing-frequency approximation of the coupling ray theory according to *Klimeš and Bulant (2012, 2016)* along the SH and SV reference rays with the linear perturbation expansions of travel times as described in this paper. They are overlaid by the **black** seismograms calculated using the Fourier pseudospectral method by *Pšenčík et al. (2012)* which is considered here as a nearly exact reference. In comparison with Fig. 8, the amplitude differences in the third (vertical) component and also in the first (radial) component (first S-wave arrival at the bottom receivers) have been corrected.

Fig. 10. Radial, transverse and vertical component of the seismograms calculated in velocity model QI4. The **red** seismograms are calculated using the prevailing-frequency approximation of the coupling ray theory according to *Klimeš and Bulant (2012, 2016)* along the anisotropic common reference rays with the quadratic perturbation expansions of travel times. They are overlaid by the **black** seismograms calculated using the Fourier pseudospectral method by *Pšenčík et al. (2012)* which is considered here as a nearly exact reference.

Fig. 11. Radial, transverse and vertical component of the seismograms calculated in velocity model QI4. The **red** seismograms are calculated using the prevailing-frequency approximation of the coupling ray theory according to *Klimeš and Bulant (2012, 2016)* along the SH and SV reference rays with the linear perturbation expansions of travel times as described in this paper. They are overlaid by the **black** seismograms calculated using the Fourier pseudospectral method by *Pšenčík et al. (2012)* which is considered here as a nearly exact reference. In comparison with Fig. 10, the amplitude differences in the third (vertical) component, and also in the second (transverse) component (first S-wave arrival at deeper receivers) and in the first (radial) component (first S-wave arrival at deeper receivers) have been corrected.

Fig. 12. Radial, transverse and vertical component of the seismograms calculated in velocity model SC1-I. The **red** seismograms are calculated using the prevailing-frequency approximation of the coupling ray theory according to *Klimeš and Bulant (2012, 2016)* along the anisotropic common reference rays with the quadratic perturbation expansions of travel times. They are overlaid by the **black** seismograms calculated using the Fourier pseudospectral method by *Pšenčík et al. (2012)* which is considered here as a nearly exact reference.

Fig. 13. Radial, transverse and vertical component of the seismograms calculated in velocity model SC1_L. The **red** seismograms are calculated using the prevailing-frequency approximation of the coupling ray theory according to *Klimeš and Bulant (2012, 2016)* along the SH and SV reference rays traced according to this paper, with the linear perturbation expansions of travel times. They are overlaid by the **black** seismograms calculated using the Fourier pseudospectral method by *Pšenčík et al. (2012)* which is considered here as a nearly exact reference. In comparison with Fig. 12, the differences of the coupling ray theory from the Fourier pseudospectral method have obviously been reduced.

Fig. 14. Radial, transverse and vertical component of the seismograms calculated in velocity model SC1-II. The **red** seismograms are calculated using the prevailing-frequency approximation of the coupling ray theory according to *Klimeš and Bulant (2012, 2016)* along the anisotropic common reference rays with the quadratic perturbation expansions of travel times. They are overlaid by the **black** seismograms calculated using the Fourier pseudospectral method by *Pšenčík et al. (2012)* which is considered here as a nearly exact reference.

Fig. 15. Radial, transverse and vertical component of the seismograms calculated in velocity model SC1-II. The **red** seismograms are calculated using the prevailing-frequency approximation of the coupling ray theory according to *Klimeš and Bulant (2012, 2016)* along the SH and SV reference rays traced according to this paper, with the linear perturbation expansions of travel times. They are overlaid by the **black** seismograms calculated using the Fourier pseudospectral method by *Pšenčík et al. (2012)* which is considered here as a nearly exact reference. In comparison with Fig. 14, the differences of the coupling ray theory from the Fourier pseudospectral method have been reduced considerably.

display split intersection singularities, we applied the selection of arrivals based on polarization according to Section 3.1 in these velocity models.

In both velocity models, we compare the prevailing-frequency approximation of the coupling ray theory along the anisotropic common reference rays with the Fourier pseudospectral method, and the prevailing-frequency approximation of the coupling ray theory along the SH and SV reference rays with the Fourier pseudospectral method. In each figure, the synthetic seismograms calculated by the prevailing-frequency approximation of the coupling ray theory and by the Fourier pseudospectral method are displayed in different colours. In each velocity model, the amplitude scaling is equal for all components and all receivers. Note that the second (transverse) component would vanish in the isotropic-ray-theory seismograms in these velocity models.

Whereas the coupling-ray-theory seismograms calculated along the anisotropic common reference rays in velocity model SC1.I (Fig. 12) display small inaccuracies, the coupling-ray-theory seismograms calculated along the SH and SV reference rays (Fig. 13) are nearly correct.

In velocity model SC1.II containing a split intersection singularity in the source–receiver plane, we observe a large inaccuracy of the coupling-ray-theory seismograms calculated along the anisotropic common reference rays (Fig. 14). The coupling-ray-theory seismograms calculated along the SH and SV reference rays (Fig. 15) represent a considerable accuracy improvement. However, some differences, mostly in amplitudes, are still present. These differences are centred around the third and fourth receivers below the surface. We can also observe an increment of amplitudes of the faster S wave in this region. In this case, the faster S wave is approximately the SV wave. We thus display the SV-wave wavefront triangulated according to the SV ray tubes in velocity model SC1.II in Figs 16 to 18. On the wavefront, we observe a belt of small geometrical spreading which intersects the receiver profile between the third and fourth receivers below the surface. This belt probably causes the increase of the zero-order ray-theory amplitudes in comparison with the correct amplitudes, similarly as in the vicinity of caustics. The resulting increase of the coupling-ray-theory amplitudes in comparison with the correct amplitudes in velocity model SC1.II may hopefully be corrected by the summation of the coupling-ray-theory Gaussian beams calculated along the SH and SV reference rays, which remains a challenge for our future research.

5. DISCUSSION AND CONCLUSIONS

When we apply the prevailing-frequency approximation of the coupling ray theory to the SH and SV reference rays in an approximately uniaxial (approximately transversely isotropic) anisotropic medium, we do not know a priori whether the proposed selection of arrivals is possible and successful.

If the proposed selection of arrivals is successful, we may expect one of the two coupling-ray-theory arrivals, obtained along each ray, to be very accurate while the other is less accurate than for a common reference ray. It has to be proved

Fig. 16. Two-point SV reference rays in velocity model SC1-II, together with a wavefront triangulated according to the ray tubes. The black lines are the edges of the computational volume for calculating Figs 14 and 15. The surface (zero depth) corresponds to the top of the computational volume for calculating Figs 14 and 15. The receivers are denoted by small red spheres. Only the receivers below the surface correspond to the receivers in the synthetic seismograms displayed in Figs 14 and 15. On the wavefront, we observe a belt of small geometrical spreading which intersects the receiver profile between the third and fourth receivers below the surface. Note that the wavefront looks like S curved, but this is just an optical illusion caused both by the curved sides of the displayed wavefront section and by the varying triangular cross-sections of the ray tubes, see Fig. 17.

Fig. 17. A side view of the wavefront section displayed in Fig. 16. We see that the wavefront is convex with just a moderate variation of its curvature.

Fig. 18. A detail of Fig. 16, displaying the belt of small geometrical spreading which intersects the receiver profile between the third and fourth receivers below the surface. This belt probably causes the increase of the zero-order ray-theory amplitudes in comparison with the correct amplitudes, similarly as in the vicinity of caustics.

theoretically that the combination of the selected accurate arrivals forms a tensor Green function within the high-frequency limit.

The numerical tests performed in approximately uniaxial elastic velocity models QI2, QI4, SC1.I and SC1.II demonstrated that the selection of arrivals was possible, and that the prevailing-frequency approximation of the coupling ray theory along the SH and SV reference rays provided a considerable improvement of the coupling-ray-theory synthetic seismograms in these approximately uniaxial (approximately transversely isotropic) velocity models. However, we had to apply the selection according to Section 3.2 in velocity models QI2 and QI4 which display no split intersection singularity, and the selection according to Section 3.1 in velocity models SC1.I and SC1.II which display split intersection singularities. The selection of arrivals thus remains an important topic for future research.

The proposed prevailing-frequency approximation of the coupling ray theory along the SH and SV reference rays has to be compared further with the exact solution and with the coupling ray theory along the common reference rays, especially in the vicinities of the kiss and intersection singularities.

Fixing the remaining inaccuracy of the zero-order ray-theory amplitudes in velocity model SC1.II using the summation of the coupling-ray-theory Gaussian

beams calculated along the SH and SV reference rays remains a challenge for our future research.

It is thus obvious that the SH and SV reference rays according to Klimeš and Bulant (2015, 2017) may represent very accurate reference rays for the coupling ray theory in approximately uniaxial (approximately transversely isotropic) anisotropic media defined by Klimeš (2015, 2016a).

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