Erratum to: A Probabilistic Proof of the Fundamental Gap Conjecture Via the Coupling by Reflection

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Erratum to: Potential Anal

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In [3, Proposition A.7], we constructed a smooth function $u_0(x) = e^{-c_2|x|^2/2}\rho^\kappa(x)$ on $\bar{\Omega}$ which admits $\log \tilde{\phi}_0$ as its modulus of log-concavity, that is, for all $x, y \in \Omega$, $x \neq y$,

$$\left\langle \nabla \log u_0(y) - \nabla \log u_0(x), \frac{y-x}{|y-x|} \right\rangle \leq 2(\log \tilde{\phi}_0)' \left(\frac{|x-y|}{2} \right).$$

However, from the choice of κ in the proof of [3, Lemma A.6], we see that the function u_0 has zero gradient on the boundary $\partial\Omega$. Therefore, it does not satisfy the assumptions of [3, Lemma A.2]. In the following, we shall prove that, for any $0 < \kappa < 1$, there exists a function $u_\kappa \in C^\infty(\bar{\Omega}, \mathbb{R}_+)$ with non-zero gradient on $\partial\Omega$, which admits $\kappa \log \tilde{\phi}_0$ as its modulus of log-concavity. This is sufficient to establish the log-concavity estimate for the ground state, see Theorem 5 below. We use the same notations as in [3]. In particular, $\rho_{\partial\Omega} : \bar{\Omega} \rightarrow \mathbb{R}_+$ is the distance function to the boundary $\partial\Omega$ and $\rho \in C^\infty(\bar{\Omega})$ coincides with $\rho_{\partial\Omega}$ on $\partial_{r_0}\Omega = \{x \in \bar{\Omega} : \rho_{\partial\Omega}(x) \leq r_0\}$. Here we may assume that $r_0 > 0$ is small enough

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such that $\rho_{\partial\Omega}$ is smooth on $\partial_{2r_0}\Omega$. ρ can be chosen as $f \circ \rho_{\partial\Omega}$, where $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a smooth increasing function such that $f(t) = t$ for $t \in [0, r_0]$ and $f(t) = 3r_0/2$ for $t \geq 2r_0$.

First, the assertion of [3, Lemma A.5] can be strengthened as follows.

Lemma 1 *For any $\theta_0 \in (0, 1)$, we can find $\varepsilon_0 \in (0, r_0]$ such that*

$$\left\langle \nabla \rho(x), \frac{y-x}{|y-x|} \right\rangle \geq \theta_0, \quad \text{for all } x, y \in \Omega \text{ with } |x-y| \geq D - \varepsilon_0.$$

Proof We consider the function

$$F(x, y) = \left\langle \nabla \rho(x), \frac{y-x}{|y-x|} \right\rangle, \quad x, y \in \bar{\Omega} \text{ and } x \neq y,$$

which is continuous on the closed set $\{(x, y) \in \bar{\Omega} \times \bar{\Omega} : |x-y| \geq D/2\}$. Let $\bar{S}$ be the closed subset of $\bar{\Omega} \times \bar{\Omega}$ consisting of pairs of points such that their distance is exactly D , i.e. $\bar{S} = \{(x, y) \in \bar{\Omega} \times \bar{\Omega} : |x-y| = D\}$. If $(x_0, y_0) \in \bar{S}$, then we have $x_0, y_0 \in \partial\Omega$ and $\nabla \rho(x_0) = \nabla \rho_{\partial\Omega}(x_0) = N(x_0) = \frac{y_0 - x_0}{|y_0 - x_0|}$, thus $F(x_0, y_0) = 1$.

For sufficiently small $\varepsilon > 0$, we define

$$F_\varepsilon = \min \{F(x, y) : x, y \in \bar{\Omega} \text{ with } |x-y| \geq D - \varepsilon\}.$$

Then F_ε increases as ε tends to 0. Note that if $x \in \partial\Omega$, then $|\nabla \rho(x)| = |N(x)| = 1$, which implies $F_\varepsilon \leq 1$. Thus the limit $\bar{\theta} := \lim_{\varepsilon \rightarrow 0} F_\varepsilon$ exists and $\bar{\theta} \leq 1$. If $\bar{\theta} < 1$, then we can find a sequence $\{(x_n, y_n)\}_{n \geq 1} \subset \bar{\Omega} \times \bar{\Omega}$ such that $|x_n - y_n| \rightarrow D$ and $F(x_n, y_n) \leq \bar{\theta}$. Extracting a subsequence if necessary, we may assume $(x_n, y_n) \rightarrow (\bar{x}, \bar{y})$ as $n \rightarrow \infty$, hence $(\bar{x}, \bar{y}) \in \bar{S}$ and $F(\bar{x}, \bar{y}) \leq \bar{\theta} < 1$. This is a contradiction. Therefore, $\lim_{\varepsilon \rightarrow 0} F_\varepsilon = 1$ which yields the desired result. $\square$

The next result is used to replace [3, Lemma A.6].

Lemma 2 *For any $\kappa \in (0, 1)$, there exists a positive constant c_2 , depending on κ , such that for all $x, y \in \Omega$, $x \neq y$,*

$$\left\langle \nabla \log \rho(y) - \nabla \log \rho(x), \frac{y-x}{|y-x|} \right\rangle \leq 2\kappa (\log \tilde{\phi}_0)' \left(\frac{|x-y|}{2} \right) + c_2 |x-y|. \quad (1)$$

Proof We fix an arbitrary $\kappa \in (0, 1)$. Take $\theta_0 \in (\kappa, 1)$ and $\alpha > 0$ such that $\kappa = 4\theta_0/(4+\alpha)$. Let $\varepsilon_0 > 0$ be determined as in Lemma 1 for this θ_0 . Set $\varepsilon_1 = \min\{\varepsilon_0, \alpha/2c_1\}$, where c_1 is given in [3, Proposition A.4].

Fix any $x, y \in \Omega$, $x \neq y$. There are two different cases. First, if $|y-x| > D - \varepsilon_1$, then we can apply Lemma 1 to get

$$\begin{aligned} \left\langle \nabla \log \rho(y) - \nabla \log \rho(x), \frac{y-x}{|y-x|} \right\rangle &= - \left[\left\langle \frac{\nabla \rho(y)}{\rho(y)}, \frac{x-y}{|x-y|} \right\rangle + \left\langle \frac{\nabla \rho(x)}{\rho(x)}, \frac{y-x}{|y-x|} \right\rangle \right] \\ &\leq -\theta_0 \left[\frac{1}{\rho(y)} + \frac{1}{\rho(x)} \right]. \end{aligned} \quad (2)$$

Suppose the straight line passing from x to y intersects the boundary $\partial\Omega$ first at the point $\hat{x}$ and then at $\hat{y}$. We have

$$\rho_{\partial\Omega}(x) + |x-y| + \rho_{\partial\Omega}(y) \leq |\hat{x}-x| + |x-y| + |y-\hat{y}| = |\hat{x}-\hat{y}| \leq D.$$

Hence $\rho_{\partial\Omega}(x) + \rho_{\partial\Omega}(y) \leq D - |x - y| < \varepsilon_1$, thus $x, y \in \partial_{\varepsilon_1}\Omega \subset \partial_{r_0}\Omega$, which implies $\rho(x) = \rho_{\partial\Omega}(x)$ and $\rho(y) = \rho_{\partial\Omega}(y)$. Therefore,

$$\rho(x) + \rho(y) \leq D - |x - y| < \varepsilon_1. \quad (3)$$

As a result,

$$(\rho(x) + \rho(y))(D - |x - y|) \geq (\rho(x) + \rho(y))^2 \geq 4\rho(x)\rho(y),$$

which, together with [3, Proposition A.4], implies

$$\begin{aligned} -\left[\frac{1}{\rho(y)} + \frac{1}{\rho(x)}\right] &\leq -\frac{4}{D - |x - y|} = -\frac{4}{4 + \alpha} \left(\frac{4}{D - |x - y|} + \frac{\alpha}{D - |x - y|} \right) \\ &\leq \frac{4}{4 + \alpha} \left[2(\log \tilde{\phi}_0)' \left(\frac{|x - y|}{2} \right) + 2c_1 - \frac{\alpha}{D - |x - y|} \right] \\ &\leq \frac{8}{4 + \alpha} (\log \tilde{\phi}_0)' \left(\frac{|x - y|}{2} \right), \end{aligned}$$

where the last inequality follows from Eq. 3 and $\varepsilon_1 \leq \alpha/2c_1$. Combining this inequality with Eq. 2, we arrive at

$$\left\langle \nabla \log \rho(y) - \nabla \log \rho(x), \frac{y - x}{|y - x|} \right\rangle \leq 2\kappa (\log \tilde{\phi}_0)' \left(\frac{|x - y|}{2} \right).$$

Therefore, we obtain (1) with $c_2 = 0$ in this case.

Next, we consider the second case that $0 < |x - y| \leq D - \varepsilon_1$. Noticing that

$$(\log \tilde{\phi}_0)''(t) = \frac{\tilde{\phi}_0 \tilde{\phi}_0'' - (\tilde{\phi}_0')^2}{\tilde{\phi}_0^2}(t) = \tilde{V}(t) - \tilde{\lambda}_0 - \left(\frac{\tilde{\phi}_0'}{\tilde{\phi}_0} \right)^2(t),$$

we can find $\bar{c} > 0$ such that

$$\left| (\log \tilde{\phi}_0)''(t) \right| \leq \bar{c}, \quad \text{for all } t \in [0, (D - \varepsilon_1)/2].$$

Moreover, $\tilde{\phi}_0'(0) = 0$ since $\tilde{\phi}_0$ is an even function. Thus, for any $t \in [0, (D - \varepsilon_1)/2]$,

$$\left| (\log \tilde{\phi}_0)'(t) \right| = \left| (\log \tilde{\phi}_0)'(t) - (\log \tilde{\phi}_0)'(0) \right| \leq \bar{c}t. \quad (4)$$

Finally, it is clear (cf. [3, Lemma A.2(4)]) that $\nabla^2 \log \rho \leq \hat{c}$ for some $\hat{c} \geq 0$; hence

$$\begin{aligned} \left\langle \nabla \log \rho(y) - \nabla \log \rho(x), \frac{y - x}{|y - x|} \right\rangle &= \frac{1}{|x - y|} \int_0^1 \left\langle \nabla^2 \log \rho|_{x+t(y-x)}(y - x), y - x \right\rangle dt \\ &\leq \hat{c}|x - y|. \end{aligned}$$

Combining the above inequality with Eq. 4, we get (1) with $c_2 = \bar{c} + \hat{c}$ in the second case. $\square$

We can now prove

Proposition 3 Fix any $\kappa \in (0, 1)$ and let $c_2 = c_2(\kappa)$ be given in Lemma 2. Define the function

$$u_\kappa(x) = e^{-c_2|x|^2/2} \rho(x), \quad x \in \bar{\Omega}.$$

Then for all $x, y \in \Omega$, $x \neq y$, we have

$$\left\langle \nabla \log u_\kappa(y) - \nabla \log u_\kappa(x), \frac{y - x}{|y - x|} \right\rangle \leq 2\kappa (\log \tilde{\phi}_0)' \left(\frac{|x - y|}{2} \right).$$

Proof By the definition of u_κ , we have

$$\nabla \log u_\kappa(x) = -c_2x + \nabla \log \rho(x).$$

Thus, it follows from Lemma 2 that

$$\begin{aligned} \left\langle \nabla \log u_\kappa(y) - \nabla \log u_\kappa(x), \frac{y-x}{|y-x|} \right\rangle &= -c_2|y-x| + \left\langle \nabla \log \rho(y) - \nabla \log \rho(x), \frac{y-x}{|y-x|} \right\rangle \\ &\leq 2\kappa(\log \tilde{\phi}_0)' \left(\frac{|x-y|}{2} \right). \end{aligned}$$

The proof is complete. $\square$

Next, analogous results of [3, Proposition A.8] still hold after small modifications.

Proposition 4 *For any κ in a small left neighborhood of 1, there exists a smooth function $\psi_\kappa \in C^\infty([0, D/2])$ satisfying $\psi_\kappa(0) = 0$ and*

- (i) $\psi_\kappa'' + 2\psi_\kappa\psi_\kappa' - \tilde{V}' = 0$ on $[0, D/2]$;
- (ii) as $\kappa \uparrow 1$, ψ_κ converges to $(\log \tilde{\phi}_0)'$ pointwise on $[0, D/2]$;
- (iii) ψ_κ is a modulus of log-concavity of u_κ constructed in Proposition 3.

Proof We may assume $\tilde{V}$ is nonnegative on $[-D/2, D/2]$, since adding a positive constant C to $\tilde{V}$ does not change the eigen-functions $\{\tilde{\phi}_i\}_{i \geq 0}$, with the corresponding eigenvalues $\{\tilde{\lambda}_i + C\}_{i \geq 0}$. We follow the idea in the proof of [1, Proposition 3.2]. Note that $\psi := (\log \tilde{\phi}_0)'$ satisfies $\psi(0) = 0$ and $\psi' = \tilde{V} - \tilde{\lambda}_0 - \psi^2$ on the interval $(-D/2, D/2)$. Letting $q = \arctan \psi$, then $q(0) = 0$ and

$$q' - (\tilde{V} - \tilde{\lambda}_0) \cos^2 q + \sin^2 q = 0 \quad \text{on } [-D/2, D/2].$$

Here the derivatives of q at the endpoints are understood as right or left derivative, respectively. For $\kappa \in (0, 1)$, we consider the first order ODE

$$\begin{aligned} \frac{\partial q}{\partial z} - \left[(\tilde{V}/\kappa) - \tilde{\lambda}_0 \right] \cos^2 q + \kappa \sin^2 q &= 0, \quad |z| \leq D/2; \\ q(0, \kappa) &= 0. \end{aligned} \tag{5}$$

By the ODE comparison theorem (cf. [2, p.23, Corollary 1]), q is strictly decreasing in κ for $z > 0$. The choice $\kappa = 1$ corresponds to $q = \arctan \psi$; hence $q(D/2, 1) = -\pi/2$ and $q(z, 1) \in (-\pi/2, \pi/2)$ for $0 \leq z < D/2$.

Because $q(D/2, \kappa)$ is strictly decreasing in κ and $q(D/2, 1) = -\pi/2$, for $\kappa < 1$ in a small neighborhood of 1, we have $q(D/2, \kappa) > -\pi/2$ and $q(z, \kappa) \in (-\pi/2, \pi/2)$ for $0 \leq z < D/2$. Set $\psi_\kappa(z) = \kappa \tan q(z, \kappa)$, $|z| \leq D/2$. Direct calculations show that ψ_κ satisfies (i). Since $\kappa < 1$,

$$\tan q(z, \kappa) > \tan q(z, 1) = \psi(z) = (\log \tilde{\phi}_0)'(z) \quad \text{for } z \in (0, D/2).$$

Thus by Proposition 3, ψ_κ is a modulus of concavity of $\log u_\kappa$ for all κ in a small left neighborhood of 1, which means (iii) holds. It remains to check the second assertion. By the continuous dependence on κ of the solution $q(z, \kappa)$ (cf. [2, p.107, Corollary]), for any $z \in (0, D/2)$, $\tan q(z, \kappa) \rightarrow \tan q(z, 1)$ as κ grows to 1, which implies the pointwise convergence of ψ_κ to $(\log \tilde{\phi}_0)'$ on $[0, D/2)$. Therefore, assertion (ii) is also verified. $\square$

Finally, we show that the result of [3, Theorem 2.6] is still true.

Theorem 5 (Sharp log-concavity estimate). *Assume that the potential V has a modulus of convexity $\tilde{V}$. Then $\log \tilde{\phi}_0$ is a modulus of concavity for $\log \phi_0$.*

Proof Fix any $\kappa \in (0, 1)$. Let $u_\kappa \in C^\infty(\bar{\Omega})$ and $\psi_\kappa \in C^\infty([0, D/2])$ be the functions constructed in Propositions 3 and 4, respectively. Applying [3, Theorem 2.4] with $u_0 = u_\kappa$ and $\psi = \psi_\kappa$, we obtain that for all $x, y \in \Omega, x \neq y$,

$$\left\langle \nabla \log u(t, x) - \nabla \log u(t, y), \frac{x - y}{|x - y|} \right\rangle \leq 2\psi_\kappa \left(\frac{|x - y|}{2} \right) \quad (6)$$

for any $t > 0$. We deduce from the approximation lemma (see [3, Lemma A.3]) that for all $x \in \Omega$,

$$\lim_{t \rightarrow \infty} \nabla \log u(t, x) = \lim_{t \rightarrow \infty} \frac{\nabla u(t, x)}{u(t, x)} = \lim_{t \rightarrow \infty} \frac{\nabla(e^{\lambda_0 t} u(t, x))}{e^{\lambda_0 t} u(t, x)} = \frac{\nabla(a_0 \phi_0(x))}{a_0 \phi_0(x)} = \nabla \log \phi_0(x).$$

Therefore, letting $t \rightarrow \infty$ in Eq. 6 yields

$$\left\langle \nabla \log \phi_0(x) - \nabla \log \phi_0(y), \frac{x - y}{|x - y|} \right\rangle \leq 2\psi_\kappa \left(\frac{|x - y|}{2} \right).$$

By Proposition 4(ii), letting $\kappa \uparrow 1$ in the above inequality yields that for all $x, y \in \Omega, x \neq y$,

$$\left\langle \nabla \log \phi_0(x) - \nabla \log \phi_0(y), \frac{x - y}{|x - y|} \right\rangle \leq 2(\log \tilde{\phi}_0)' \left(\frac{|x - y|}{2} \right).$$

The proof is complete. $\square$

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References

1. Andrews, B., Clutterbuck, J.: Proof of the fundamental gap conjecture. *J. Amer. Math. Soc.* **24**(3), 899–916 (2011)
2. Birkhoff, G., Rota, G.-C.: Ordinary Differential Equations. 2nd edn. Blaisdell Publishing Co. Ginn and Co., Waltham, Mass.-Toronto, Ont.-London (1969)
3. Gong, F.-Z., Li, H.-Q., Luo, D.-J.: A probabilistic proof of the fundamental gap conjecture via the coupling by reflection. *Potential Anal.* doi:[10.1007/s11118-015-9476-3](https://doi.org/10.1007/s11118-015-9476-3)