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# Rare event asymptotics for a random walk in the quarter plane

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**Abstract** This paper presents a novel technique for deriving asymptotic expressions for the occurrence of rare events for a random walk in the quarter plane. In particular, we study a tandem queue with Poisson arrivals, exponential service times and coupled processors. The service rate for one queue is only a fraction of the global service rate when the other queue is non-empty; when one queue is empty, the other queue has full service rate. The bivariate generating function of the queue lengths gives rise to a functional equation. In order to derive asymptotic expressions for large queue lengths, we combine the kernel method for functional equations with boundary value problems and singularity analysis.

**Keywords** Boundary value problems · Random walks in the quarter plane · Rare events · Queueing theory · Singularity analysis · Tail decay rate · Large deviations

**Mathematics Subject Classification (2000)** 60K25 · 60F10

## 1 Introduction

Stationary distributions of two-dimensional one-step random walks in the quarter plane can be obtained by solving functional equations. Malyshev pioneered this general problem in the 1970's, and the theory has advanced since via its use in applications like lattice path counting and two-server queueing models. The idea of re-

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ducing the functional equation for the generating function to a standard Riemann–Hilbert boundary value problem stems from the work of Fayolle and Iasnogorodski [10] on two parallel $M/M/1$ queues with *coupled processors* (the service speed of a server depends on whether or not the other server is busy). Extensive treatments of the boundary value technique for functional equations can be found in Cohen and Boxma [5] and Fayolle, Iasnogorodski and Malyshev [11]. This technique concerns sophisticated complex analysis, Riemann surfaces and various boundary value problems.

This paper presents a novel technique for deriving asymptotic estimates for the occurrence of certain types of rare events in a random walk in the quarter plane. We believe that our technique might prove useful in the analysis of a wider class of random walks, but in this paper we concentrate on a tandem queue with Poisson arrivals, exponential service times and coupled processors. Denote by $N_1$ and $N_2$ the stationary number of customers in the first and second queue. The generating function $P(x,y)=\mathbb{E}(x^{N_1}y^{N_2})$ then satisfies the functional equation

$$h_1(x,y)P(x,y)=h_2(x,y)P(x,0)+h_3(x,y)P(0,y)+h_4(x,y)P(0,0), \qquad (1)$$

where the functions $h_j$ are quadratic polynomials in $x$ and $y$. Equation (1) cannot be solved directly for $P(x,y)$, because it contains other unknown functions $P(x,0)$ and $P(0,y)$. The classical approach is then to consider the roots of the kernel $h_1(x,y)$ w.r.t. one of the variables $x$, $y$. Substituting such roots into (1) yields additional relations between the unknown functions $P(x,0)$ and $P(0,y)$. These relations give rise to boundary value problems whose solutions lead to a specification of $P(x,0)$ and $P(0,y)$ and hence $P(x,y)$. For the tandem queue with coupled processors this was done in [28, 32]. The formal solution obtained, however, is too complicated to invert for the stationary distribution. We shall look at this inversion problem, and derive asymptotic expressions for the marginal distributions.

To analyze $\mathbb{P}(N_1=n)$, for large $n$, we need to extract information from the generating function $P(x,1)=\sum_{n=0}^{\infty}\mathbb{P}(N_1=n)x^n$. We shall employ the functional equation to determine the dominant (closest to the origin) singularities of the functions $P(x,0)$ and $P(x,1)$. Let $\xi$ denote the dominant singularity of $P(x,1)$. Large deviations theory typically focusses on *rough* tail asymptotics of the form (the pole $\xi$ is positive)

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(N_1=n)=-\log\xi.$$

We shall derive the *exact* tail asymptotics, which requires the investigation of $P(x,1)$ in the neighborhood of its dominant singularity $\xi$.

## 1.1 Singularity analysis

In [28, 32] solutions for $P(x,0)$ and $P(0,y)$ were derived that are valid only in certain parts of the complex planes. In this paper we provide complete solutions to $P(x,0)$ and $P(0,y)$ that are in fact the analytic continuations to the entire complex planes of the solutions in [28, 32]. The technique of investigating a function near its dominant singularity to obtain asymptotic expressions for its coefficients is known as *singularity analysis* and has a long history in areas of mathematics like analysis, combinatorics and number theory; for an elaborate exposition see Flajolet and Sedgewick [12]. In most cases the generating function is univariate and explicit, and extracting information on the coefficients boils down to the (asymptotic) evaluation of univariate contour integrals. The work on obtaining asymptotics from multivariate generating functions has been strongly motivated by recursively defined combinatorial structures like trees, see e.g. [9, 12, 13], and specific random walks or queueing models [4, 14, 15, 19]. One of the central ideas in multivariate asymptotics is to exploit a functional equation to reduce multivariate problems to univariate contour integrals. In contrast to most functional equations that are subject to multivariate asymptotics (see [27] for an overview), our functional equation (1) does not allow for a closed-form solution, which complicates considerably the application of singularity analysis. Our method can be considered as a contribution to the technique of singularity analysis for bivariate generating functions.

Related work was done in [19] for two parallel $M/M/1$ queues with coupled processors, also leading to rare event probabilities. However, this latter model can be reduced to a Dirichlet problem (the boundary value problem has a boundary which is a circle, and the problem is solved by using the Poisson kernel; see [10, 19]). In the present paper, the boundary is a general smooth closed contour and we use a Riemann–Hilbert formulation, which allows us to directly extend the function outside the domain delineated by the boundary. In this respect, the problem considered in the present paper is more general than the one considered in [19], and the approach that we take might prove useful for many models that fall into the class of random walks in the quarter plane.

## 1.2 Alternative methods

There are at least two alternative techniques to derive tail asymptotics for random walks in the quarter plane, perhaps the one with the longest tradition being *large deviations theory*. Seminal work in this area was done by Borovkov and Mogul'skii [2]. For so-called 0-partially homogeneous chains, of which our random walk in the quarter plane is a special case, in [2] (see also [21]) both the rough and the exact asymptotics have been considered. However, the decay rate is not made explicit, and it is not clear how to obtain results for marginal distributions. For determining the rough decay rate one needs to identify the local rate function and solve a variational problem. This variational problem, under additional assumptions, can be reduced to an optimization problem (see e.g. [30]), but requires a case-specific approach.

For the large deviations approach, it is necessary to incorporate the boundary effects, which requires finding an optimal path that minimizes a cost function. For many models, this gives rise to multiple regimes, where each regime corresponds to a certain *most likely path*. For a modified Jackson network, in which one server may help the other server, Foley and McDonald [17] were able to find most of these regimes using the large deviations approach. See also [1, 16, 18].

The second alternative techniques is the matrix-analytic method. Initially, the matrix-geometric method targeted at deriving the so-called *boundary condition*, under which the asymptotics show geometric behavior; see [20, 25, 31]. This boundary condition plays a crucial role in the large deviations approach too, and is naturally the subject of much recent work [20, 22, 23, 25, 29]. Geometric decay requires the dominant singularity to be a pole, whereas it could be a singularity of a different nature like a branch point. The recent work of Miyazawa [24] greatly enlarges the scope of applicability of the matrix-analytic techniques, because it is no longer restricted to the boundary condition. For the general class of skip-free random walks in the quarter plane, Miyazawa characterizes both the rough and exact asymptotics of the tail decay rates, for coordinate directions and marginal distributions. Among other things, it is shown in [24] that the matrix-analytic methods can be used to determine all asymptotic regimes for the modified Jackson network in [17].

### 1.3 Contributions and outline of the paper

The tandem queue with coupled processors, which we chose as our vehicle to present the asymptotic technique, is of independent interest. It arises as a natural model for bandwidth sharing of Internet capacity that is based on reservation procedures (see [7, 28, 32]). The two processors are coupled such that the speed of processor $i$ is $\mu_i$ when the other processor is busy, and $\mu_i^*$ when the other processor is idle. This coupling became extremely popular in the last decade due to its relation to the Generalized Processor Sharing (GPS) discipline ($\mu_1^* = \mu_2^* = \mu_1 + \mu_2$), the prevalent discipline for bandwidth sharing in packet networks. See [3] for an overview on GPS. The different asymptotic regimes identified in this paper yield structural insights on the impact of GPS on rare events in a tandem queue.

In the present paper, we make the following contributions:

- We provide in Propositions 3 and 4 exact solutions to $P(x, 0)$ and $P(0, y)$, in terms of meromorphic functions, which are valid in the entire complex $x$ and $y$ planes cut along some segments. The solutions follow from analytic continuations through the functional equation (1).
- We determine the domain of analyticity of the functions $P(x, 1)$ and $P(1, y)$. A crucial role is fulfilled by the resultant of the functions $h_1$ and $h_2$. The domains of analyticity lead to exact asymptotic expressions for $\mathbb{P}(N_1 = n)$ and $\mathbb{P}(N_2 = n)$.
- The parameter values determine the nature of the dominant singularities of $P(x, 1)$ and $P(1, y)$. This gives rise to several different asymptotic regimes. Asymptotic estimates for the probabilities of large queue lengths are obtained using Laplace's method and Darboux's method. Proposition 5 distinguishes four different regimes for queue 1, and Proposition 6 shows that there are three different regimes for queue 2.

Section 2 contains the model description and an extensive analysis of the zero-pairs of the kernel $h_1$ in (1). In particular, various analytic continuations of these zero-pairs are constructed, which identify some of the singularities of the function $P(x, 0)$ and $P(0, y)$. Further singularities are identified in Sect. 3 by considering the resultant of $h_1$ and $h_2$. In Sect. 4 we formulate $P(x, 0)$ and $P(0, y)$ in terms of boundary value problems. The solutions to these boundary value problems yield solutions to $P(x, 0)$ and $P(0, y)$ in terms of meromorphic functions, with a clear description of their singularities. In Sect. 5 this knowledge is used to obtain a complete characterization of the exact asymptotics for the marginal distributions $\mathbb{P}(N_1 = n)$ and $\mathbb{P}(N_2 = n)$.

## 2 Model description and preliminary properties

Consider a two-stage tandem queue, where jobs arrive at queue 1 according to a Poisson process with rate $\lambda$, demanding service at both queues before leaving the system. Each job requires an exponential amount of work with parameter $\nu_j$ at queue $j$. The global service rate is set to one. The service rate for one queue is only a fraction ($p$ for queue 1 and $1-p$ for queue 2) of the global service rate when the other queue is non-empty; when one queue is empty, the other queue has full service rate. Therefore, when both queues are non-empty, the departure rates at queue 1 and 2 are $\nu_1 p$ and $\nu_2(1-p)$, respectively.

When one of the queues is empty, the departure rate of the nonempty queue $j$ is temporarily increased to $\nu_j$. With $N_j(t)$ the number of jobs in queue $j$ at time $t$, the two-dimensional process $\{(N_1(t), N_2(t)), t \geq 0\}$ is a Markov process, and upon uniformization, a random walk in the quarter plane.

The ergodicity condition under which this Markov process has a unique stationary distribution is given by

$$\rho = \frac{\lambda}{\nu_1} + \frac{\lambda}{\nu_2} < 1. \tag{2}$$

This can be explained by the fact that, independent of $p$, the two stations together always work at capacity 1 (if there is work in the system), and that $\lambda/\nu_1 + \lambda/\nu_2$ equals the amount of work brought into the system per time unit. We henceforth assume that the ergodicity condition is satisfied.

Denote the joint stationary probabilities by

$$\mathbb{P}(N_1 = n, N_2 = k) = \lim_{t\to\infty} \mathbb{P}\big(N_1(t) = n, N_2(t) = k\big)$$

and let $P(x, y)$ represent the bivariate generating function

$$P(x, y) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \mathbb{P}(N_1 = n, N_2 = k)x^n y^k.$$

From the balance equations it follows (see [28]) that $P(x, y)$ satisfies the functional equation (1) with

$$\begin{aligned}
h_1(x, y) &= \big(\lambda + p\nu_1 + (1-p)\nu_2\big)xy - \lambda x^2 y - p\nu_1 y^2 - (1-p)\nu_2 x,\\
h_2(x, y) &= (1-p)\big(\nu_1 y(y-x) + \nu_2 x(y-1)\big),\\
h_3(x, y) &= -\frac{p}{1-p} h_2(x, y),\\
h_4(x, y) &= \nu_2 x(y-1) - h_2(x, y).
\end{aligned}$$

We have $P(0, 0) = 1 - \rho$ by (2).

### 2.1 Zero-pairs of the kernel

Let $\hat{r} = 1 + 1/r_1 + 1/r_2$ with $r_1 = \lambda/(p\nu_1)$ and $r_2 = \lambda/((1-p)\nu_2)$. With this notation, the equation $h_1(x, y) = 0$ in $y$ has two roots:

$$X_\pm(y) = \frac{1}{2y}\big((\hat{r}y - 1/r_2) \pm \sqrt{D_2(y)}\big),$$

where

$$D_2(y) = (\hat{r}y - 1/r_2)^2 - 4y^3/r_1. \tag{3}$$

The functions $X_\pm(y)$ are well defined for $y \in \mathbb{R} \setminus \{0\}$ as long as $D_2(y) \geq 0$. It is easily checked that $\lim_{y\to 0} X_+(y) = 0$ (the point 0 is a removable singularity for the function $X_+(y)$) and $\lim_{y\to 0} X_-(y) = -\infty$ (the point 0 is a singularity for the function $X_-(y)$). In addition, as shown in [32], the discriminant $D_2(y)$ has three roots in $\mathbb{R}$. These three roots are denoted by $y_1$, $y_2$ and $y_3$ and are such that $0 < y_1 < y_2 \leq 1 < y_3$. We have $D_2(y) > 0$ for $y \in (-\infty, y_1) \cup (y_2, y_3)$ and $D_2(y) < 0$ for $y \in (y_1, y_2) \cup (y_3, \infty)$.

Similarly, the equation $h_1(x, y) = 0$ in $x$ has two roots:

$$Y_\pm(x) = \frac{r_1}{2}\big((\hat{r} - x)x \pm \sqrt{D_1(x)}\big),$$

where

$$D_1(x) = \big((\hat{r} - x)x\big)^2 - 4x/(r_1 r_2). \tag{4}$$

The functions $Y_\pm(x)$ are well defined for $x \in \mathbb{R}$ as long as the discriminant $D_1(x) \geq 0$. As shown in [32], the discriminant $D_1(x)$ has four real roots $x_1 = 0 < x_2 \leq 1 < x_3 < x_4$. We have $D_1(x) > 0$ for $x \in (-\infty, x_1) \cup (x_2, x_3) \cup (x_4, \infty)$ and $D_1(x) < 0$ for $x \in (x_1, x_2) \cup (x_3, x_4)$.

In the next section we investigate how to analytically continue the functions $Y_\pm(x)$ in $\mathbb{C} \setminus ([x_1, x_2] \cup [x_3, x_4])$ and $X_\pm(y)$ in $\mathbb{C} \setminus ([y_1, y_2] \cup [y_3, \infty))$.

### 2.2 Analytic continuation

In the following, we assume that for $z \in \mathbb{C}$, $\arg(z) \in (-\pi, \pi]$, and we take the determination of the square root such that $\sqrt{x^2} = x$ if $x \geq 0$ and $\sqrt{-1} = i$. The couple $(X_+(y), (-\infty, y_1))$ defines a germ of analytic function. We first investigate how this germ can be analytically continued in the complex plane deprived of the segments $[y_1, y_2]$ and $[y_3, \infty)$. Let $z^+ = \Re(z) + i|\Im(z)|$.

**Lemma 1** *The function*

$$X^*(y) = \begin{cases} X_+(y) & \text{when } y \in \{z : \Re(z) \leq y_2, \Im(D_2(z^+)) < 0\} \cup (-\infty, y_1), \\ X_-(y) & \text{otherwise}, \end{cases} \tag{5}$$

*defined in* $\mathbb{C} \setminus ([y_1, y_2] \cup [y_3, \infty))$, *is analytic.*

*Proof* Let $y = u + iv$ with $u, v \in \mathbb{R}$. We have $D_2(y) = \Re(D_2(y)) + i\Im(D_2(y))$ with

$$\Re\big(D_2(y)\big) = \left(\hat{r}u - \frac{1}{r_2}\right)^2 - \hat{r}^2 v^2 - \frac{4}{r_1}\big(u^3 - 3uv^2\big),$$
$$\Im\big(D_2(y)\big) = v\left(\frac{4}{r_1}v^2 - \left(\frac{12}{r_1}u^2 - 2\hat{r}^2 u + \frac{2}{r_2}\right)\right).$$

The imaginary part vanishes for $u$ and $v$ satisfying the equation

$$\frac{4}{r_1}v^2 = \frac{12}{r_1}u^2 - 2\hat{r}^2 u + \frac{2}{r_2}. \tag{6}$$

For sufficiently large $u$, the right-hand side of the above equation is positive. If we assume that this term does not cancel for $u$ describing the whole of $\mathbb{R}$, then we can define two curves in $\mathbb{C}$ along which the imaginary part of $D_2(y)$ vanishes: one curve lies entirely in the positive half-plane $\{y : \Im(y) > 0\}$ and the other curve lies entirely in the negative half-plane $\{y : \Im(y) < 0\}$.

Along one of these curves, the sign of the real part $\Re(D_2(y))$ is constant since the imaginary and real parts cancel only for $y \in \mathbb{R}$ (namely for $y$ equal to one of the roots $y_1$, $y_2$ and $y_3$). For the curve in the upper half-plane we have $v^2 \sim 3u^2$ for $|u| \to +\infty$. But in this case, we would have $\Re(D_2(y)) \sim 32u^3/r_1$, which contradicts the fact that $\Re(D_2(y))$ should keep the same sign as $u$ describes the whole of $\mathbb{R}$. Hence, the polynomial in the right-hand side of (6) has roots in $\mathbb{R}$, which are positive since the value of this polynomial at 0 is $2/r_2 > 0$. Let $y_1^*$ and $y_2^*$ denote these roots with $0 < y_1^* \le y_2^*$.

Equation (6) defines two hyperbolic branches as depicted in Fig. 1. The left branch intersects the real axis at point $y_1^*$ and for a point $y$ on this branch such that $\Im(y) \neq 0$, $\Re(D_2(y)) < 0$. By continuity of the real part, which is a polynomial in $u$ and $v$, we have $\Re(D_2(y_1^*)) \le 0$ and hence $y_1 \le y_1^* \le y_2$. The right branch intersects the real axis at point $y_2^*$. For $y$ on this branch such that $\Im(y) \neq 0$, $\Re(D_2(y)) > 0$ and by continuity of the real part, we have $\Re(D_2(y_2^*)) \ge 0$, which implies that $y_2 \le y_2^* \le y_3$.

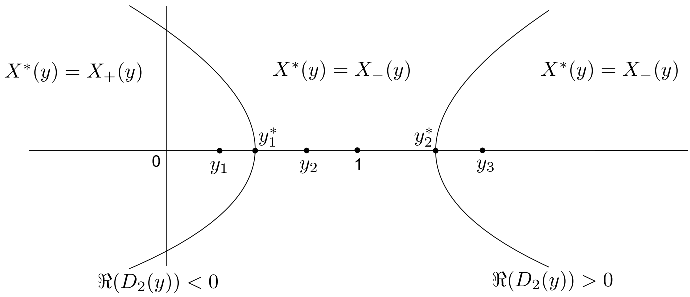



**Fig. 1** Branches on which $\Im(D_2(y)) = 0$

The function $X_+(y)$ is analytic in the domain $\{y : \Re(y) \leq y_1^*, \Im(D_2(y^+)) < 0\} \cup (-\infty, y_1)$. The function $X_-(y)$ is analytic in the complementary domain of the closure of this set in $\mathbb{C} \setminus ([y_1, y_2] \cup [y_3, \infty))$. To show that the function $X^*(y)$ is analytic in the whole of $\mathbb{C}$ deprived of the segments $[y_1, y_2]$ and $[y_3, \infty)$, by Moreira's theorem, it is sufficient to show that this function is continuous on the branch $\{y : \Im(D_2(y)) = 0,\ \Re(D_2(y)) \leq 0\}$ separating the two above domains. But this is straightforwardly checked from the choice of the determination of the square root. □

By using exactly the same arguments as in the proof of Lemma 1, we can prove the following result.

**Lemma 2** *The function*

$$X_*(y) = \begin{cases} X_-(y) & \text{when } y \in \{z : \Re(z) \leq y_2, \Im(D_2(z^+)) < 0\} \cup (-\infty, y_1), \\ X_+(y) & \text{otherwise}, \end{cases} \tag{7}$$

*defined in* $\mathbb{C} \setminus ([y_1, y_2] \cup [y_3, \infty))$, *is analytic.*

We now turn to the functions $Y_\pm(x)$. First note that $Y_\pm(0) = 0$. As shown in [32], when $x$ is close to the segment $[x_1, x_2]$, $Y_\pm(x)$ is close to a contour $\partial D_y$ in the $y$-plane included in the half-plane $\{y : \Re(y) \geq 0\}$. In particular, the point 0 lies in $\partial D_y$. In addition, when $y$ is close to the segment $[y_1, y_2]$, $X(y)$ is in the $x$-plane close to a contour $\partial D_x$ surrounding the point 0. The contours $\partial D_x$ and $\partial D_y$ delineate bounded open domains in the $x$-plane deprived of the segment $[x_1, x_2]$ and the $y$-plane deprived of the segment $[y_1, y_2]$ denoted by $D_x$ and $D_y$, respectively. Since our ultimate goal is to establish a conformal mapping between these two domains, and since $Y_\pm(-i\varepsilon) \sim \pm(\cos(\pi/4) + i\sin(\pi/4))\sqrt{\varepsilon/(r_1 r_2)}$ for small $\varepsilon > 0$, we are led to pick the function $Y_+(x)$ as a candidate for the desired conformal mapping because $Y_+(-i\varepsilon) \in D_y$ while $Y_-(-i\varepsilon) \notin D_y$ for sufficiently small $\varepsilon > 0$.

**Lemma 3** *The function*

$$Y^*(x) = \begin{cases} Y_+(x) & \text{when } x \in \{z : \Re(z) \leq x_2, \Im(D_1(z^+)) < 0\} \cup (-\infty, x_1), \\ Y_+(x) & \text{when } x \in \{z : \Re(z) \geq x_3, \Im(D_2(z^+)) > 0\} \cup (x_4, \infty), \\ Y_-(x) & \text{otherwise}, \end{cases} \tag{8}$$

*defined in* $\mathbb{C} \setminus ([x_1, x_2] \cup [x_3, x_4])$, *is analytic.*

*Proof* Let $x = u + iv$ with $u, v \in \mathbb{R}$. We have $D_1(x) = \Re(D_1(x)) + i\Im(D_1(x))$ with

$$\Re\big(D_1(x)\big) = \big((\hat{r} - u)u + v^2\big)^2 - v^2(\hat{r} - 2u)^2 - \frac{4u}{r_1 r_2},$$

$$\Im\big(D_1(x)\big) = 2v\left((\hat{r} - 2u)v^2 + u(\hat{r} - u)(\hat{r} - 2u) - \frac{2}{r_1 r_2}\right).$$

Note that $\Im(D_1(x)) = 0$ if $(u, v)$ satisfies

$$(2u - \hat{r})v^2 = u(2u - \hat{r})(u - \hat{r}) - \frac{2}{r_1 r_2}. \tag{9}$$

Let $d_1(u)$ be the polynomial in the right-hand side of the above equation. This polynomial is of degree 3 and has at least one real root ($u_1$ say). Since $\lim_{u\to+\infty} d_1(u) = +\infty$ and $d_1(\hat{r}) = -2/(r_1 r_2) < 0$, $u_1 > \hat{r}$. The polynomial $d_1(u)$ can then be decomposed as $d_1(u) = (u - u_1)d_{11}(u)$. If the polynomial $d_{11}(u)$ had no real roots, then this polynomial would be positive in the whole of $\mathbb{R}$ since $d_1(u)$ is positive for large $u$. When $u < \hat{r}/2$, (9) would have two roots, namely

$$v = \pm\sqrt{\frac{(u - u_1)d_{11}(u)}{2u - \hat{r}}}.$$

We would then obtain two curves, one in the half-plane $\{x : \Im(x) > 0\}$ and the other in the half-plane $\{x : \Im(x) < 0\}$. Along each of these curves, the sign of $\Re(D_1(x))$ should be constant (see the arguments in the proof of Lemma 1). But when $u \to -\infty$, $v^2 \sim u^2$ and then $\Re(D_1(x)) < 0$, and when $u \to \hat{r}/2$, $v^2 \sim -2/(r_1 r_2(2u - \hat{r}))$ and $\Re(D_1(x)) > 0$, which contradicts the fact that the sign of $\Re(D_1(x))$ should be constant along the curves $\Im(D_1(x)) = 0$. As a consequence, the polynomial $d_1(u)$ has three real roots. Let us denote these roots by $x_1^*, x_2^*$ and $x_3^*$ with $x_1^* \le x_2^* \le x_3^*$. Their product is equal to $1/(r_1 r_2)$ and since one of them is positive, the two others have the same sign.

We already know that $x_3^* > \hat{r}$. If $x_1^* \ge \hat{r}/2$, then (9) defines two curves for $u \le \hat{r}/2$, one is included in the upper half-plane and the other in the lower half-plane, which is not possible for the same reasons as above. Hence, $x_1^* \le \hat{r}/2$. This also implies that $x_2^* < \hat{r}/2$ since $d_1(\hat{r}/2) = -2/(r_1 r_2) < 0$. Hence, we have

$$0 \le x_1^* \le x_2^* < \hat{r}/2 < \hat{r} < x_3^*.$$

Let us consider the three curves defined by

$$v = \pm\sqrt{\frac{(u - x_1^*)(u - x_2^*)(u - x_3^*)}{2u - \hat{r}}} \quad \text{when } u \le x_1^* \quad \text{or } x_2^* \le u < \hat{r}/2 \quad \text{or } u \ge x_3^*.$$

See Fig. 2.

For the curve defined for $u \le x_1^*$ it is easily checked that $\Re(D_1(x)) < 0$ when $v \ne 0$ and by continuity we deduce that $\Re(D_1(x)) \le 0$. This implies that $x_1 \le x_1^* \le x_2$. Similar arguments show that $x_3 \le x_3^* \le x_4$. For the curve defined for $x_2 \le u < \hat{r}/2$, we have $\Re(D_1(x)) > 0$ when $v \ne 0$ and hence $\Re(D_1(x)) \ge 0$ all along the curve. This implies that $x_2 \le x_2^* \le x_3$. We finally have the ordering

$$x_1 \le x_1^* \le x_2 \le x_2^* < \hat{r}/2 < x_3 \le x_3^* \le x_4.$$

Note that it is easily checked that $x_3 > \hat{r}/2$. Indeed, if we assume that $x_3 \le \hat{r}/2 \le x_3^* \le x_4$, we would have $D_1(\hat{r}/2) \le 0$ and then $D_1(x)$ would be nonpositive for all

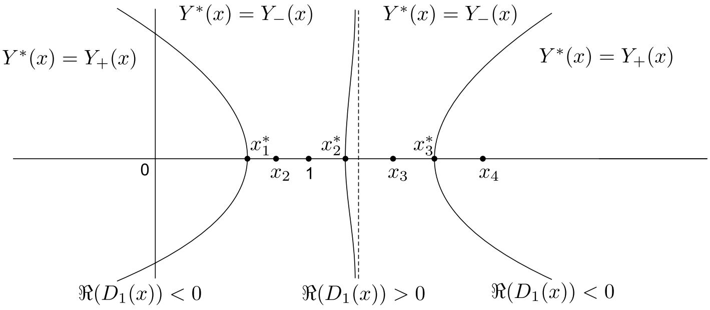



**Fig. 2** Branches on which $\Im(D_1(x)) = 0$

$x \geq \hat{r}/2$ since the term $(\hat{r} - x)x$ is maximal at the point $\hat{r}/2$; this is clearly not possible.

By invoking the same arguments as in the proof of Lemma 1, it is easily checked that the function $Y^*(x)$ defined by (8) is analytic in the complex plane deprived of the segments $[x_1, x_2]$ and $[x_3, x_4]$. □

By similar arguments, we can prove the following result.

**Lemma 4** *The function*

$$Y_*(x) = \begin{cases} Y_-(x) & \textit{when } x \in \{z : \Re(z) \leq x_2, \Im(D_1(z^+)) < 0\} \cup (-\infty, x_1), \\ Y_-(x) & \textit{when } x \in \{z : \Re(z) \geq x_3, \Im(D_2(z^+)) > 0\} \cup (x_4, \infty), \\ Y_+(x) & \textit{otherwise}, \end{cases} \tag{10}$$

*defined in* $\mathbb{C} \setminus ([x_1, x_2] \cup [x_3, x_4])$, *is analytic*.

To conclude this section, let us examine the images of the contours $\partial D_x$ and $\partial D_y$ by the analytic functions $Y^*$ and $X^*$, respectively. First note that for $x \in \mathbb{C} \setminus ([x_1, x_2] \cap [x_3, x_4])$, $X^*(Y^*(x)) = x$ and for $y \in \mathbb{C} \setminus ([y_1, y_2] \cap [y_3, \infty))$, $Y^*(X^*(y)) = y$. To prove the first equality, consider $x \in (-\infty, 0)$ sufficiently close to 0, so that $Y^*(x) = Y_+(x) \sim \sqrt{-r_1 x / r_2}$ and $X^*(Y^*(x)) = X_+(Y^*(x)) \sim x$. It follows that the equality $X^*(Y^*(x)) = x$ holds for a neighborhood of 0 and since the function $X^*(Y^*(x))$ is analytic in $\mathbb{C} \setminus ([x_1, x_2] \cap [x_3, x_4])$ this equality holds for the whole of $\mathbb{C} \setminus ([x_1, x_2] \cap [x_3, x_4])$. Similar arguments can be invoked to prove the second equality.

**Corollary 1** *We have* $X^*(\partial D_y) \subset [x_1, x_2]$ *and* $Y^*(\partial D_x) \subset [y_1, y_2]$.

*Proof* Consider $y \in \partial D_y$ (the case of $\partial D_x$ is similar). By construction, there exists an $x \in [x_1, x_2]$ such that

$$y = Y_+(x + 0i), \qquad \bar{y} = Y_+(x - 0i), \qquad y = Y_-(x - 0i), \qquad \bar{y} = Y_-(x + 0i).$$

Note that we use the notation $x + 0i$ (resp. $x - 0i$) to designate the limit of a sequence in the upper (resp. lower) half-plane converging to $x \in \mathbb{R}$. From the definition of

$Y^*(x)$, the determination of this function at the point $x \pm 0i$ is either $Y_+(x \pm 0i)$ or $Y_-(x \pm 0i)$. It follows that $y = Y^*(x + \varepsilon 0i)$ where $\varepsilon = \pm 1$, depending on the determination of $Y^*(x)$. It follows that $X^*(y) = X^*(Y^*(x + \varepsilon 0i)) = x \in [x_1, x_2]$. Hence, $X^*(\partial D_y) \subset [x_1, x_2]$. □

### 2.3 Conformal mappings

We are now able to establish the conformal mappings which will play a crucial role in the derivation of the boundary functions $P(0, y)$ and $P(x, 0)$.

**Proposition 1** *The function $X^*(y)$ is a conformal mapping from $D_y$ onto $D_x$. The reciprocal function is $Y^*(x)$.*

*Proof* As noted before, when $y$ is in $D_y$ and sufficiently close to 0, $X_+(y) \equiv X^*(y) \in D_x$. Since the set $D_y$ is an open and simply connected domain and since $X^*(y)$ is an analytic function, $X^*(D_y) \cap D_x$ is a non-null, open and simply connected domain included in $D_x$.

If $D_x$ is not a subset of $X^*(D_y)$, consider the complementary set $X^*(D_y)^c \cap D_x \neq \emptyset$ in $D_x$. Let $x$ be a point on the boundary between this set and $X^*(D_y) \cap D_x$. There exist a sequence $(x_n)$ in $X^*(D_y) \cap D_x$, and a sequence $(x'_n)$ in the interior of $X^*(D_y)^c \cap D_x$ both converging to $x$. Since $(x_n)$ is in $X^*(D_y) \cap D_x$, there exists a sequence $(y_n)$ in $D_y$ such that $X^*(y_n) = x_n$. Moreover, as we have $X^*(Y^*(x)) = x$ for all $x$ in the $x$-plane deprived of the segments $[x_1, x_2]$ and $[x_3, x_4]$, and $Y^*(X^*(y)) = y$ for all $y$ in the $y$-plane deprived of the segments $[y_1, y_2]$ and $[y_3, \infty)$, the sequences $(y_n)$ and $(Y^*(x_n))$ converge to the same point. But by definition the points $Y^*(x_n)$ lie outside the domain $D_y$. It follows that these two sequences converge to a point on $\partial D_y$. By Corollary 1, this implies that $x \in [x_1, x_2]$, which is not possible. It follows that $D_x \subset X^*(D_y)$.

If the above inclusion is strict, we consider a point $x$ on the boundary $\partial D_x$. There should exist a point $y$ in $D_y$ such that $X^*(y) = x$ but this is not possible since $y$ should be in $[y_1, y_2]$ since $Y^*(\partial D_x) \subset [y_1, y_2]$. It follows that $X^*(D_y) = D_x$. In addition, the function $X^*(y)$ is one to one since $Y^*(X^*(y)) = y$. It follows that this function is a conformal mapping from $D_y$ onto $D_x$ and the reciprocal function is $Y^*$. □

The conformal mappings $X^*$ and $Y^*$ between the domains $D_x \setminus [x_1, x_2]$ and $D_y \setminus [y_1, y_2]$ are illustrated in Fig. 3. While $X^*$ maps $D_y \setminus [y_1, y_2]$ onto $D_x \setminus [x_1, x_2]$, the set $X_*(D_y \setminus [y_1, y_2])$ is an open domain surrounding $D_x$ in the $x$-plane. Similarly, $Y_*(D_x \setminus [x_1, x_2])$ is an open domain surrounding $D_y$ in the $y$-plane.

It is worth noting that $X^*(\xi) \to x \in \partial D_x$ from inside $D_x$ when $\xi \to y \in [y_1, y_2]$. Similarly, $Y^*(\xi) \to y \in \partial D_y$ from inside $D_y$ when $\xi \to x \in [x_1, x_2]$. We also have $X_*(\xi) \to x \in \partial D_x$ from outside $D_x$ when $\xi \to y \in [y_1, y_2]$ and $Y_*(\xi) \to y \in \partial D_y$ from outside $D_y$ when $\xi \to x \in [x_1, x_2]$.

## 3 Intersection points of the curves $h_1(x, y) = 0$ and $h_2(x, y) = 0$

When $h_1(x, y) = 0$, we see from (1) that we can express $P(x, 0)$ (resp. $P(0, y)$) as a function of $P(0, y)$ (resp. $P(x, 0)$) and $h_4(x, y)$, where the function $h_2(x, y)$

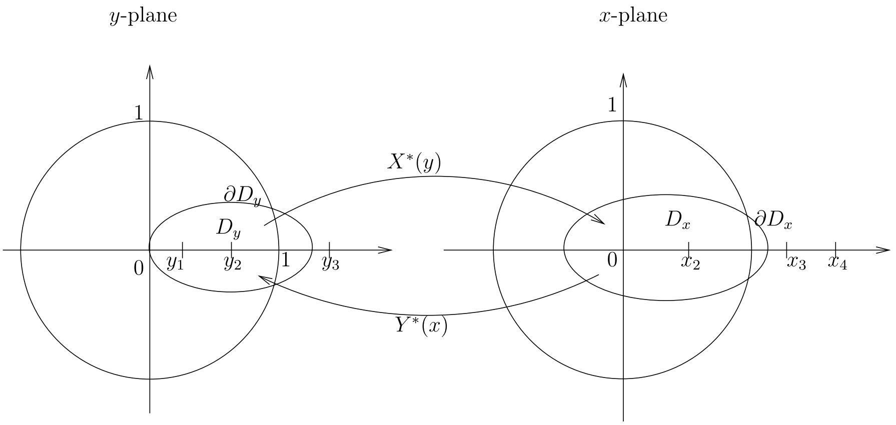



**Fig. 3** Fundamental domains $D_y$ and $D_x$

appears in the denominator. The common solutions of the equations $h_1(x, y) = 0$ and $h_2(x, y) = 0$ are then potential singularities for the functions $P(x, 0)$ and $P(0, y)$.

## 3.1 The common roots in variable $y$

Let $y \in \mathbb{C} \setminus ([y_1, y_2] \cup [y_3, \infty))$ and $h_1(x, y) = 0$, $x = X_\pm(y)$. If in addition $h_2(x, y) = 0$, then $y$ is a root of the resultant in $x$ of the two polynomials $h_1(x, y)$ and $h_2(x, y)$ (see the Appendix); this resultant, denoted by $Q_x(y)$, is a polynomial of degree 5 in $y$, which has at most four distinct zeros in $\mathbb{C}$. The point 0 is a double root. Another trivial root is 1 since $h_1(1, 1) = 0$ and $h_2(1, 1) = 0$. As shown in the Appendix, the resultant $Q_x(y)$ can actually be decomposed as

$$Q_x(y) = c_x y^2 (y - 1) \mathcal{Q}_x(y),$$

where $\mathcal{Q}_x(y)$ is the quadratic polynomial

$$\mathcal{Q}_x(y) = \lambda \nu_1 y^2 + \nu_2(\nu_2 - \nu_1 + \lambda) y - \nu_2^2 \tag{11}$$

and $c_x$ is a constant.

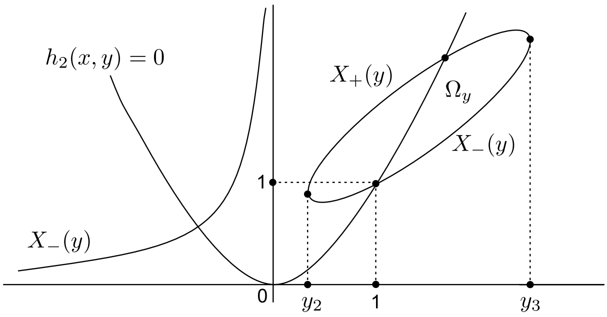
When $y$ describes the segment $[y_2, y_3]$, the curves $y \to x = X_\pm(y)$ describe the contour of a closed domain $\Omega_y$ in the $(y, x)$-plane as illustrated in Fig. 4. The contour $\partial \Omega_y$ of $\Omega_y$ contains the point $(1, 1)$.

When $h_2(x, y) = 0$,

$$x = \frac{\nu_1 y^2}{(\nu_1 - \nu_2) y + \nu_2}. \tag{12}$$

As illustrated in Fig. 4, when $r_1 < 1$, the hyperbolic branch defined by (12) intersects the branch $x = X_-(y)$ at some point with a negative abscissa. The same observation is true when $r_1 \geq 1$. It follows that the resultant $Q_y(x)$ has four real roots and the

**Fig. 4** Intersection points of the functions $X_\pm(y)$ and the curve $h_2(x, y) = 0$ when $r_1 < 1$




quadratic polynomial $\mathcal{Q}_x(y)$ has two real roots, one is negative and the other is in $[y_2, y_3]$. The positive root is (with $\rho_i = \lambda/\nu_i$)

$$y^* = \frac{\rho_1}{2\rho_2}\left(\frac{1}{\rho_1} - \frac{1}{\rho_2} - 1 + \sqrt{\left(\frac{1}{\rho_1} - \frac{1}{\rho_2} - 1\right)^2 + \frac{4}{\rho_1}}\right) \tag{13}$$

and the negative root is

$$y_* = \frac{\rho_1}{2\rho_2}\left(\frac{1}{\rho_1} - \frac{1}{\rho_2} - 1 - \sqrt{\left(\frac{1}{\rho_1} - \frac{1}{\rho_2} - 1\right)^2 + \frac{4}{\rho_1}}\right).$$

It is worth noting that $y^*$ does not depend on $p$. From the Appendix, we know that $y^* \in (1, y_3]$.

## 3.2 The common roots in variable $x$

The resultant in $x$ of the polynomials $h_1(x, y)$ and $h_2(x, y)$ is a polynomial of degree 5 with trivial roots 0 and 1 (0 is a double root). If $x \neq 0$ and $(x, y)$ is an intersection point of the curves $h_1(x, y) = 0$ and $h_2(x, y) = 0$, then

$$y = \frac{\nu_2}{\lambda + \nu_2 - \lambda x}. \tag{14}$$

For $x \in [x_2, x_3]$, the curves $y = Y_\pm(x)$ delineate a closed domain $\Omega_x$ such that its contour $\partial\Omega_x$ contains the point $(1, 1)$. Note that if $r_1 < r_2$, then $Y_+(1) = 1$ and if $r_1 > r_2$, then $Y_-(1) = 1$.

The hyperbolic branch defined by (14) intersects the branch $y = Y_-(x)$ or $y = Y_+(x)$ at a point with abscissa $x > x_4$. It follows that the resultant in $y$ of the polynomials $h_1(x, y)$ and $h_2(x, y)$, denoted by $Q_y(x)$, can be decomposed as

$$Q_y(x) = c_y x^2 (x - 1)\mathcal{Q}_y(x),$$

where $c_y$ is a constant and

$$\mathcal{Q}_y(x) = \lambda^2 x^2 - (\lambda + \nu_1 + \nu_2)\lambda x + \nu_1 \nu_2. \tag{15}$$

The roots $x^*$ and $x_*$ are given by

$$x^* = \frac{1}{2}\left(1 + \frac{1}{\rho_1} + \frac{1}{\rho_2} - \sqrt{\left(1 + \frac{1}{\rho_1} + \frac{1}{\rho_2}\right)^2 - \frac{4}{\rho_1\rho_2}}\right) \tag{16}$$

and

$$x_* = \frac{1}{2}\left(1 + \frac{1}{\rho_1} + \frac{1}{\rho_2} + \sqrt{\left(1 + \frac{1}{\rho_1} + \frac{1}{\rho_2}\right)^2 - \frac{4}{\rho_1\rho_2}}\right),$$

and are such that $x^* \le x_3 < x_4 \le x_*$. In addition, we know that $x^* > 1$ and hence $x^* \in (1, x_3]$. The variable $x^*$ does not depend on the probability $p$.

From the above observations, we deduce the following result.

**Proposition 2** *The equation $Q_y(X^*(y)) = 0$ has a solution in $(-\infty, y_3]$, which is necessarily equal to $y^* \in (1, y_3]$, if and only if $x^* = X_-(y^*)$.*

*Similarly, the equation $Q_x(Y^*(x)) = 0$ has a solution in $(-\infty, x_3]$, which is necessarily equal to $x^* \in (1, x_3]$, if and only if $y^* = Y_-(x^*)$.*

It is worth noting that we can have $x^* = X^*(y^*)$ only if $1 = X^*(1)$, that is $r_1 \le 1$. Similarly, we can have $y^* = Y^*(x^*)$ only if $1 = Y^*(1)$, that is $r_1 \ge r_2$.

## 4 Boundary value problems

We first determine the function $P(x, 0)$; the derivation of the function $P(0, y)$ can be done in the exact same fashion.

**Proposition 3** *The function $P(x, 0)$ is given by*

$$P(x,0) = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_{\partial D_x} \dfrac{g_x(z)}{z-x}dz & \text{for } x \in D_x, \\ g_x(x) + \dfrac{1}{2\pi i}\displaystyle\int_{C_x} \dfrac{g_x(z)}{z-x}dz & \text{for } x \in \mathbb{C} \setminus D_x, \end{cases} \tag{17}$$

*where $C_x$ is a contour in $D_x$ surrounding the slit $[x_1, x_2]$ and such that the function $g_x$ given by*

$$g_x(x) = (1-\rho)\frac{\nu_2 Y^*(x)(p\nu_1 Y^*(x) - \lambda x^2)}{(1-p)x Q_x(Y^*(x))}$$

*is analytic in the strip delineated by the contours $C_x$ and $\partial D_x$. The function $P(x, 0)$ is a meromorphic function in $\mathbb{C} \setminus [x_3, x_4]$ with singularities at the solutions to the equation $Q_x(Y^*(x)) = 0$, if they exist.*

*Proof* From the analysis carried out in Sect. 2, we know that for $y$ in a neighborhood $V_y(0)$ of $0^+$, $X^*(y)$ is close to 0 in $D_x(0, 1)$ (the unit disk in the $x$-plane). For $y \in$

$V_y(0)$, we deduce from (1) that

$$h_2\big(X^*(y), y\big)P\big(X^*(y), 0\big) + h_3\big(X^*(y), y\big)P(0, y) + h_4\big(X^*(y), y\big)P(0, 0) = 0,$$

which implies that

$$P\big(X^*(y), 0\big) = \frac{p}{1-p}P(0, y) - (1-\rho)\frac{h_4(X^*(y), y)}{h_2(X^*(y), y)}.$$

Note that $h_2(X^*(y), y) = 0$ if and only if $Q_y(X^*(y)) = 0$, which has only real solutions (see Sect. 3). From Proposition 2, this equation has a solution in $(-\infty, y_3]$ if and only if $x^* = X^*(y^*)$, which is then the unique solution and which is in $(1, y_3]$. If $\alpha = Y^*(x_2) \le 1$, the domain $D_y$ is included in the unit disk $D_y(0, 1)$ and in that case the function $h_4(X^*(y), y)/h_2(X^*(y), y)$ has no singularities in $D_y$. If $\alpha > 1$, then $r_1 > r_2$. In this case, $x^*$ is not equal to $X^*(y^*)$ and the function $h_4(X^*(y), y)/h_2(X^*(y), y)$ has no singularities in $D_y$. Hence, by using the same arguments as in [32], we deduce that the function $P(x, 0)$ can be analytically continued to the domain $D_x$. (We use the fact that the function $P(x, 0)$ can be expanded in a power series of $x$ at the point 0 with positive coefficients and $P(0, y_2) < \infty$, which implies that $P(x, 0)$ is analytic in the disk with center 0 and radius $X^*(y_2)$ containing $D_x$.)

Now, if we use the function $X_*(y)$, we obtain a meromorphic function in a domain surrounding from outside the domain $D_x$. If we take $y$ in a sufficiently small neighborhood of $[y_1, y_2]$ we can analytically define $P(x, 0)$ in an outer neighborhood of $D_x$.

Consider $x_0 \in \partial D_x$. Then there exists $y_0 \in [y_1, y_2]$ such that $X^*(y) \to x_0$ from inside when $y \to y_0$. In that case, $X_*(y) \to \bar{x}_0$ from outside. Let us define the interior (resp. exterior) limit $P_i(x, 0)$ (resp. $P_e(x, 0)$) of the function $P(x, 0)$ with respect to the contour $\partial D_y$ by

$$P_i(x_0, 0) = \lim_{x \to x_0, x \in D_x} P(x, 0) \left(\text{resp. } P_e(x_0, 0) = \lim_{x \to x_0, x \in \mathbb{C} \setminus D_x} P(x, 0)\right).$$

We then deduce from the above observation that for $x \in \partial D_y$ and $y = Y^*(x)$

$$P_i(x, 0) = \frac{pP(0, y)}{1-p} - (1-\rho)\frac{h_4(x, y)}{h_2(x, y)}, \qquad P_e(x, 0) = \frac{pP(0, y)}{1-p} - (1-\rho)\frac{h_4(\bar{x}, y)}{h_2(\bar{x}, y)},$$

since $P(\cdot, 0)$, $h_2$ and $h_4$ have real coefficients. Hence, we arrive at the fact that for $x \in \partial D_x$ and $y = Y^*(x)$

$$P_i(x, 0) - P_e(x, 0) = -2i(1-\rho)\Im\left(\frac{h_4(x, y)}{h_2(x, y)}\right).$$

Note that for $x \in \partial D_x$, we have $x\bar{x} = y/r_1 = Y^*(x)/r_1$ since $x$ and $\bar{x}$ are the two solutions to (1) in $x$. In addition, from the Appendix, we know that the resultant $Q_x(y)$ can be written as

$$Q_x(y) = p_x(x, y)h_1(x, y) + q_x(x, y)h_2(x, y),$$

where $p_x(x,y)$ and $q_x(x,y)$ are polynomials in $x$ and $y$. For $y = Y^*(x)$, we have $h_1(x,y) = 0$ and then

$$Q_x(y) = q_x(x,y)h_2(x,y).$$

Simple computations show that

$$\frac{h_4(x,y)}{h_2(x,y)} = -1 + \frac{\nu_2 x(y-1)}{h_2(x,y)}$$

and

$$q_x(x,y) = \lambda y b_1(y)x - \big(\lambda(1-p)\nu_1 y^3 + a_1(y)b_1(y)\big),$$

where

$$a_1(y) = \big(\lambda + p\nu_1 + (1-p)\nu_2\big)y - (1-p)\nu_2,$$
$$b_1(y) = (1-p)\big((\nu_2 - \nu_1)y - \nu_2\big).$$

Hence, for $x \in \partial D_y$ and $y = Y^*(x)$, we have

$$\Im\left(\frac{h_4(x,y)}{h_2(x,y)}\right) = \Im\left(\frac{\nu_2 x(y-1)(\lambda y b_1(y)x - (\lambda(1-p)\nu_1 y^3 + a_1(y)b_1(y)))}{-\nu_1(1-p)^2 y^2 (y-1) Q_x(y)}\right).$$

By using the fact that $\lambda y x^2 - a_1(y)x = -p\nu_1 y^2$, we have

$$\Im\left(\frac{h_4(x,y)}{h_2(x,y)}\right) = \Im\left(\frac{\nu_2(p\nu_1 b_1(y) + \lambda(1-p)\nu_1 y x)}{\nu_1(1-p)^2 Q_x(y)}\right)$$

and then

$$\Im\left(\frac{h_4(x,y)}{h_2(x,y)}\right) = \frac{\nu_2\lambda y \Im(x)}{(1-p)Q_x(y)} = \frac{\nu_2 \lambda y(r_1 x^2 - y)}{2ir_1 x(1-p)Q_x(y)}.$$

We finally arrive at the classical Riemann–Hilbert problem. For $x \in \partial D_x$,

$$P_i(x,0) - P_e(x,0) = (1-\rho)\frac{\nu_2 Y^*(x)(p\nu_1 Y^*(x) - \lambda x^2)}{x(1-p)Q_x(Y^*(x))} = g_x(x).$$

The solution to this Riemann–Hilbert problem is given by

$$P(x,0) = \frac{1}{2\pi i}\int_{\partial D_x} \frac{g_x(z)}{z-x}dz \quad \text{for } x \notin \partial D_x.$$

The above formula defines an analytic function in $D_x$. For $x \in \mathbb{C} \setminus D_x$, let us pick a closed contour $C_x$ in $D_x$ surrounding the slit $[x_1, x_2]$ and so that the function $g_x$ is analytic in the strip delineated by the contours $\partial D_x$ and $C_x$. Then, we have

$$\frac{1}{2\pi i}\int_{\partial D_x} \frac{g_x(z)}{z-x}dz = g_x(x) + \frac{1}{2\pi i}\int_{C_x} \frac{g_x(z)}{z-x}dz.$$

The function on the right-hand side of the above equation defines a meromorphic function in $\mathbb{C} \setminus [x_3, x_4]$. □

We can replace the integrals appearing in (17) with integrals along the segment $[y_1, y_2]$. We then obtain elliptic integrals. Since these integrals do not appear as simple combinations of Jacobi elliptic functions, we do not further investigate the connection between the function $P(x, 0)$ and elliptic functions. Finally, it is worth noting that the radius of convergence of the function $P(x, 0)$ is equal to either $x_3$ or else $x^*$ if $y^* = Y^*(x^*)$.

By adapting the above proof to the function $P(0, y)$, we obtain the following result.

**Proposition 4** *The function $P(0, y)$ is given by*

$$P(0, y) = \begin{cases} \dfrac{1}{2\pi i} \displaystyle\int_{\partial D_y} \frac{g_y(z)}{z - y} dz & \text{for } y \in D_y, \\ g_y(y) + \dfrac{1}{2\pi i} \displaystyle\int_{C_y} \frac{g_y(z)}{z - y} dz & \text{for } y \in \mathbb{C} \setminus D_y, \end{cases}$$

*where $C_y$ is a closed contour in $D_y$ surrounding the slit $[y_1, y_2]$ such that the function $g_y$ given by*

$$g_y(y) = (1 - \rho) \frac{\lambda(p\nu_1 y^2 - (1 - p)\nu_2 X^*(y))}{py\mathcal{Q}_y(X^*(y))}$$

*is analytic in the strip delineated by the contours $C_y$ and $\partial D_y$. The function $P(0, y)$ is a meromorphic function in $\mathbb{C} \setminus [y_3, y_4]$ with singularities at the solutions to the equation $\mathcal{Q}_y(X^*(y)) = 0$, if they exist.*

*Proof* Denote by $P_i(0, y)$ and $P_e(0, y)$ the interior and exterior limits of the function $P(0, y)$ with respect to the contour $\partial D_y$. We have for $y \in \partial D_y$ and $x = X^*(y)$

$$P_i(0, y) - P_e(0, y) = 2i(1 - \rho) \frac{1 - p}{p} \Im\left(\frac{h_4(x, y)}{h_2(x, y)}\right).$$

We have $Q_y(x) = q_y(x, y)h_2(x, y)$ for $x = x^*(y)$ with

$$\begin{aligned} q_y(x, y) = (1 - p)\nu_1\big[&-yp\nu_1\big(p(\nu_2 - \nu_1)x + \alpha_1(x)\big) \\ &+ p\alpha_1(x)(\nu_2 - \nu_1)x + \alpha_1(x)^2 - p\nu_1\nu_2 x\big]. \end{aligned}$$

Then

$$\Im\left(\frac{h_4(x, y)}{h_2(x, y)}\right) = \frac{\nu_2 x}{Q_y(x)} \Im\big((y - 1)q_y(x, y)\big) = \frac{\lambda(p\nu_1 y^2 - (1 - p)\nu_2 x)}{2i(1 - p)y\mathcal{Q}_y(x)},$$

which implies that

$$P_i(0, y) - P_e(0, y) = (1 - \rho) \frac{\lambda(p\nu_1 y^2 - (1 - p)\nu_2 x)}{py\mathcal{Q}_y(x)}$$

$$= (1-\rho)\frac{\lambda(p\nu_1 y^2 - (1-p)\nu_2 X^*(y))}{py\mathcal{Q}_y(X^*(y))} = g_y(y).$$

Note that 0 is a removable singularity of the function $g_y(y)$ since $X^*(y) \sim -r_2 y^2/r_1$ when $y \to 0$. □

## 5 Asymptotic analysis

We derive in this section the tail of the distribution of the numbers of customers in the first and the second queue. For this purpose, we consider the generating functions $P(x,1)$ and $P(1,y)$, which satisfy

$$P(x,1) = \sum_{n=0}^{\infty} \mathbb{P}(N_1 = n)x^n \quad \text{and} \quad P(1,y) = \sum_{n=0}^{\infty} \mathbb{P}(N_2 = n)y^n,$$

where $N_1$ and $N_2$ are the numbers of customers in the first and the second queue, respectively. From (1), we have

$$P(x,1) = \nu_1 \frac{(1-p)P(x,0) - pP(0,1) - (1-p)(1-\rho)}{\lambda x - p\nu_1}$$

and

$$P(1,y) = \frac{(\nu_1 y + \nu_2)((1-p)P(1,0) - pP(0,y) - (1-p)(1-\rho)) + \nu_2(1-\rho)}{(1-p)\nu_2 - p\nu_1 y}.$$

Note that the normalizing condition $P(1,1) = 1$ implies that

$$(1-p)P(1,0) - pP(0,1) = (1-p)(1-\rho) + \rho_1 - p. \tag{18}$$

**Lemma 5** *If $r_2 \le 1$, then*

$$(1-p)P\bigl(r_1^{-1},0\bigr) - pP(0,1) - (1-p)(1-\rho) = 0, \tag{19}$$

*which implies that the point $1/r_1$ is a removable singularity for the function $P(x,1)$. If $r_2 > 1$ (and then $r_1 \le 1$ by the stability condition (2)), we have*

$$(1-p)P\bigl(r_1^{-1},0\bigr) - pP(0,1) - (1-p)(1-\rho) < 0 \tag{20}$$

*and the point $1/r_1$ is a nonremovable singularity for the function $P(x,1)$.*

*Proof* We know that $P(x,0)$ is a meromorphic function in the disk with center 0 and radius $x_3$, with a unique potential singularity at point $x^*$. Equation (1) implies for $x \ne x^*$, when $x^*$ is a singularity for $P(x,0)$,

$$h_2\bigl(x, Y^*(x)\bigr)P(x,0) + h_3\bigl(x, Y^*(x)\bigr)P\bigl(0, Y^*(x)\bigr) + h_4\bigl(x, Y^*(x)\bigr) = 0. \tag{21}$$

When $r_2 \leq 1$, we have $Y^*(1/r_1) = 1$ and the above equation implies (19). When $r_2 > 1$ (and hence $r_1 \leq 1$), we have $Y^*(1/r_1) = 1/r_2 < 1$, and (21) implies

$$\begin{aligned}(1-p)P(1/r_1, 0) - pP(0, 1/r_2) - (1-p)(1-\rho)\\ = (1-\rho)\frac{\frac{\nu_2}{r_1}(1-\frac{1}{r_2})}{\frac{\nu_1}{r_2}(\frac{1}{r_2}-\frac{1}{r_1}) + \frac{\nu_2}{r_1}(\frac{1}{r_2}-1)} < 0.\end{aligned}$$

Since $P(0, 1/r_2) \leq P(0, 1)$, Inequality (20) follows. □

Similar arguments yield the following result for the function $P(1, y)$; the proof is omitted.

**Lemma 6** *We have*

$$(1-p)P(1,0) - pP(0, r_1/r_2) - (1-p)(1-\rho) + p(1-\rho) = 0, \tag{22}$$

*and hence the point $r_1/r_2$ is a removable singularity for the function $P(1, y)$.*

By using the two lemmas above, we are now able to determine the tails of the probability distributions of the random variables $N_1$ and $N_2$.

**Proposition 5** *For given system parameters $\lambda$, $\nu_1$, $\nu_2$ and $p$, the exact asymptotics of $\mathbb{P}(N_1 = n)$ as $n \to \infty$ are:*

I *If $y^* = Y^*(x^*)$ and $x^* < x_3$, which can occur only if $r_1 \geq r_2$, then*

$$\mathbb{P}(N_1 = n) \sim \kappa_1^{(1)} \left(\frac{1}{x^*}\right)^n. \tag{23}$$

II *If $y^* \neq Y^*(x^*)$ and $r_2 > 1$ (and then $r_1 \leq 1$),*

$$\mathbb{P}(N_1 = n) \sim \kappa_2^{(1)} (r_1)^n. \tag{24}$$

III *If $y^* \neq Y^*(x^*)$ and $r_2 \leq 1$, $1/r_1$ is a removable singularity for $P(x, 1)$ and we have*

$$\mathbb{P}(N_1 = n) \sim \kappa_3^{(1)} \frac{1}{n\sqrt{n}} \left(\frac{1}{x_3}\right)^n. \tag{25}$$

IV *If $y^* = Y^*(x_3)$ and $x^* = x_3$,*

$$\mathbb{P}(N_1 = n) \sim \kappa_4^{(1)} \frac{1}{\sqrt{n}} \left(\frac{1}{x^*}\right)^n, \tag{26}$$

*with $x_3$ the third largest real root of $D_1(x)$ in (4), and $Y^*(x)$, $y^*$, $x^*$ as in (8), (13), (16), respectively. Further, $r_i = \lambda/(p\nu_i)$, $\rho = 1 - \lambda/\nu_1 - \lambda/\nu_2$ and*

$$\kappa_1^{(1)} = \frac{\nu_1\nu_2(1-\rho)((1-p)\nu_2 x^* - p\nu_1 (y^*)^2)}{(\lambda x^* - p\nu_1)(\nu_2^2 + \lambda\nu_1(y^*)^2)x^*},$$

$$\kappa_2^{(1)} = P(0,1) + \frac{1-p}{p}\big(1-\rho-P(r_1^{-1},0)\big),$$

$$\kappa_3^{(1)} = \frac{(1-\rho)\lambda\nu_1\nu_2}{4\sqrt{\pi}(\lambda x_3 - p\nu_1)} \frac{\frac{\lambda^2(1-p)}{p\nu_2}x_3^2 + 2\lambda x_3 - (p\lambda+\nu_1)}{\mathcal{Q}_y(x_3)\mathcal{Q}_y^*(x_3)}\sqrt{x_3}\tau_x,$$

$$\kappa_4^{(1)} = \frac{(1-\rho)\lambda\nu_1\nu_2}{2\sqrt{\pi}(\lambda x_3 - p\nu_1)} \frac{\frac{\lambda^2(1-p)}{p\nu_2}x_3^2 + 2\lambda x_3 - (p\lambda+\nu_1)}{\sqrt{x_3}\mathcal{Q}_y'(x_3)\mathcal{Q}_y^*(x_3)}\tau_x,$$

*with* $\mathcal{Q}_y(x)$ *as in* (15),

$$\tau_x = \sqrt{(x_3-x_1)(x_3-x_2)(x_4-x_3)} \tag{27}$$

*and*

$$\mathcal{Q}_y^*(x) = \frac{1}{\lambda^2}\left(x - \frac{p\nu_1 y^*}{x^*}\right)\left(x - \frac{p\nu_1 y_*}{x_*}\right).$$

*Proof* Note first that we always have $1/r_1 \le x_3$ since

$$D_1(1/r_1) = (1-1/r_2)^2/r_1^2 \ge 0.$$

In case I, note that if $r_2 \le 1$, $1/r_1$ is a removable singularity for the function $P(x,1)$. If $r_2 > 1$, then $x^* < 1/r_1 \le x_3$ since

$$Q_y(1/r_1) = \nu_1\lambda\big(1/r_2 - p - (1-p)/r_1\big) < 0.$$

This implies that $x^*$ is the singularity with the smallest modulus. The residue of the function $P(x,0)$ at point $x^*$ is equal to

$$(1-\rho)\frac{\nu_2 y^*(p\nu_1 y^* - \lambda(x^*)^2)}{(1-p)x^*\mathcal{Q}_x'(y^*)\frac{\partial Y^*}{\partial x}|_{x=x^*}}.$$

Since $h_1(x, Y^*(x)) = 0$, we deduce that

$$\left.\frac{\partial Y^*}{\partial x}\right|_{x=x^*} = -\frac{\frac{\partial h_1}{\partial x}(x^*,y^*)}{\frac{\partial h_1}{\partial y}(x^*,y^*)} = \frac{(y^*)^2(p\nu_1 y^* - \lambda(x^*)^2)}{x^*(p\nu_1(y^*)^2 - (1-p)\nu_2 x^*)}.$$

A direct application of Darboux's method then yields (23).

In case II, the point $r_1 \le 1$ is the pole with the smallest modulus for the function $P(x,1)$ and Darboux's method yields (24).

In case III, the function $P(x,1)$ has no singularities in the disk $D(0,x_3)$ with center 0 and radius $x_3$. The function $P(x,0)$ can then be represented, for $|x| < x_3$, as

$$P(x,0) = \frac{1}{2i\pi}\int_{C(x_3)} \frac{g_x(z)}{z-x}dz,$$

where $C(x_3)$ is the circle with center 0 and radius $x_3$. By using (19), we have

$$P(x,1) = \nu_1(1-p)\frac{P(x,0) - P(1/r_1,0)}{\lambda x - p\nu_1} = \frac{1}{2i\pi}\int_{C(x_3)} \frac{h_x(z)}{z-x}dz,$$

where

$$h_x(z) = \nu_1(1-p)\frac{g_x(z)}{\lambda z - p\nu_1}.$$

As shown in Sect. 3, the point $x_*$ may be a pole of the function $h_x$. Let $\text{Res}(h_x; x_*)$ denote the residue of the function $h_x$ at $x_*$. By deforming the integration contour so as to encompass the segment $[x_3, x_4]$, and since $|h_x(z)| < K_x/|z|$ for some constant $K_x > 0$ when $|z| \to \infty$, we deduce that

$$P(x,1) = \frac{1}{2i\pi}\int_{x_3}^{x_4} \frac{h_x(z+0i) - h_x(z-0i)}{z-x}dz + \frac{\text{Res}(h_x; x_*)}{x - x_*}$$

and then

$$P(x,1) = \frac{-1}{\pi}\int_{x_3}^{x_4} \frac{(1-\rho)\nu_1\nu_2}{\xi(\lambda\xi - p\nu_1)(\xi - x)}\Im\left(\frac{Y^*(\xi)(\lambda\xi^2 - p\nu_1 Y^*(\xi))}{Q_x(Y^*(\xi))}\right)d\xi$$
$$+ \frac{\text{Res}(h_x; x_*)}{x - x_*}.$$

We have

$$\Im\left(\frac{Y^*(\xi)(\lambda\xi^2 - p\nu_1 Y^*(\xi))}{Q_x(Y^*(\xi))}\right) = \frac{\Im(Y^*(\xi)(\lambda\xi^2 - p\nu_1 Y^*(\xi))Q_x(\overline{Y^*(\xi)}))}{Q_x(Y^*(\xi))Q_x(\overline{Y^*(\xi)})}.$$

When $\xi \in [x_3, x_4]$, the relation

$$\overline{Y^*}(\xi) = Y_*(\xi)$$

holds and tedious computations show that $Q_x(Y^*(\xi))Q_x(Y_*(\xi))$ is a quadratic polynomial in $\xi$. In particular, we get

$$Q_x\big(Y^*(\xi)\big)Q_x\big(Y_*(\xi)\big) = (\lambda\nu_1)^2\big(Y^*(\xi) - y^*\big)\big(Y^*(\xi) - y_*\big)\big(Y_*(\xi) - y^*\big)\big(Y_*(\xi) - y_*\big).$$

By definition, we know that the above quantity vanishes for $x$ equal to $x^*$ or $x_*$. More precisely, in case III, we have $Y_*(x^*) = y^*$. In addition, $Y_*(x_*)$ or $Y^*(x_*)$ is equal to $y_*$. Finally, we note that if $x$ is such that $h_1(x,y) = 0$ then $p\nu_1 y/(\lambda x)$ is also such that $h_1(x,y) = 0$. This implies that the four roots of the polynomial $Q_x(Y^*(\xi))Q_x(Y_*(\xi))$ are $x_*$, $x^*$, $p\nu_1 y^*/(\lambda x^*)$ and $p\nu_1 y_*/(\lambda x_*)$. Hence,

$$Q_x\big(Y^*(\xi)\big)Q_x\big(Y_*(\xi)\big) = -\frac{\lambda^3\nu_2^2}{p^2\nu_1}(\xi - x_*)(\xi - x^*)\left(\xi - \frac{p\nu_1 y^*}{\lambda x^*}\right)\left(\xi - \frac{p\nu_1 y_*}{\lambda x_*}\right)$$
$$= -\frac{\lambda\nu_2^2}{p^2\nu_1}Q_y(\xi)Q_y^*(\xi),$$

where the polynomial $\mathcal{Q}_y^*(x)$ is defined by (5).

Moreover, we have

$$\Im\Big(Y^*(\xi)\big(p\nu_1 Y^*(\xi)-\lambda\xi^2\big)\mathcal{Q}_x\big(\overline{Y^*(\xi)}\big)\Big)$$
$$=\frac{\lambda}{2p\nu_1}\left(-\frac{\lambda^2\nu_2(1-p)}{p}\xi^3+\nu_2^2(p\lambda+\nu_1)\xi-2\nu_2^2\lambda\xi^2\right)\sqrt{-D_1(\xi)}.$$

It follows that

$$P(x,1)=\frac{1}{\pi}\int_{x_3}^{x_4}\frac{H_x(\xi)}{\xi-x}d\xi+\frac{\mathrm{Res}(h_x;x_*)}{x-x_*},$$

where

$$H_x(\xi)=\frac{(1-\rho)\nu_1}{p\nu_1-\lambda\xi}\frac{\lambda^2(1-p)\xi^2+2p\lambda\nu_2\xi-p\nu_2(p\lambda+\nu_1)}{2\mathcal{Q}_y(\xi)\mathcal{Q}_y^*(\xi)}\sqrt{-D_1(\xi)}.$$

Then

$$\mathbb{P}(N_1=n)=\frac{1}{\pi}\int_{x_3}^{x_4}\frac{H_x(\xi)}{\xi}e^{-n\log\xi}d\xi-\frac{\mathrm{Res}(h_x;x_*)}{(x_*)^{n+1}}. \tag{28}$$

In the neighborhood of $x_3$, we have

$$-\log\xi=-\log x_3-\frac{1}{x_3}(\xi-x_3)+o(\xi-x_3)$$

and

$$\frac{H_x(\xi)}{\pi\xi}=k_3^{(1)}\sqrt{\xi-x_3}+o\big(\sqrt{\xi-x_3}\big),$$

where

$$k_3^{(1)}=\frac{(1-\rho)\nu_1}{2\pi(p\nu_1-\lambda x_3)}\frac{\lambda^2(1-p)x_3^2+2p\nu_2\lambda x_3-p\nu_2(p\lambda+\nu_1)}{x_3\mathcal{Q}_y(x_3)\mathcal{Q}_y^*(x_3)}\tau_x,$$

with $\tau_x$ as in (27). A direct application of Laplace's method [6, 8] then yields

$$\mathbb{P}(N_1=n)\sim k_3^{(1)}\Gamma(3/2)\frac{1}{n^{3/2}}\left(\frac{1}{x_3}\right)^{n-\frac{3}{2}}$$

when $n\to\infty$. Since $\Gamma(3/2)=\sqrt{\pi}/2$, (25) follows.

In case IV, we have for $\xi$ in the neighborhood of $x_3$

$$\mathcal{Q}_y(\xi)=\mathcal{Q}_y'(x_3)(\xi-x_3)+o(\xi-x_3)$$

and then

$$\frac{H_x(\xi)}{2\pi\xi}=k_4^{(1)}(\xi-x_3)^{-1/2}+o\big((\xi-x_3)^{-1/2}\big),$$

where

$$k_4^{(1)} = \frac{(1-\rho)\nu_1}{2\pi(p\nu_1 - \lambda x_3)} \frac{\lambda^2(1-p)x_3^2 + 2p\nu_2\lambda x_3 - p\nu_2(p\lambda+\nu_1)}{x_3 \mathcal{Q}_y'(x_3)\mathcal{Q}_y^*(x_3)} \tau_x.$$

Laplace's method then yields

$$\mathbb{P}(N_1 = n) \sim k_4^{(1)} \Gamma(1/2) \frac{1}{n^{1/2}} \left(\frac{1}{x_3}\right)^{n-\frac{1}{2}}$$

and by using the fact that $\Gamma(1/2) = \sqrt{\pi}$, (26) follows. □

*Remark* When we set $p = 0$ we give full priority to queue 2 and the functional equation greatly simplifies due to $h_3(x, y) = 0$. Then, for $\zeta(x) = \nu_2/(\lambda + \nu_2 - \lambda x)$, we see that $h_1(x, \zeta(x)) = 0$ and hence

$$\begin{aligned} P(x,0) &= -\frac{h_4(x,\zeta(x))P(0,0)}{h_2(x,\zeta(x))} = \frac{(\nu_1\nu_2 - \lambda\nu_1 x)(1-\rho)}{\mathcal{Q}_y(x)} \\ &= \frac{(\nu_1\nu_2 - \lambda\nu_1 x)(1-\rho)}{\lambda^2(x-x_*)(x-x^*)} = \frac{c_1}{x-x_*} + \frac{c_2}{x-x^*}, \end{aligned}$$

with

$$c_1 = \frac{(\nu_1\nu_2 - \lambda\nu_1 x_*)(1-\rho)}{\lambda^2(x_* - x^*)}, \qquad c_2 = \frac{(\nu_1\nu_2 - \lambda\nu_1 x^*)(1-\rho)}{\lambda^2(x_* - x^*)}.$$

This gives

$$P(x,1) = \frac{\nu_1}{\lambda x}\left[\frac{c_1}{x-x_*} + \frac{c_2}{x-x^*} - (1-\rho)\right]$$

and

$$\mathbb{P}(N_1 = n) \sim \frac{\nu_1^2\lambda x^* - \nu_1^2\nu_2}{\lambda^3(x^*-x_*)(x^*)^2}(1-\rho)\left(\frac{1}{x^*}\right)^n.$$

Note that this agrees with regime I in Proposition 5 if

$$\frac{\nu_1(\lambda x^* - \nu_2)}{\lambda^2(x^*-x_*)x^*} = \frac{\nu_2^2}{\nu_2^2 + \lambda\nu_1(y_*)^2},$$

which can indeed be shown to be true.

For the second queue, we first note by using Lemma 6 that the point $r_1/r_2$ is always a removable singularity for the function $P(1, y)$.

**Proposition 6** *For given system parameters $\lambda$, $\nu_1$, $\nu_2$ and $p$, the exact asymptotics for $\mathbb{P}(N_2 = n)$ as $n \to \infty$ are*:

I *If* $x^* = X^*(y^*)$ *and* $y^* < y_3$, *which can occur only when* $r_1 \leq 1$, *then*

$$\mathbb{P}(N_2 = n) \sim \kappa_1^{(2)} \left(\frac{1}{y^*}\right)^n. \tag{29}$$

II *If* $x^* \neq X^*(y^*)$, *then*

$$\mathbb{P}(N_2 = n) \sim \kappa_2^{(2)} \frac{1}{n\sqrt{n}} \left(\frac{1}{y_3}\right)^n. \tag{30}$$

III *If* $x^* = X^*(y^*)$ *and* $y^* = y_3$,

$$\mathbb{P}(N_2 = n) \sim \kappa_3^{(2)} \frac{1}{\sqrt{n}} \left(\frac{1}{y^*}\right)^n. \tag{31}$$

*with* $y_3$ *the third largest real root of* $D_2(y)$ *in* (3), *and* $X^*(y)$, $y^*$, $x^*$ *as in* (5), (13), (16), *respectively. Further,* $r_i = \lambda/(p\nu_i)$, $\rho = 1 - \lambda/\nu_1 - \lambda/\nu_2$ *and*

$$\kappa_1^{(2)} = \frac{(1-\rho)\lambda(\nu_1 y^* + \nu_2)(p\nu_1 y^* - \lambda(x^*)^2)}{((1-p)\nu_2 - p\nu_1 y^*)x^* \mathcal{Q}'_y(x^*)},$$

$$\kappa_2^{(2)} = \frac{(1-\rho)(\nu_2 + \nu_1 y_3)(\lambda p(p\nu_2 + (1-p)\nu_1)y_3^2 + 2\lambda p(1-p)\nu_2 y_3 - (1-p)\nu_2^2)}{2\sqrt{\pi}\, p^2(p\nu_1 y_3 - (1-p)\nu_2)\mathcal{Q}_x(y_3)\mathcal{Q}^*_x(y_3)} \sqrt{y_3}\,\tau_y,$$

$$\kappa_3^{(2)} = \frac{(1-\rho)(\nu_2 + \nu_1 y_3)(\lambda p(p\nu_2 + (1-p)\nu_1)y_3^2 + 2\lambda p(1-p)\nu_2 y_3 - (1-p)\nu_2^2)}{\sqrt{\pi}\sqrt{y_3}\, p^2(p\nu_1 y_3 - (1-p)\nu_2)\mathcal{Q}'_x(y_3)\mathcal{Q}^*_x(y_3)} \tau_y,$$

*with* $\tau_y = \sqrt{p\nu_1(y_3 - y_1)(y_3 - y_2)/\lambda}$,

$$\mathcal{Q}^*_x(y) = \left(y - \frac{(1-p)\nu_2 x^*}{p\nu_1 y^*}\right)\left(y - \frac{(1-p)\nu_2 x_*}{p\nu_1 y_*}\right), \tag{32}$$

*and* $\mathcal{Q}_x(y)$, $\mathcal{Q}_y(x)$ *as in* (11), (15), *respectively.*

*Proof* In case I, $y^*$ is the pole with the smallest modulus for the function $P(1, y)$ and a direct application of Darboux's method yields

$$\mathbb{P}(N_2 = n) \sim \frac{(1-\rho)\lambda}{(1-p)\nu_2 - p\nu_1 y^*} \frac{(1-p)\nu_2 x^* - p\nu_1 (y^*)^2}{(y^*)^2 \mathcal{Q}'_y(x^*) \frac{\partial X^*}{\partial y}|_{y=y^*}} \left(\frac{1}{y^*}\right)^n$$

and (29) follows.

In case II, the function $P(1, y)$ is analytic in the disk with center 0 and radius $y_3$ and we have

$$P(0, y) = \frac{1}{2i\pi} \int_{C(y_3)} \frac{g_y(z)}{z - y} dz,$$

where $C(y_3)$ is the circle with center 0 and radius $y_3$. By using (22), we have

$$P(1, y) = 1 - \rho + \frac{(\nu_1 y + \nu_2)p(P(0, r_1/r_2) - P(0, y))}{(1-p)\nu_2 - p\nu_1 y}$$

$$= 1 - \rho + \frac{1}{2i\pi}\int_{C(y_3)} \frac{h_y(z)}{z-x}dz,$$

where

$$h_y(z) = \frac{p(\nu_2+\nu_1 z)g_y(z)}{p\nu_1 z-(1-p)\nu_2}.$$

By deforming the integration contour along the segment $[y_3,\infty)$, and since the function $h_y(z)$ is such that $|h_y(z)| < K_y/|z|$ for some constant $K_y > 0$ when $|z| \to \infty$, we deduce that

$$P(1,y) = (1-\rho) + \frac{1}{2i\pi}\int_{y_3}^{\infty} \frac{h_y(z+0i)-h_y(z-0i)}{z-y}dz$$

and then

$$\begin{aligned} &P(1,y)\\ &\quad= (1-\rho) + \frac{-1}{\pi}\int_{y_3}^{\infty} \frac{(1-\rho)\lambda(\nu_2+\nu_1 z)}{z(p\nu_1 z-(1-p)\nu_2)}\Im\left(\frac{(p\nu_1 y^2-(1-p)\nu_2 X^*(y))}{\mathcal{Q}_y(X^*(y))}\right)dz. \end{aligned}$$

There holds

$$\Im\left(\frac{(p\nu_1 y^2-(1-p)\nu_2 X^*(y))}{\mathcal{Q}_y(X^*(y))}\right) = \frac{\Im((p\nu_1 y^2-(1-p)\nu_2 X^*(y))\mathcal{Q}_y(\overline{X^*(y)}))}{\mathcal{Q}_y(X^*(y))\mathcal{Q}_y(\overline{X^*(y)})}.$$

When $z \in [y_3,\infty)$, we have

$$\overline{X^*}(z) = X_*(z).$$

It is easily checked that the function $z \to z^2\mathcal{Q}_y(X^*(z))\mathcal{Q}_y(X_*(z))$ is a quadratic polynomial in $z$. By definition, we know that this polynomial vanishes for $y$ equal to $y^*$ or $y_*$. More precisely, in case II, we have $X_*(y^*) = x^*$. In addition, $X_*(y_*)$ or $X^*(y_*)$ is equal to $x_*$. If $y$ is such that $h_1(x,y)=0$ then $(1-p)\nu_2 x/(p\nu_1 y)$ is also such that $h_1(x,y)=0$. This implies that the four roots of the polynomial $z^2\mathcal{Q}_y(X^*(z))\mathcal{Q}_y(X_*(z))$ are $y_*$, $y^*$, $(1-p)\nu_2 x^*/(p\nu_1 y^*)$ and $(1-p)\nu_2 x_*/(p\nu_1 y_*)$. Hence,

$$\begin{aligned} &z^2\mathcal{Q}_y\big(X^*(z)\big)\mathcal{Q}_y\big(X_*(z)\big)\\ &\quad= \lambda^2 p^2\nu_1^2(z-y_*)(z-y^*)\left(z-\frac{(1-p)\nu_2 x^*}{p\nu_1 y^*}\right)\left(z-\frac{(1-p)\nu_2 x_*}{p\nu_1 y_*}\right) \end{aligned}$$

and then

$$z^2\mathcal{Q}_y\big(X^*(z)\big)\mathcal{Q}_y\big(X_*(z)\big) = \lambda\nu_1 p^2\mathcal{Q}_x(z)\mathcal{Q}_x^*(z),$$

where the polynomial $\mathcal{Q}_x^*(z)$ as defined in (32).

Moreover, we have in the neighborhood of $y_3$

$$\Im\Big(\big(p\nu_1 y^2-(1-p)\nu_2 X^*(y)\big)\mathcal{Q}_y\big(\overline{X^*(y)}\big)\Big)$$
$$=-\frac{\nu_1}{2y}\big(\lambda p\big(p\nu_2+(1-p)\nu_1\big)y^2+2\lambda p(1-p)\nu_2 y-(1-p)\nu_2^2\big)\sqrt{-D_2(y)}.$$

It follows that

$$P(1,y)=(1-\rho)+\frac{1}{\pi}\int_{y_3}^{\infty}\frac{H_y(z)}{z-x}dz,$$

where the function $H_y(z)$ is defined for $z$ in the neighborhood of $y_3$ by

$$H_y(z)=\frac{(1-\rho)(\nu_2+\nu_1 z)\sqrt{-D_2(z)}}{2p^2(p\nu_1 z-(1-p)\nu_2)\mathcal{Q}_x(z)\mathcal{Q}_x^*(z)}$$
$$\times\big(\lambda p\big(p\nu_2+(1-p)\nu_1\big)z^2+2\lambda p(1-p)\nu_2 z-(1-p)\nu_2^2\big).$$

Then, for $n\geq 1$,

$$\mathbb{P}(N_2=n)=\frac{1}{\pi}\int_{y_3}^{\infty}\frac{H_y(z)}{z}e^{-n\log z}\,dz. \tag{33}$$

In the neighborhood of $y_3$, we have

$$-\log z=-\log y_3-\frac{1}{y_3}(z-y_3)+o(z-y_3)$$

and

$$\frac{H_y(z)}{\pi z}=k_2^{(2)}\sqrt{z-y_3}+o\big(\sqrt{z-y_3}\big),$$

where

$$k_2^{(2)}=\frac{(1-\rho)(\nu_2+\nu_1 y_3)(\lambda p(p\nu_2+(1-p)\nu_1)y_3^2+2\lambda p(1-p)\nu_2 y_3-(1-p)\nu_2^2)}{2\pi y_3 p^2(p\nu_1 y_3-(1-p)\nu_2)\mathcal{Q}_x(y_3)\mathcal{Q}_x^*(y_3)}$$
$$\times\sqrt{4p\nu_1(y_3-y_1)(y_3-y_2)/\lambda}.$$

A direct application of Laplace's method then yields

$$\mathbb{P}(N_2=n)\sim k_2^{(2)}\Gamma(3/2)\frac{1}{n^{3/2}}\left(\frac{1}{y_3}\right)^{n-\frac{3}{2}}$$

when $n\to\infty$. Since $\Gamma(3/2)=\sqrt{\pi}/2$, (30) follows.

In case III, we have for $z$ in the neighborhood of $y_3$

$$\mathcal{Q}_x(z)=\mathcal{Q}_x'(y_3)(z-y_3)+o\big((z-y_3)\big)$$

and then

$$\frac{H_y(z)}{2\pi z}=k_3^{(2)}(z-y_3)^{-1/2}+o\big((z-y_3)^{-1/2}\big),$$

where

$$k_3^{(2)} = \frac{(1-\rho)(\nu_2+\nu_1 y_3)(\lambda p(p\nu_2+(1-p)\nu_1)y_3^2+2\lambda p(1-p)\nu_2 y_3-(1-p)\nu_2^2)}{2\pi y_3 p^2(p\nu_1 y_3-(1-p)\nu_2)\mathcal{Q}_x'(y_3)\mathcal{Q}_x^*(y_3)} \times \sqrt{4p\nu_1(y_3-y_1)(y_3-y_2)/\lambda}.$$

Laplace's method then yields

$$\mathbb{P}(N_2=n) \sim k_3^{(2)}\Gamma(1/2)\frac{1}{n^{1/2}}\left(\frac{1}{y_3}\right)^{n-\frac{1}{2}}$$

and with $\Gamma(1/2)=\sqrt{\pi}$ (31) follows. □

### 5.1 Numerical examples

We shall now compare the asymptotic estimates in Propositions 5 and 6 against results obtained by numerical calculations. Truncating the state space by bounding one of the queue lengths leads to a Markov process on an infinite strip, better known as a Quasi-Birth-Death (QBD) process. For these processes, fast numerical algorithms are available (see [26]). All numerical results presented were obtained by imposing an upper bound on the second queue of 500 (so that the truncation effect should be negligible).

For a first scenario we take $\lambda = 1.5$, $\rho_1 = .4$ and $\rho_2 = .3$. Figure 5 compares $X^*(y^*)$ with $x^*$ and $Y^*(x^*)$ with $y^*$, when $p$ varies. For example, we see that for $p < .6$, $Y^*(x^*) = y^*$. For $p = .5$, we have regime (23) for queue 1 and regime (30) for queue 2. Results for this case are presented in Table 1. Note that (23) converges fast to the true (numerical) value. The convergence of the branch point asymptotics (30) seems slower, in particular the convergence of the last column in Table 1 to the value $\kappa_2^{(2)} = 20.7454$. In order to demonstrate that $\kappa_2^{(2)}$ is indeed the leading constant, we

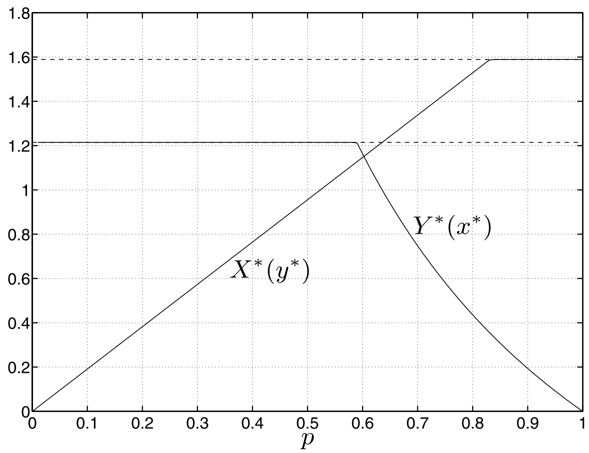



**Fig. 5** Comparisons of $x^*$, $X^*(y^*)$ and $y^*$, $Y^*(x^*)$ when $p$ varies. Here, $\lambda = 1.5$, $\rho_1 = .4$, $\rho_2 = .3$ and $x^* = 1.5890$, $y^* = 1.2146$

**Table 1** Illustration of (23) and (30) for $\lambda = 1.5$, $\rho_1 = .4$, $\rho_2 = .3$, $p = .5$. In this case $x^* = 1.5890$, $X^*(y^*) = 0.9555$, $y^* = Y^*(x^*) = 1.2146$. We find that $\kappa_2^{(2)} = 20.7454$

| $n$ | $\mathbb{P}(N_1 = n)$ | $\kappa_1^{(1)}(x^*)^{-n}$ | $\mathbb{P}(N_2 = n)$ | $n^{-3/2}(y_3)^{-n}$ | $\frac{\mathbb{P}(N_2=n)}{n^{-3/2}(y_3)^{-n}}$ |
|---|---|---|---|---|---|
| 5 | 2.8301e−002 | 2.4891e−002 | 1.0567e−002 | 5.1414e−003 | 2.0553e+000 |
| 10 | 2.5852e−003 | 2.4569e−003 | 3.6384e−004 | 1.0449e−004 | 3.4821e+000 |
| 15 | 2.4842e−004 | 2.4252e−004 | 1.5032e−005 | 3.2693e−006 | 4.5978e+000 |
| 20 | 2.4237e−005 | 2.3938e−005 | 6.7391e−007 | 1.2206e−007 | 5.5210e+000 |
| 50 | 2.2151e−011 | 2.2140e−011 | 1.0132e−014 | 1.1140e−015 | 9.0958e+000 |
| 100 | 1.9438e−021 | 1.9438e−021 | 1.8829e−027 | 1.5511e−028 | 1.2139e+001 |
| 200 | 1.4983e−041 | 1.4983e−041 | 1.2762e−052 | 8.5067e−054 | 1.5002e+001 |
| 300 | 1.1549e−061 | 1.1549e−061 | 1.1804e−077 | 7.1825e−079 | 1.6434e+001 |

**Table 2** Comparison of (33) and (30) for $\lambda = 1.5$, $\rho_1 = .4$, $\rho_2 = .3$, $p = .5$ and $\kappa_2^{(2)} = 20.7454$

| $n$ | (33) | $n^{-3/2}(y_3)^{-n}$ | $\frac{\mathbb{P}(N_2=n)}{n^{-3/2}(y_3)^{-n}}$ |
|---|---|---|---|
| $10^2$ | 1.8301e−27 | 1.5509e−28 | 1.1801e+1 |
| $10^3$ | 4.8227e−252 | 2.5453e−253 | 1.8947e+1 |
| $10^4$ | 2.3446e−2486 | 1.1415e−2487 | 2.0540e+1 |
| $10^5$ | 2.4607e−24816 | 1.1873e−24817 | 2.0725e+1 |
| $10^6$ | 1.1550e−248102 | 5.5682e−248104 | 2.0743e+1 |
| $10^7$ | 1.8797e−2480952 | 9.0611e−2480954 | 2.0745e+1 |

**Table 3** Illustration of (25) and (30) for $\lambda = 1.5$, $\rho_1 = .4$, $\rho_2 = .3$, $p = .65$. In this case $x^* = 1.5890$, $X^*(y^*) = 1.2421$, $y^* = 1.2146$ and $Y^*(x^*) = 0.9392$. We find that $\kappa_3^{(1)} = 81.6727$ and $\kappa_2^{(2)} = 3.7799$

| $n$ | $\mathbb{P}(N_1 = n)$ | $n^{-3/2}(x_3)^{-n}$ | $\frac{\mathbb{P}(N_1=n)}{n^{-3/2}(x_3)^{-n}}$ | $\mathbb{P}(N_2 = n)$ | $n^{-3/2}(y_3)^{-n}$ | $\frac{\mathbb{P}(N_2=n)}{n^{-3/2}(y_3)^{-n}}$ |
|---|---|---|---|---|---|---|
| 5 | 2.0854e−002 | 7.4520e−003 | 2.7985e+000 | 2.6154e−002 | 2.7103e−002 | 9.6499e−001 |
| 10 | 1.2811e−003 | 2.1951e−004 | 5.8359e+000 | 4.1746e−003 | 2.9037e−003 | 1.4377e+000 |
| 15 | 8.6268e−005 | 9.9552e−006 | 8.6656e+000 | 8.3828e−004 | 4.7896e−004 | 1.7502e+000 |
| 20 | 6.0730e−006 | 5.3873e−007 | 1.1273e+001 | 1.8669e−004 | 9.4268e−005 | 1.9804e+000 |
| 50 | 1.0651e−012 | 4.5586e−014 | 2.3364e+001 | 4.9780e−008 | 1.8464e−008 | 2.6961e+000 |
| 100 | 9.3290e−024 | 2.5976e−025 | 3.5914e+001 | 1.3411e−013 | 4.2613e−014 | 3.1472e+000 |
| 200 | 1.1821e−045 | 2.3856e−047 | 4.9552e+001 | 2.2323e−024 | 6.4202e−025 | 3.4770e+000 |
| 300 | 1.9248e−067 | 3.3732e−069 | 5.7061e+001 | 5.3829e−035 | 1.4892e−035 | 3.6146e+000 |

compare (30) against the integral representation (33) (omitting the residue term); see Table 2. Indeed, this confirms the correctness of $\kappa_2^{(2)} = 20.7454$.

Results for $p = .65$ are presented in Table 3 in which case we have regime (25) for queue 1 and regime (30) for queue 2. Note again the slow convergence to the asymptotic constants $\kappa_3^{(1)}$ and $\kappa_2^{(2)}$.

**Table 4** Illustration of (24) and (29) for $\lambda = 1$, $\rho_1 = .1$, $\rho_2 = .85$, $p = .3$. In this case $x^* = X^*(y^*) = 1.0581$, $y^* = 1.0520$ and $Y^*(x^*) = 0.2761$

| $n$ | $\mathbb{P}(N_1 = n)$ | $\kappa_2^{(1)}(r_1)^n$ | rel. error | $\mathbb{P}(N_2 = n)$ | $\kappa_1^{(2)}(y^*)^{-n}$ |
|---|---|---|---|---|---|
| 5 | 2.5017e−003 | 2.1008e−003 | 1.1909 | 3.7599e−002 | 3.7227e−002 |
| 10 | 1.0008e−005 | 8.6452e−006 | 1.1577 | 2.8912e−002 | 2.8894e−002 |
| 15 | 4.0423e−008 | 3.5577e−008 | 1.1362 | 2.2428e−002 | 2.2427e−002 |
| 20 | 1.6403e−010 | 1.4641e−010 | 1.1204 | 1.7407e−002 | 1.7407e−002 |
| 50 | 7.6060e−025 | 7.1109e−025 | 1.0696 | 3.8058e−003 | 3.8058e−003 |
| 100 | 1.0262e−048 | 9.9052e−049 | 1.0360 | 3.0199e−004 | 3.0199e−004 |
| 200 | 1.9451e−096 | 1.9219e−096 | 1.0120 | 1.9014e−006 | 1.9014e−006 |
| 300 | 3.7427e−144 | 3.7292e−144 | 1.0036 | 1.1972e−008 | 1.1972e−008 |

Table 4 illustrates some results for $\lambda = 1.5$, $\rho_1 = .2$, $\rho_2 = .4$ and $p = .4$, in which case we have regimes (24) and (29).



## Appendix: The resultant of the polynomials $h_1$ and $h_2$

Generally speaking, when we have two polynomials in two variables, say,

$$f_1(x, y) = a_0(y) + a_1(y)x + \cdots + a_n(y)x^n,$$
$$f_2(x, y) = b_0(y) + b_1(y)x + \cdots + b_m(y)x^m,$$

the resultant of the polynomials $f_1$ and $f_2$ with respect to $x$ is the determinant $\text{Res}_x(f_1, f_2)$ of the matrix

$$\begin{pmatrix} a_n & \cdots & a_0 & 0 & \cdots & \cdots \\ 0 & a_n & \cdots & a_0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & 0 & a_n & \cdots & a_0 \\ b_m & \cdots & b_0 & 0 & \cdots & \cdots \\ 0 & b_m & \cdots & b_0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & 0 & b_m & \cdots & b_0 \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} m \text{ rows} \\ \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} n \text{ rows} \end{matrix}$$

which is a polynomial in $y$. The polynomials $f_1$ and $f_2$ have a common nontrivial root $(x_0, y_0)$ if and only if the resultant with respect to $x$ is 0 at $y_0$. This leads to the resolution of a polynomial equation. Note that by adding to the $(m+n)$th column, the $i$th column multiplied by $x^{m+n-i}$ for $0 \le i < n+m$, $\text{Res}_x(f_1, f_2)$ is equal to the determinant of the matrix

$$\begin{pmatrix} a_n & \cdots & a_0 & 0 & \cdots & x^{m-1} f_1 \\ 0 & a_n & \cdots & a_0 & 0 & x^{m-2} f_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & 0 & a_n & \cdots & f_1 \\ b_m & \cdots & b_0 & 0 & \cdots & x^{n-1} f_2 \\ 0 & b_m & \cdots & b_0 & 0 & x^{n-2} f_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & 0 & b_m & \cdots & f_2 \end{pmatrix},$$

which can written as $p(x,y) f_1(x,y) + q(x,y) f_2(x,y)$, where $p$ and $q$ are polynomials in variables $x$ and $y$.

## A.1 Resultant in $x$

In the case of the polynomials $h_1(x,y)$ and $h_2(x,y)$, the resultant in $x$, denoted by $Q_x(y)$, is the determinant of the matrix

$$\begin{pmatrix} -\lambda y & a_1(y) & -p\nu_1 y^2 \\ b_1(y) & (1-p)\nu_1 y^2 & 0 \\ 0 & b_1(y) & (1-p)\nu_1 y^2 \end{pmatrix},$$

where $a_1(y) = (\lambda + p\nu_1 + (1-p)\nu_2)y - (1-p)\nu_2$ and $b_1(y) = (1-p)((\nu_2 - \nu_1)y - \nu_2)$. Straightforward computations show that

$$Q_x(y) = -\nu_1 (1-p)^2 y^2 (y-1) \mathcal{Q}_x(y),$$

where

$$\mathcal{Q}_x(y) = \lambda \nu_1 y^2 + \nu_2(\nu_2 - \nu_1 + \lambda) y - \nu_2^2.$$

It is easily checked that the quadratic polynomial $\mathcal{Q}_x(y)$ has two roots of opposite sign, as mentioned in Sect. 3. The positive root is

$$y^* = \frac{\nu_2}{2\lambda\nu_1}\left(-(\nu_2 - \nu_1 + \lambda) + \sqrt{(\nu_2 - \nu_1 + \lambda)^2 + 4\lambda\nu_1}\right)$$

and the negative root is

$$y_* = \frac{\nu_2}{2\lambda\nu_1}\left(-(\nu_2 - \nu_1 + \lambda) - \sqrt{(\nu_2 - \nu_1 + \lambda)^2 + 4\lambda\nu_1}\right).$$

In addition, the value of this polynomial at the point 1 is equal to $\lambda(\nu_1 + \nu_2) - \nu_1\nu_2 = \nu_1\nu_2(\rho_1 + \rho_2 - 1) < 0$, which implies that $y^* > 1$.

### A.2 Resultant in $y$

The resultant in $y$ of the polynomials $h_1(x,y)$ and $h_2(x,y)$ is denoted by $Q_y(x)$ and is equal to the determinant of the matrix

$$\begin{pmatrix} -p\nu_1 & \alpha_1(x) & -(1-p)\nu_2 x & 0 \\ 0 & -p\nu_1 & \alpha_1(x) & -(1-p)\nu_2 x \\ (1-p)\nu_1 & (1-p)(\nu_2-\nu_1)x & -\nu_2(1-p)x & 0 \\ 0 & (1-p)\nu_1 & (1-p)(\nu_2-\nu_1)x & -\nu_2(1-p)x \end{pmatrix},$$

where $\alpha_1(x) = x(\lambda + p\nu_1 + (1-p)\nu_2 - \lambda x)$. Straightforward computations show that

$$Q_y(x) = -\nu_2\nu_1(1-p)^2 x^2 (x-1)\mathcal{Q}_y(x)$$

with $\mathcal{Q}_y(x) = \lambda^2 x^2 - (\lambda + \nu_1 + \nu_2)\lambda x + \nu_1\nu_2$. The quadratic polynomial $\mathcal{Q}_y(x)$ has two positive roots equal to

$$x^* = \frac{\lambda + \nu_1 + \nu_2 - \sqrt{(\lambda+\nu_1+\nu_2)^2 - 4\nu_1\nu_2}}{2\lambda}$$

and

$$x_* = \frac{\lambda + \nu_1 + \nu_2 + \sqrt{(\lambda+\nu_1+\nu_2)^2 - 4\nu_1\nu_2}}{2\lambda}$$

with $x^* < x_*$, and since $\mathcal{Q}_y(1) = \nu_1\nu_2(1-\rho_1-\rho_2) > 0$, $x^* > 1$.

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