The idle period of the finite G/M/1 queue with an interpretation in risk theory

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Abstract We consider a G/M/1 queue with restricted accessibility in the sense that the maximal workload is bounded by 1. If the current workload V_t of the queue plus the service time of an arriving customer exceeds 1, only $1 - V_t$ of the service requirement is accepted. We are interested in the distribution of the idle period, which can be interpreted as the deficit at ruin for a risk reserve process R_t in the compound Poisson risk model. For this risk process a special dividend strategy applies, where the insurance company pays out all the income whenever R_t reaches level 1. In the queueing context we further introduce a set-up time $a \in [0, 1]$. At the end of every idle period, an arriving customer has to wait for a time units until the server is ready to serve it.

Keywords Finite G/M/1 · Finite M/G/1 · Workload · Idle period · Sample path analysis · Level crossing · Risk process · Deficit at ruin

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1 Introduction

Queues with workload restrictions have been studied extensively and appear in various settings and under different synonyms in the mathematical literature: “queues

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Fig. 1 The workload process V_t and the associated risk reserve process R_t . The idle period of the finite G/M/1 queue coincides with the deficit at ruin of the risk process

with restricted accessibility,” “finite-buffer queues,” “uniformly bounded virtual waiting time,” “limited queueing waiting time,” “finite dam,” etc. [7, 13, 15, 16, 19, 25–29, 31, 36]. The current paper can be seen as a continuation of [1], with the following extra feature. We investigate a G/M/1 queue with restricted accessibility in the sense that the workload (virtual waiting time) V_t is bounded by 1. If V_t plus the service time of an arriving customer exceeds 1, only $1 - V_t$ of the service requirement is accepted.

This paper focusses on the study of I , the idle period of the finite queue, i.e. the duration of the period between the time when the queue becomes empty and the next customer arrives. The random variable I can also be interpreted as the deficit at ruin of a modified risk reserve process R_t in the compound Poisson case with a constant barrier strategy (see Fig. 1). When the risk reserve process reaches level 1, dividends are paid out with constant rate equal to 1, so that R_t stays constant until the next claim occurs. Risk models with similar dividend strategies have been intensively studied in the insurance risk literature—often with a focus on optimality of the chosen barrier. We refer to [2, 3, 6, 17, 23], where risk models with different dividend strategies are treated. Models with constant dividend barrier have been investigated in [22] and [24]. In [24] an integro-differential equation for the Gerber–Shiu discounted penalty function is found. Using this result the moments of the deficit at ruin, which is identical to our idle period I , are determined. In our study we focus on the law of I and find expressions for the Laplace–Stieltjes transform (LST) and distribution, the latter in the case with set-up time. We introduce two different methods to find these quantities. In [22] the model [24] was extended to hypoexponential claim inter-arrival times. For a general introduction to risk models, see [5] or [32].

After introducing notation in Sect. 2, we give a short overview of the G/M/1 queueing model and related useful results in Sect. 3. An auxiliary result for the conditional distribution of idle period, given the workload process reaches level 1, is derived in Sect. 4. In the following sections we present two new methods to derive relations for the LST and probability distribution function of I ; both methods are

based on a sample path analysis. The first method, demonstrated in Sect. 5, uses the idea of collecting subsequent overshoots over level 1 of the original G/M/1 workload process to form the service times of a new queue that can again be identified as a G/M/1 queue. Using this construction, the LST of the idle period of the finite queue without set-up time is derived. For the special case of an M/M/1 queue, the result is then checked in Sect. 6. The second method is presented in Sect. 7. It is based on the observation that the idle period of the original queue can be seen as the overshoot of the workload process in a dual M/G/1 queue. We use this observation and regeneration theory to construct a modified M/G/1-type process and derive a formula for the distribution of the idle period in the case with set-up time.

Level-crossing and sample-path based arguments similar to the ones used in the present paper can be found also in [9, 20].

2 Preliminaries and notation

We have already mentioned the workload process V_t , the set-up time a and the idle period I of the finite G/M/1 queue. Whenever it is important to distinguish the successive idle periods, we write $I_1, I_2, \dots$ for the first, second, and the subsequent idle periods; otherwise we use a generic random variable I . Note that $I_1, I_2, \dots$ form an i.i.d. sequence of random variables.

Let $S_1, S_2, \dots$ denote the inter-arrival times of the customers and let F_S be the distribution function of S_1 , with $1/\mu = E(S_1)$. Let $Z_1, Z_2, \dots$ denote the exponential service times, having mean $E(Z_1) = 1/\lambda$.

We let $\rho = \lambda/\mu$, so that $1/\rho$ is the traffic intensity of the G/M/1 queue. Since we are concerned with the finite queue, we can ignore all stability issues and investigate both $\rho > 1$, when the standard G/M/1 queue is stable, and $\rho \leq 1$, when it is unstable.

If not otherwise stated, we denote the LST $E(e^{-sX})$ of a random variable X by $\phi_X(s)$. The convolution of two probability distribution functions F and G or two functions f and g is denoted by $F * G$ and $f * g$. F^{j*} and f^{j*} denote the j -fold convolution of F and f respectively.

3 The standard G/M/1 queue

It is instructive to first review the standard G/M/1 queue and introduce some known relevant results for the standard case with no set-up time, i.e. $a = 0$. Let $\widehat{V}_t$ denote the workload process of this queue and let $\widehat{T}$ denote the idle period. The LST of $\widehat{T}$ is then given by

$$\phi_{\widehat{T}}(s) = \lambda \cdot \frac{z - \phi_S(s)}{s - \lambda(1 - z)}, \quad (1)$$

where z is the smallest positive root of $z = \phi_S(\lambda(1 - z))$ (see [30], p. 35, and [1]) and ϕ_S is the LST of S . An inversion is possible for $\rho \geq 1$, when Lagrange's theorem

yields (see [34])

$$z = \sum_{j=1}^{\infty} \frac{\lambda^{j-1}}{j!} \int_0^{\infty} x^{j-1} e^{-\lambda x} dF_S^{j*}(x).$$

In the case that $\rho < 1$ we have $z = 1$, so that (1) reduces to

$$\phi_{\hat{T}}(s) = \lambda \cdot \frac{1 - \phi_S(s)}{s}. \quad (2)$$

The distribution function of $\hat{T}$ is then given by

$$F_{\hat{T}}(x) = \lambda \int_0^x (1 - F_S(u)) du$$

(cf. (4) in [18] for the risk process context). Note that $(1 - \phi_S(s))/\mu s$ is the LST of the limit of the residual lifetime in a renewal process with epochs having distribution F_S and that $\phi_{\hat{T}}(s)$ is a transform of a defective probability distribution function; in particular, $P(\hat{T} < \infty) = \phi_{\hat{T}}(0+) = \rho$.

In the sequel the so-called dual M/G/1 queue is obtained by interchanging the inter-arrival and service times, so that $Z_1, Z_2, \dots$ denote the inter-arrival times and the variables $S_1, S_2, \dots$ become the successive service times of the dual M/G/1 queue. Note that when $\rho > 1$, the G/M/1 queue is stable, while the dual M/G/1 queue is not. On the other hand, if $\rho < 1$ then the M/G/1 queue is stable, but not the G/M/1 queue. In the latter case, the workload process $\hat{W}_t$ of the dual M/G/1 queue has a stationary distribution $\hat{F}$ with LST

$$\phi_{\hat{F}}(s) = \frac{1 - \rho}{1 - \phi_{\hat{T}}(s)}, \quad (3)$$

which is the transform version of the Pollaczek–Khintchine formula.

Let π be the probability that $\hat{V}_t$ up-crosses level 1 during a busy period and let η be the probability that starting in 1, the process hits 0 before it returns to 1. It has been shown that if $\rho < 1$ then

$$\pi = \frac{1 - \rho}{\hat{F}(1)} \quad (4)$$

and

$$\eta = 1 - \frac{\hat{F} * F_S(1)}{\hat{F}(1)} = \frac{\hat{f}(1)}{\lambda \hat{F}(1)}, \quad (5)$$

where $\hat{f}(x)$ is the density of $\hat{F}(x)$ for $x > 0$ and the second step of (5) follows from the Pollaczek–Khintchine formula. These formulas can be found in [1, 12, 15, 26, 33, 35]. Note that in the notation of [33] the r.h.s. of (4) is given by $1 - q$ while the r.h.s. of (5) is denoted by $1 - p$. In the notation of [1], $\theta_1(0, 1)$ represents the π in (4) and $P(T_1^- < T_x^+)$ is η in (5) (also note that in [1] the symbols μ

Fig. 2 Visualization of the proof of Theorem 1

and λ have a reversed meaning). For $\rho > 1$ the probability π , the probability that the cycle maximum is larger than 1, is given in Theorem 4 of [1].

Further references and results about the standard G/M/1 queue can be found in [1, 4, 10, 15, 21].

4 Conditional idle period

We derive a formula for the conditional distribution of the idle period of the standard G/M/1 queue, given the event that the workload process exceeds level 1 during a busy period; or equivalently: we present an expression for the distribution of the deficit at ruin of the risk reserve process R_t , given that some dividends were paid out.

Let $\hat{V}^{\max}$ be the maximum of $\hat{V}_t$ during the first busy cycle and let H be the conditional distribution function of the idle period $\hat{T}$, given the event $\{\hat{V}^{\max} \geq 1\}$.

Theorem 1 *The conditional distribution H is equal to the residual lifetime distribution at time $t = 1$ of a renewal process with renewal times having the same distribution as the idle period $\hat{T}$.*

Proof By tracing Fig. 2 for a typical sample path of $\hat{V}_t$, let T^* and T_1 denote the last up- and down-crossing times of level 1 before the idle period starts. T_1 is the endpoint of an excess period of $\hat{V}_t$ over level 1. The time X_1 from T_1 to the next arrival has the same distribution as that of the G/M/1 idle period, since the interval $T_1 - T^*$ can be seen as the busy period of a G/M/1 queue (indicated by a gray area). At time $T_1 + X_1$ another busy period of a G/M/1 queue starts; it ends at time T_2 . Again, the distribution of X_2 is the same as that of an idle period and we see that this property also holds for $X_3, X_4, \dots$. Thus, the sequence $X_1, X_2, \dots$ is i.i.d. and forms a renewal process. Let $\hat{T} = X_1 + X_2 + \dots + X_k - 1$, where $k = \inf\{k | X_1 + \dots + X_k > 1\}$; $\hat{T}$ can be seen as its residual lifetime at time $t = 1$ of the renewal process. $\square$

Expressions for the distribution of the residual lifetime of a renewal process can be found in [8] and [15]. We note that H is a solution of the renewal equation ([4], p. 143)

$$H(x) = (F_{\hat{T}}(1+x) - F_{\hat{T}}(1)) + F_{\hat{T}} * H(x). \quad (6)$$

Fig. 3 Construction used in the proof of Theorem 2

5 The idle period without set-up time

In this section we let the set-up time $a = 0$. We are interested in the LST of the idle period I of the finite G/M/1 queue.

Theorem 2 *The LST of I is given by*

$$\phi_I(s) = \frac{\phi_L(s)}{\pi + (1 - \pi)\phi_L(s)}, \quad (7)$$

with

$$\phi_L(s) = \begin{cases} z_0 - (z_0 - 1 + \frac{s}{\lambda\eta})\phi_D(s), & \rho \geq 1 \\ 1 - \frac{s}{\lambda\eta}\phi_D(s), & \rho < 1 \end{cases} \quad (8)$$

where z_0 is the smallest positive root of $z_0 = \phi_L(\lambda\eta(1 - z_0))$ and

$$\phi_D(s) = \frac{\pi\phi_H(s)}{1 - \phi_{\hat{I}}(s) + \pi\phi_H(s)}. \quad (9)$$

Here $\phi_H(s) = \int_0^\infty e^{-su} dH(u)$, where H is the conditional distribution of the idle period $\hat{I}$, given the event $\{\hat{V}^{\max} \geq 1\}$.

Before we prove Theorem 2, we note that the LST $\phi_{\hat{I}}$ is given in (1) and that ϕ_H can in principle be calculated from Theorem 1. For $\rho \geq 1$, (8) is implicit in the sense that $\phi_L(s)$ is given in terms of the root z_0 , which itself can be determined only when ϕ_L is known.

Proof The dashed line in Fig. 3(a) shows the standard G/M/1 workload process and the solid line shows the finite G/M/1 queue. Let K_1 be the number of overshoots $U_{1,1}, U_{1,2}, \dots, U_{1,K_1}$ of level 1 during the first busy period R_1 of the finite queue. Note that $\pi = P(K_1 > 0)$ and $\eta = P(K_1 = 1 | K_1 > 0)$. We construct a third queue as follows. We collect the overshoots $U_{1,1}, U_{1,2}, \dots, U_{1,K_1}$ during R_1 and let them be

processed during the idle period I_1 by a new server. As soon as a new customer arrives, this new server stops working and collects the overshoots $U_{2,1}, U_{2,2}, \dots, U_{2,K_2}$ during R_2 of the finite queue. The server then continues working during the second idle period, and so on. Once all the overshoots are processed, the server becomes idle until a busy period with overshoots ends. Finally we remove the inactive periods $R_1, R_2, \dots$, so that we end up with a situation as shown in Fig. 3(b). Let L_k, A_k and D_k denote the inter-arrival times, service times and the idle periods of this new queue. Note that all three variables can be represented as geometric sums. Indeed, we have

$$L_1 = \sum_{i=1}^M I_i, \quad (10)$$

where M is a geometrically distributed random variable with support $M \geq 1$ and $P(M = 1) = \pi$, representing the number of idle periods of the finite queue during a cycle of the standard queue. From (10) it follows that

$$\phi_L(s) = \sum_{i=1}^{\infty} \phi_I(s)^i \pi (1 - \pi)^{i-1} = \frac{\pi \phi_I(s)}{1 - (1 - \pi) \phi_I(s)},$$

and then relation (7) follows immediately.

The first service time of the new queue A_1 consists of the cumulated overshoots during the first busy period of the finite queue,

$$A_1 = \sum_{i=1}^K U_{1,i}, \quad (11)$$

where the variable K denotes the number of overshoots of level 1 during a cycle of the finite queue. K is also geometric with $K \geq 1$ and $P(K = 1) = \eta$. Since the U_i are exponential with LST $\phi_U(s) = \lambda/(\lambda + s)$, we obtain

$$\phi_A(s) = \frac{\lambda}{\lambda + s} \frac{\eta}{1 - (1 - \eta) \frac{\lambda}{\lambda + s}} = \frac{\lambda \eta}{s + \lambda \eta},$$

so that the A_i are exponentially distributed with rate $\lambda \eta$.

It follows that the new queue is again a G/M/1 queue with service rate $\lambda \eta$. We can apply formula (1) for the law of its idle period D_1 . Hence, replacing λ by $\lambda \eta$, z by z_0 and ϕ_S by ϕ_L in (1), we obtain

$$\phi_D(s) = \lambda \eta \cdot \frac{z_0 - \phi_L(s)}{s - \lambda \eta (1 - z_0)},$$

and (8) follows. But D_1 can also be seen as the sum of the idle periods of the standard queue (see Fig. 3(b)),

$$D_1 = \hat{I}_1 + \sum_{i=1}^{M-1} \hat{I}_{i+1}. \quad (12)$$

The LST of $\hat{I}_1$ is ϕ_H , the transform of the conditional distribution of the idle period of the standard G/M/1 queue, given level 1 is reached during the first busy period. The variables $\hat{I}_2, \hat{I}_3, \dots$ are i.i.d. and have the same distribution as the idle period of the standard G/M/1 queue conditioned on the event that level 1 was not reached during the last busy period. Letting ϕ^+ denote the LST of $\hat{I}_2$, we thus have

$$\phi_D(s) = \phi_H(s) \cdot \frac{\pi}{1 - (1 - \pi)\phi^+(s)}.$$

From the law of total probability we obtain $\phi_{\hat{I}}(s) = (1 - \pi)\phi^+(s) + \pi\phi_H(s)$ and hence (8) follows. $\square$

6 The M/M/1 special case

In this section we check Theorem 2 for the M/M/1 case, where the inter-arrival distribution is exponential with mean $1/\mu$. We assume that $\rho < 1$. Then $\phi_S(s) = \mu/(\mu + s)$ and the smallest positive root of $z = \phi_S(\lambda(1 - z))$ is simply $z = 1$. Consequently we have

$$\phi_{\hat{I}}(s) = \lambda \cdot \frac{1 - \phi_S(s)}{s} = \rho \frac{\mu}{\mu + s},$$

for the idle period of the standard G/M/1 queue. According to (3), the LST of the stationary distribution of $\hat{W}$ is given by

$$\phi_{\hat{F}}(s) = \frac{1 - \rho}{1 - \phi_{\hat{I}}(s)} = \frac{1 - \rho}{1 - \rho \frac{\mu}{\mu + s}}$$

which is the LST of $\hat{F}(x) = 1 - \frac{\lambda}{\mu} e^{-(\mu - \lambda)x}$. Hence it follows that the probabilities η and π are given by

$$\eta = \frac{\mu - \lambda}{\mu e^{(\mu - \lambda)} - \lambda} = \pi e^{-(\mu - \lambda)} \quad \text{and} \quad \pi = \frac{\mu - \lambda}{\mu - \lambda e^{-(\mu - \lambda)}} = \eta e^{(\mu - \lambda)}.$$

To find the transform ϕ_H of Z , the residual lifetime from Theorem 1, let ℓ be the number of finite renewals in a renewal process $X_1, X_2, \dots$ with defective inter-arrival distribution $F_{\hat{I}}$ and let $S = \sum_{i=1}^{\ell} X_i$. The random variable ℓ has a geometric distribution with $P(\ell = 0) = 1 - \rho$, so that S is the sum of ℓ exponential random variables with rate μ and thus exponentially distributed with rate $\mu(1 - \rho) = \mu - \lambda$. It follows that

$$P(Z \leq x) = P(S > 1, S - 1 \leq x, \ell > 0) = P(S > 1, X \leq x, \ell > 0)$$

where X is exponential with mean $1/\mu$, independent of S . Hence

$$P(Z \leq x) = \rho P(S > 1 | \ell > 0) (1 - e^{-\mu x}) = \rho e^{-(\mu - \lambda)} (1 - e^{-\mu x})$$

with transform $\phi_H(s) = e^{-(\mu-\lambda)\frac{\lambda}{\mu+s}}$. It follows from (9) that

$$\phi_D(s) = \frac{\pi \phi_H(s)}{1 - \phi_I(s) + \pi \phi_H(s)} = \frac{\lambda \eta}{s + \mu \pi}.$$

Equations (8) and (7) yield

$$\phi_L(s) = 1 - \frac{s}{\lambda \eta} \phi_D(s) = \frac{\mu \pi}{\mu \pi + s}$$

and then

$$\phi_I(s) = \frac{\phi_L(s)}{\pi + (1 - \pi)\phi_L(s)} = \frac{\mu}{s + \mu}.$$

Hence I is exponentially distributed, as expected by the lack of memory property.

7 Idle period with set-up time

We now consider the idle period of the finite G/M/1 queue, with set-up time $a \in (0, 1]$, i.e. after each busy period the workload starts with an initial value being equal in distribution to $Z_1 + a$.

The upper diagram in Fig. 4 shows the workload process V_t , together with the first cycle C_1 of that queue.

From V_t we construct a new process $\bar{R}_t$, representing the time elapsed since the arrival of the customer being served. $\bar{R}_t$ is obtained from the risk reserve process R_t by removing the time intervals where $R_t = 1$ (a similar construction can be found in [9]).

Next we define the process $W_t = 1 - \bar{R}_{a+t}$ for $0 \leq t < C_1 - a$, yielding a first cycle for W_t . In other words, we remove the set-up times and flip the process $\bar{R}_t$. At time $t = C_1 - a$ we let the process W_t restart at $1 - a$ and define the second cycle of W_t via $W_t = 1 - \bar{R}_{2a+t}$ for $C_1 - a \leq t < C_2 - 2a$. Continuing in this way, cycle by cycle, we define a regenerative process W_t for $t \geq 0$.

By construction, the idle periods $I_1, I_2, \dots$ of the finite G/M/1 queue are identical to the overshoots of the process W_t over level 1. Since the overflows occur only once in a cycle, it follows that the I_i are i.i.d. with common distribution function denoted by F_I .

Being a regenerative process with finite cycle mean, the process W_t is stable in the sense that W_t converges in distribution to some W as $t \rightarrow \infty$. Let f denote the density of the distribution of W .

Lemma 3 *The density function f fulfills the equation*

$$f(x) = \begin{cases} ch(x) + \rho h * f(x), & 0 \leq x < 1 - a, \\ ch(x) - d + \rho h * f(x), & 1 - a \leq x < 1, \\ ch(x) + \rho \int_0^1 h(x-y)f(y)dy, & x \geq 1, \end{cases} \quad (13)$$

Fig. 4 Construction of the processes $\bar{R}$ and W

where $c = f(0)/\mu$, $d = ch(1) + \rho \int_0^1 h(1-y)f(y)dy$ and $h(x) = \mu(1 - F_S(x))$ is the equilibrium density of the inter-arrival distribution F_S .

Proof Let $D_x(C)$ and $U_x(C)$ denote the number of down- and up-crossings of level x by W_t during the first cycle C . By level crossing theory [11, 14, 26] the long-run average number of down-crossings is given by $E(D_x(C))/E(C) = f(x)$, for all $x \geq 0$. By similar reasoning as in [26] we find that for $x < 1 - a$ the average number of up-crossings is given by

$$\frac{E(U_x(C))}{E(C)} = f(0)(1 - F_S(x)) + \lambda \int_0^x (1 - F_S(x-u))f(u)du = ch(x) + \rho h * f(x),$$

and equating the two averages leads to the first line in (13).

For $x \geq 1$ the number of up- and down-crossings is again equal and the average number of up-crossings is $f(0)(1 - F_S(x)) + \lambda \int_0^1 (1 - F_S(x-u))f(u)du$, since there are no jumps from above level 1.

If $x \in [1 - a, 1)$ then $D_x(C) = U_x(C) - 1$, since after crossing level 1 the process never returns to $[0, x)$ during the cycle. Hence

$$\begin{aligned} \frac{E(D_x(C))}{E(C)} &= \frac{E(U_x(C))}{E(C)} - \frac{1}{E(C)} \\ &= f(0)(1 - F_S(x)) + \lambda \int_0^x (1 - F_S(x - u)) f(u) du - \frac{1}{E(C)}. \end{aligned}$$

Now, since W_t crosses level 1 exactly once every cycle, we have $f(1+) = 1/E(C)$. But $f(1+) = d$ and hence the second equation in (13) follows. $\square$

Solving for the renewal equation $f(x) = ch(x) + \rho h(x) * f$ in $[0, 1 - a)$ (i.e. finding a solution f that fulfills the equation only on $[0, 1 - a)$) we obtain

$$f(x) = ch * \Delta(x), \quad x \in [0, 1 - a),$$

where $\Delta(x) = \sum_{n=0}^{\infty} \rho^n h^{*n}(x)$ and h^{*n} is the n -fold convolution of h with itself. Similarly for $x \in [1 - a, 1)$, we have the renewal equation $f(x) = (ch(x) - d) + \rho h * f(x)$. Letting $\hat{d}(x) = d1_{\{x \geq 1\}}$ we get

$$\begin{aligned} f(x) &= (ch - \hat{d}) * \Delta(x) = ch * \Delta(x) - d \int_0^x 1_{\{x-u \geq 1\}} d\Delta(u) \\ &= ch * \Delta(x) - d\Delta((x - 1)-). \end{aligned} \tag{14}$$

Finally for the interval $[1, \infty)$ we have

$$f(x) = ch(x) + \rho \int_0^1 h(x - y)f(y) dy,$$

where $f(y)$ is already known for $y \in [0, 1)$. To find the two constants c and d , note that from $d = ch(1) + \rho h * f(1-)$ and (13) it follows that $f(1-) = 0$. By using (14) we obtain $ch * \Delta(1) = d\Delta(0)$ and since $\Delta(0) = 1$,

$$\frac{d}{c} = h * \Delta(1).$$

The constants can then be determined from the normalizing condition

$$\int_0^{\infty} f(u) du = 1.$$

We now prove the main result of this section, relating the distribution of the idle period I of the finite G/M/1 queue to the stationary density f of W_t .

Theorem 4 *The distribution function of I is given by*

$$F_I(x) = 1 - \frac{f(1+x)}{f(1)}, \tag{15}$$

where f and F denote the density and distribution of $W = \lim_{t \rightarrow \infty} W_t$ (the latter limit is defined in terms of weak convergence).

Proof The conditional density f_W^c of $W - 1$, given that $W > 1$, is

$$f_W^c(x) = \frac{f(1+x)}{1-F(1)}.$$

By looking at the renewal process $I_1, I_2, \dots$ we conclude that f_W^c is also the density of the (equilibrium) forward recurrence times of that process. Hence $f_W^c(x) = (1 - F_I(x))/E(I)$, and

$$F_I(x) = 1 - E(I) \cdot f_W^c(x) = 1 - \frac{f(1+x)}{(1-F(1))/E(I)}.$$

By Lemma 3, $f(1) = 1/E(C)$ and by renewal theory, $1 - F(1) = E(I)/E(C)$. Consequently, $\frac{1-F(1)}{E(I)} = f(1)$ and thus (15) follows. $\square$

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