Erratum to: Wireless three-hop networks with stealing II: exact solutions through boundary value problems

Fabrice Guillemin · Charles Knessl ·
Johan S. H. van Leeuwen

Published online: 28 August 2014
© Springer Science+Business Media New York 2014

Erratum to: Queueing Syst (2013) 74:235–272 DOI 10.1007/s11134-012-9332-8

Contrary to what we claimed in [5], the solution to the Riemann–Hilbert problem (4) considered in [5] for some domain D_x is in general not the restriction to D_x of the solution to the modified Riemann–Hilbert problem (6) in [5]. This occurs only when D_x is a circle, which is not the case considered in that paper. To solve problem (4), we have to consider the conformal mapping $\gamma_x(x)$ from D_x onto the unit disk D . We are thus led to consider a modified version of the Riemann–Hilbert problem (5) formulated in that paper.

The online version of the original article can be found under doi:[10.1007/s11134-012-9332-8](https://doi.org/10.1007/s11134-012-9332-8).

F. Guillemin
Orange Labs, 2 Avenue Pierre Marzin, 22300 Lannion, France
e-mail: Fabrice.Guillemin@orange-ftgroup.com

C. Knessl
Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago,
815 South Morgan Street, Chicago, IL 60607-7045, USA
e-mail: knessl@uic.edu

J. S. H. van Leeuwen (✉)
Department of Mathematics and Computer Science, Eindhoven University of Technology,
Room HG 9.13, P.O. Box 513, 5600 MB Eindhoven, The Netherlands
e-mail: j.s.h.v.leeuwen@tue.nl

1 Modified boundary value problem

To compute the function $P(x, 0)$, by using arguments similar to those in [5], we consider the function $P_x(x)$ given by Eq. (20) in [5], namely

$$P_x(x) = P(x, 0) - \frac{R_x}{x - x^*} + P(0, 0).$$

The function $P_x(x)$ is analytic in D_x and satisfies for $x \in \partial D_x$ and $y = Y^*(x)$

$$\Re \left(i \frac{h_2(x, y)}{h_3(x, y)} P_x(x) \right) = \Im \left(\frac{R_x h_2(x, y)}{h_3(x, y)(x - x^*)} \right). \quad (1)$$

To solve this problem, we consider the conformal mapping $\gamma_x(x)$ from D_x onto the unit disk. This conformal mapping can be chosen to preserve the symmetry with respect to the horizontal axis and to satisfy $\gamma_x(X^+(y_1)) = -1$ and $\gamma_x(X^+(y_2)) = 1$. Moreover, by imposing the condition $\gamma_x(0) = 0$, the conformal mapping $\gamma_x(x)$ is unique.

We are then led to consider the following problem on the unit circle: The function $P_x(c_x(u))$ is analytic in the unit disk D and satisfies for u on the unit circle C

$$\Re \left(i \frac{h_2(c_x(u), Y^*(c_x(u)))}{h_3(c_x(u), Y^*(c_x(u)))} P_x(c_x(u)) \right) = \Im \left(\frac{R_x h_2(c_x(u), Y^*(c_x(u)))}{h_3(c_x(u), Y^*(c_x(u)))(c_x(u) - x^*)} \right),$$

where $c_x(u)$ is the inverse of $\gamma_x(x)$ and maps the unit disk onto the domain D_x .

The above Riemann–Hilbert problem is of the following form: Find a function which is analytic in D , continuous in the closure $\overline{D}$ of the unit disk, and satisfies

$$\Re(ia(u)f(u)) = c(u).$$

Since, by construction, the function $f(u)$ is real on the real axis, it is possible to define by the reflection principle [3] the function $g(u) = \overline{f(1/\overline{u})}$, which is analytic in $\mathbb{C} \setminus \overline{D}$. The function $f(u)$ (resp. $g(u)$) is the restriction to D (resp. $\mathbb{C} \setminus \overline{D}$) of the sectionally analytic function $F(u)$, which satisfies the following Riemann–Hilbert problem: Find a sectionally analytic function $F(u)$ with respect to the unit circle, bounded at infinity ($F(\infty) = f(0)$) and such that for $u \in C$

$$a(u)F^i(u) - \overline{a(u)}F^e(u) = -2ic(u),$$

where $F^i(u)$ (resp. $F^e(u)$) is the interior (resp. exterior) limit of the function $F(u)$ on the unit circle.

The solution to this problem, when it exists, is given by [2]

$$F(u) = \frac{\phi^i(u)}{2i\pi} \int_C \frac{C(z)}{\phi^{(i)}(z)(z - u)} dz + Q(u)\phi(u),$$

where $Q(u)$ is a polynomial, which can be determined by using the conditions at infinity, the function $C(u)$ is given by

$$C(u) = -2i \frac{c(u)}{a(u)},$$

and the function $\phi(u)$ is defined by

$$\phi(u) = \begin{cases} \exp\left(\frac{1}{2i\pi} \int_C \log\left(z^{-\kappa} \frac{\overline{a(z)}}{a(z)}\right) \frac{dz}{z-u}\right), & u \in D, \\ \frac{1}{u^\kappa} \exp\left(\frac{1}{2i\pi} \int_C \log\left(z^{-\kappa} \frac{\overline{a(z)}}{a(z)}\right) \frac{dz}{z-u}\right), & u \in \mathbb{C} \setminus \overline{D}, \end{cases}$$

with κ denoting the index of the Riemann–Hilbert problem and $\phi^{(i)}(u)$ being the interior limit of the function $\phi(u)$ on the unit circle. When $\kappa < 0$, the solution to the Riemann–Hilbert problem exists and is unique if and only if for $k = 0, \dots, |\kappa| - 1$

$$\int_C \frac{z^k C(z)}{\phi^{(i)}(z)} dz = 0;$$

in that case, the polynomial $P(u) \equiv 0$. When $\kappa = 0$, the solution is unique and $P(u)$ is a constant.

2 Conformal map

We determine in this section the conformal mapping $\gamma_x(x)$ from the domain D_x onto the unit disk.

Let $w_x(x)$ be the conformal gluing function for the shift transformation $t_x(x)$ and the contour ∂D_x , where

$$t_x(x) \stackrel{\text{def}}{=} \bar{x} = \frac{Y^*(x)}{(1-p)x}.$$

The function $w_x(x)$ is solution to the Carleman problem

$$w_x(t_x(x)) = w_x(x)$$

for $x \in \partial D_x$. As proved in [8], the function $w_x(x)$ conformally transforms the domain D_x onto the complex plane deprived of the positively oriented arc L_x joining the points $w_x(X^+(y_2))$ and $w_x(X^+(y_1))$.

For the case under consideration, the conformal gluing function $w_x(t)$ is given in [1] and [7, Remark 16] by

$$w_x(t) = \frac{t}{(t - x_2) \left(t - \sqrt{\frac{1+p}{(1-p)^2 x_2}} \right)^2}.$$

It is worth noting that $w_x(x)$ is real for $x \in \partial D_x$ and that

$$w_x(X^+(y_1)) < w_x(X^+(y_2)).$$

Using the fact that for reals $a < b$ the function $v(a, b; x)$ defined by

$$v(a, b; x) = \frac{\sqrt{\frac{x-a}{x-b}} - 1}{\sqrt{\frac{x-a}{x-b}} + 1}$$

conformally maps the complex plane deprived of the segment $[a, b]$ onto the unit disk, the conformal mapping $\gamma_x(x)$ is given by

$$\gamma_x(x) = v(w_x(X^+(y_1)), w_x(X^+(y_2)); w_x(x)).$$

The function $w_x(x)$ takes value in $[w_x(X^+(y_1)), w_x(X^+(y_2))]$ for $x \in [x_3, x_4]$. It can then be shown that the equation with degree three $w_x(t) = w_x(x)$ for $x \in [x_3, x_4]$ has three solutions, namely x , $Y^*(x)/((1-p)x)$ and $Y_*(x)/((1-p)x)$. Since $Y^\pm([x_3, x_4]) = \partial D_x$, we deduce that the conformal mapping $\gamma_x(x)$ can be analytically continued in the whole of $\mathbb{C} \setminus [x_3, x_4]$ with an algebraic singularity at point x_3 .

3 Solution of the corrected boundary value problem for $P(x, 0)$

By using the above results, Eq. (34) in [5] should then read for $x \in D_x$ (when $p \leq p^*$, the index $\kappa_x = 0$)

$$\varphi_x^{(1)}(x) = \exp \left(\frac{1}{2i\pi} \int_{\partial D_x} \frac{\log(A_x(z))}{z-x} k_x(x, z) dz \right), \quad (2)$$

where the kernel $k_x(x, z)$ is given by

$$k_x(x, z) = \frac{(z-x)\gamma'_x(z)}{\gamma_x(z) - \gamma_x(x)}.$$

The kernel $k_x(x, z)$ is defined for $x \in D_x$ and when $x = z \in D_x$, $k_x(x, z) = 1$.

The function $\varphi_x^{(1)}(x)$ can be continued by setting

$$\varphi_x^{(1)}(x) = \begin{cases} \exp\left(\frac{1}{2i\pi} \int_{\partial D_x} \frac{\log(A_x(z))}{z-x} k_x(x, z) dz\right), & \text{for } x \in D_x, \\ A_x(x) \exp\left(\frac{1}{2i\pi} \int_{\partial D_x} \frac{\log(A_x(z))}{z-x} k_x(x, z) dz\right), & \text{for } x \in \overline{D_x}^c \cap V_x \cap D(0, x_3), \end{cases} \quad (3)$$

where V_x is the domain of analyticity of the function $\log(A_x(x))$.

Similarly, Eq. (37) in [5] should read

$$\varphi_x^{(2)}(x) = \begin{cases} \exp\left(\frac{1}{2i\pi} \int_{C_x} \frac{\log(\mathcal{A}_x(z))}{z-x} k_x(x, z) dz\right), & \text{for } x \in \overline{D_x}, \\ \mathcal{A}_x(x) \exp\left(\frac{1}{2i\pi} \int_{\partial D_x} \frac{\log(\mathcal{A}_x(z))}{z-x} k_x(x, z) dz\right), & \text{for } x \in \overline{D_x}^c \cap \mathcal{V}_x \cap D(0, x_3). \end{cases} \quad (4)$$

Eq. (35) is erroneous.

The function $P(x, 0)$ is eventually given by

$$P(x, 0) = \frac{R_x}{x - x^*} + B_x(x) + \frac{\varphi_x(x)}{2i\pi} \int_{\partial D_x} \frac{B_x(z) - B_x(x)}{\phi_x^i(z)(z-x)} k_x(x, y) dz - P(0, 0) + K_x \varphi_x(x), \quad (5)$$

where

$$\varphi_x(x) = \begin{cases} \varphi_x^{(1)}(x), & \text{when } p \leq p^*, \\ \varphi_x^{(2)}(x), & \text{when } p > p^*, \end{cases} \quad (6)$$

with $\varphi_x^{(1)}(x)$ as in (3) and $\varphi_x^{(2)}(x)$ as in (4), K_x is some constant, and the function $B_x(x)$ defined by Eq. (31) in [5].

4 Solution of the corrected boundary value problem for $P(0, y)$

For determining the function $P(0, y)$, the conformal mapping $\gamma_y(y)$ from the domain D_y onto the unit disk is required, which is given by (see [1, 10])

$$\gamma_y(y) = \frac{y k_y \left(\frac{\eta}{1+p} \right) - (1+p) \eta k_y \left(\frac{y}{1+p} \right)}{y k_y \left(\frac{\eta}{1+p} \right) + (1+p) \eta k_y \left(\frac{y}{1+p} \right)},$$

where

$$k_y(y) = (\alpha - y) \sqrt{\frac{1-p}{1+p} - (1-p)^2 \alpha^2 y}$$

with $\alpha = Y^*(x_2)/(1+p)$ and η is any constant in $(0, \alpha)$. In the following, we take $\eta = \alpha/2$. By using similar arguments as those for the conformal mapping $\gamma_x(x)$, we can prove that the conformal mapping $\gamma_y(y)$ is analytic in $D(0, y_3)$ with continuous limits on the boundary.

Equation (51) in [5] should read

$$\phi_y(y) = \exp \left(\frac{1}{2i\pi} \int_{\partial D_y} \frac{\log(A_y(z))}{z-y} k_y(y, z) dz \right), \quad (7)$$

where the kernel $k_y(y, z)$ is defined by

$$k_y(y, z) = \frac{(z-y)\gamma'_y(z)}{\gamma_y(z) - \gamma_y(y)}.$$

The interior limit of $\phi_y(y)$ on the contour ∂D_y is defined by

$$\phi_y^{(i)}(y) = \exp \left(\frac{1}{2} \log(A_y(y)) + \frac{1}{2i\pi} \oint_{\partial D_y} \frac{\log(A_y(z))}{z-y} k_y(y, z) dz \right). \quad (8)$$

The restriction to D_y of the function $\phi_y(y)$ coincides with the restriction to D_y of the function $\varphi_y(y)$ defined by

$$\varphi_y(y) = \begin{cases} \phi_y(y), & \text{if } y \in D_y, \\ A_y(y)\phi_y(y), & \text{if } y \notin D_y, \end{cases} \quad (9)$$

which is a meromorphic function in $\mathbb{C} \setminus [y_3, \infty)$ with a pole at the point y^{**} .

The function $P(0, y)$ is given in $\mathbb{C} \setminus [y_3, y_4]$ by

$$P(0, y) = \frac{R_y}{y - y^{**}} - P(0, 0) + B_y(y) + \frac{\varphi_y(y)}{2i\pi} \int_{\partial D_y} \frac{B_y(z) - B_y(y)}{\phi_y^i(z)(z-y)} k_y(z, y) dz + K_y \varphi_y(y), \quad (10)$$

where $B_y(y)$ is given by Eq. (50) in [5] and K_y is a constant as in Eq. (56) in [5].

5 Asymptotic analysis

In spite of these modifications, the asymptotic analysis given in Sect. 6 of [5] is correct; the factors κ_1 and κ_2 should be computed by using the new formulas giving $P(0, y^*)$ and $P(0, 0)$ depending on the conformal mapping $\gamma_y(y)$.

The same flaw occurs in [6] but the asymptotic analysis is also correct.

References

1. Blanc, J.P.C.: The relaxation time of two queues systems in series. *Commun. Statist. Stochastic Models* **1**, 1–16 (1985)
2. Dautray, R., Lions, J.L.: *Analyse Mathématique et Calcul Numérique pour les Sciences et les Techniques*. Masson, Paris (1995)
3. Dieudonne, J.: *Calcul Infinitésimal*. Hermann, Paris (1980)
4. Fayolle, G., Iasnogorodski, R., Malyshev, V.: *Random Walks in the Quarter Plane*. Springer-Verlag, New York (1999)
5. Guillemin, F., van Leeuwaarden, J.S.H.: Rare event asymptotic for a random walk in the quarter plane. *Queueing Syst.* **67**, 1–32 (2011)
6. Guillemin, F., Knessl, C., van Leeuwaarden, J.S.H.: Wireless three-hop networks with stealing II: exact solutions through boundary value problems. *Queueing Syst.* **74**, 235–272 (2013)
7. Kurkova, I., Raschel, K.: Random walks in $\mathbb{Z}_+^2$ with non-zero drift absorbed at the axes. *Bull. Soc. Math. France* **139**, 341–387 (2011)
8. Litvinchuk, G.: *Solvability of Boundary Value Problems and Singular Integral Equations with Shift*. Springer, Dordrecht (2000)
9. Muschelischwili, N.I.: *Singuläre Integralgleichungen*. Akademie Verlag, Berlin (1965)
10. van Leeuwaarden, J. S. H., Resing, J. A. C.: A tandem queue with coupled processors: computational issues. *Queueing Syst.* **51**(1–2), 29–52 (2005)