$L^{(\alpha)}$ -Conjugates on Parabolic Bergman Spaces

Yôsuke Hishikawa · Masaharu Nishio ·
Masahiro Yamada

Received: 18 November 2011 / Accepted: 12 June 2013 / Published online: 22 June 2013
© The Author(s) 2013. This article is published with open access at SpringerLink.com

Abstract The parabolic Bergman space is the set of all L^p -solutions of the parabolic operator $L^{(\alpha)}$. In this paper, we define $L^{(\alpha)}$ -conjugates by using fractional derivatives, which are the extension of harmonic conjugates. We study several properties of $L^{(\alpha)}$ -conjugates on parabolic Bergman spaces.

Keywords Parabolic operator of fractional order · Heat equation · Bergman space · Conjugate function

Mathematics Subject Classifications (2010) Primary 35K05; Secondary 26A33 · 26D15

1 Introduction

Let H be the upper half-space of the $(n + 1)$ -dimensional Euclidean space $\mathbb{R}^{n+1}$ ($n \geq 1$), that is, $H = \{(x, t) \in \mathbb{R}^{n+1}; x \in \mathbb{R}^n, t > 0\}$. For $0 < \alpha \leq 1$, the parabolic operator $L^{(\alpha)}$ is defined by

$$L^{(\alpha)} = \partial_t + (-\Delta_x)^\alpha,$$

Y. Hishikawa (✉)

Department of General Education, Gifu National College of Technology,
Kamimakuwa 2236–2, Gifu 501–0495, Japan
e-mail: yosuke-h@gifu-nct.ac.jp

M. Nishio

Department of Mathematics, Osaka City University,
Sugimoto, Sumiyoshi 3–3–138, Osaka 558–8585, Japan
e-mail: nishio@sci.osaka-cu.ac.jp

M. Yamada

Department of Mathematics, Faculty of Education, Gifu University,
Yanagido 1–1, Gifu 501–1193, Japan
e-mail: yamada33@gifu-u.ac.jp

where $\partial_t = \partial/\partial t$ and Δ_x is the Laplacian with respect to x . Let $C(H)$ be the set of all real-valued continuous functions on H . For a positive integer k , $C^k(H)$ denotes the set of all k times continuously differentiable functions on H , and put $C^\infty(H) = \cap_k C^k(H)$. Furthermore, let $C_c^\infty(H)$ be the set of all functions in $C^\infty(H)$ with compact support. A function $u \in C(H)$ is said to be α -parabolic if $L^{(\alpha)}u = 0$ in the sense of distributions (for details, see Section 2). For $1 \leq p < \infty$ and $\lambda > -1$, $L^p(\lambda)$ is the set of all Lebesgue measurable functions f on H which satisfy

$$\|f\|_{L^p(\lambda)} := \left(\int_H |f(x, t)|^p t^\lambda dV(x, t) \right)^{\frac{1}{p}} < \infty,$$

where dV is the Lebesgue volume measure on H . The parabolic Bergman space $\mathbf{b}_\alpha^p(\lambda)$ is the set of all α -parabolic functions u on H which belong to $L^p(\lambda)$. We remark that $\mathbf{b}_\alpha^p(\lambda)$ is a Banach space with the norm $\|\cdot\|_{L^p(\lambda)}$, and $\mathbf{b}_{1/2}^p(\lambda)$ coincides with the harmonic Bergman space of Ramey and Yi [7]. Also, we note that $\mathbf{b}_\alpha^p(\lambda) = \{0\}$ when $\lambda \leq -1$ (see Proposition 4.3 of [3]).

Our aim of this paper is the study of $L^{(\alpha)}$ -conjugates on parabolic Bergman spaces. In [7], Ramey and Yi study harmonic conjugates on harmonic Bergman spaces. In [3], the authors study α -parabolic conjugate functions on parabolic Bergman spaces, which are the extension of harmonic conjugates of Ramey and Yi. However, our extension of [3] is not satisfactory, because there exist parabolic Bergman functions which do not have α -parabolic conjugate functions (see Theorem A, below). Hence, we consider to improve our extension in [3]. Therefore, in this paper, we introduce the new notion of conjugacy which is called $L^{(\alpha)}$ -conjugate, and investigate properties of $L^{(\alpha)}$ -conjugates on parabolic Bergman spaces.

We give some notations. For a real number κ , let $\mathcal{D}_t^\kappa = (-\partial_t)^\kappa$ be the fractional differential operator, and $\mathcal{FC}^\kappa$ the class of functions φ on $\mathbb{R}_+ = (0, \infty)$ such that $\mathcal{D}_t^\kappa \varphi$ is well-defined (for the explicit definitions of $\mathcal{D}_t^\kappa$ and $\mathcal{FC}^\kappa$, see Section 2). Let $\partial_j = \partial/\partial x_j$. And let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For a multi-index $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}_0^n$, let $\partial_x^\beta = \partial^{|\beta|}/\partial x_1^{\beta_1} \dots \partial x_n^{\beta_n}$, where $|\beta| = \beta_1 + \dots + \beta_n$. We begin with describing the definition of harmonic conjugates. Following [6], harmonic conjugates will be defined by the generalized Cauchy-Riemann equations below.

Definition A [6] Let u be a function on H . An n -tuple $(v_1, \dots, v_n)$ is called a harmonic conjugate of u if an $(n+1)$ -tuple $(v_1, \dots, v_n, u)$ satisfies the following generalized Cauchy-Riemann equations;

$$\partial_j v_k = \partial_k v_j \quad (1 \leq j, k \leq n), \quad (1.1)$$

$$\partial_j u = -\mathcal{D}_t v_j \quad (1 \leq j \leq n), \quad (1.2)$$

and

$$\mathcal{D}_t u = \sum_{j=1}^n \partial_j v_j. \quad (1.3)$$

In our previous paper [3], we extended the notion of harmonic conjugates by using fractional derivatives. In the following definition, we describe previous extension in [3].

Definition B (Definition 1 of [3]) Let u be a function on H . An n -tuple $(v_1, \dots, v_n)$ is called an α -parabolic conjugate function of u if $v_j \in C^1(H)$ and an $(n+1)$ -tuple $(v_1, \dots, v_n, u)$ satisfies Eqs. 1.1, 1.2, and the following equation;

$$\mathcal{D}_t^{\frac{1}{\alpha}-1} u = \sum_{j=1}^n \partial_j v_j. \quad (1.3')$$

The following theorem is the main result of [3].

Theorem A (Theorem 1 of [3]). Let $0 < \alpha \leq 1$, $1 \leq p < \infty$, $\lambda > -1$, and $u \in \mathbf{b}_\alpha^p(\lambda)$. If α , p , and λ satisfy the condition $\eta = p(\frac{1}{2\alpha} - 1) + \lambda > -1$, then there exists a unique α -parabolic conjugate function $(v_1, \dots, v_n)$ of u such that $v_j \in \mathbf{b}_\alpha^p(\eta)$. Also, there exists a constant $C > 0$ independent of u such that

$$C^{-1} \|u\|_{L^p(\lambda)} \leq \sum_{j=1}^n \|v_j\|_{L^p(\eta)} \leq C \|u\|_{L^p(\lambda)}.$$

We note that similar statements in Theorem A can not hold for the case $\eta = (\frac{1}{2\alpha} - 1)p + \lambda \leq -1$, because $\mathbf{b}_\alpha^p(\eta) = \{0\}$ when $\eta \leq -1$. For example, if $\alpha = 1$, $p = 2$, and $\lambda = 0$, then η does not satisfy the condition $\eta = (\frac{1}{2\alpha} - 1)p + \lambda > -1$. However, when $\alpha = 1$ and $n = 1$, Kochneff and Sagher introduced conjugate temperatures of heat solutions in [4]. An idea of [4] is effective in our investigation. Now, we introduce new notion of conjugacy which are called $L^{(\alpha)}$ -conjugates.

Definition 1 Let u be a function on H . An n -tuple $(v_1, \dots, v_n)$ is called an $L^{(\alpha)}$ -conjugate of u if $v_j(x, \cdot), u(x, \cdot) \in \mathcal{FC}^{\frac{1}{2\alpha}}$ and an $(n+1)$ -tuple $(v_1, \dots, v_n, u)$ satisfies the following equations;

$$\partial_j v_k = \partial_k v_j \quad (1 \leq j, k \leq n), \quad (\text{N.1})$$

$$\partial_j u = -\mathcal{D}_t^{\frac{1}{2\alpha}} v_j \quad (1 \leq j \leq n), \quad (\text{N.2})$$

and

$$\mathcal{D}_t^{\frac{1}{2\alpha}} u = \sum_{j=1}^n \partial_j v_j. \quad (\text{N.3})$$

We note that when $\alpha = 1/2$, Definition 1 coincides with Definition A. The following theorem is the main result of this paper.

Theorem 1 Let $0 < \alpha \leq 1$, $1 \leq p < \infty$, $\lambda > -1$, and $u \in \mathbf{b}_\alpha^p(\lambda)$. Then there exists a unique $L^{(\alpha)}$ -conjugate $(v_1, \dots, v_n)$ of u such that $v_j \in \mathbf{b}_\alpha^p(\lambda)$. Also, there exists a constant $C > 0$ independent of u such that

$$C^{-1} \|u\|_{L^p(\lambda)} \leq \sum_{j=1}^n \|v_j\|_{L^p(\lambda)} \leq C \|u\|_{L^p(\lambda)}. \quad (1.4)$$

Theorem 1 asserts that the $L^{(\alpha)}$ -conjugate of $u \in \mathbf{b}_\alpha^p(\lambda)$ always exists for all $0 < \alpha \leq 1$, $1 \leq p < \infty$, and $\lambda > -1$. Furthermore, for $u \in \mathbf{b}_\alpha^p(\lambda)$, the functions v_j belong to the same Banach space $\mathbf{b}_\alpha^p(\lambda)$.

In [3], we give the estimates of tangential derivative norms on $\mathbf{b}_\alpha^p(\lambda)$ by using Theorem A. We also obtain the same estimates by using Theorem 1, which are the following.

Theorem 2 *Let $0 < \alpha \leq 1$, $1 \leq p < \infty$, $\lambda > -1$, and $u \in \mathbf{b}_\alpha^p(\lambda)$. Then, for each $m \in \mathbb{N}_0$, there exists a constant $C > 0$ independent of u such that*

$$C^{-1} \|u\|_{L^p(\lambda)} \leq \sum_{|\gamma|=m} \left\| t^{\frac{m}{2\alpha}} \partial_x^\gamma u \right\|_{L^p(\lambda)} \leq C \|u\|_{L^p(\lambda)}. \quad (1.5)$$

Also, we give the inversion theorem.

Theorem 3 *Let $0 < \alpha \leq 1$, $1 \leq p < \infty$, and $\lambda > -1$. Suppose an n -tuple $(v_1, \dots, v_n)$ satisfies $v_j \in \mathbf{b}_\alpha^p(\lambda)$ and Eq. N.1. Then, there exists a unique function $u \in \mathbf{b}_\alpha^p(\lambda)$ such that $(v_1, \dots, v_n)$ is the $L^{(\alpha)}$ -conjugate of u . Also, there exists a constant $C > 0$ independent of $v_1, \dots, v_n$ such that*

$$C^{-1} \sum_{j=1}^n \|v_j\|_{L^p(\lambda)} \leq \|u\|_{L^p(\lambda)} \leq C \sum_{j=1}^n \|v_j\|_{L^p(\lambda)}. \quad (1.6)$$

This paper is constructed as follows. In Section 2, we present basic properties of parabolic Bergman functions. In Section 3, we give the proof of Theorems 1 and 2. In Section 4, we give the proof of Theorem 3. In Section 5, we give more consideration for $L^{(\alpha)}$ -conjugates. As an application, we study isomorphisms on parabolic Bergman spaces.

Throughout this paper, we will denote by C a positive constant whose value may not necessarily be the same at each occurrence.

2 Preliminaries

In this section, we present basic properties of parabolic Bergman functions. We give the definition of α -parabolic functions. We describe the operator $(-\Delta_x)^\alpha$. Since the case $\alpha = 1$ is trivial, we consider the case $0 < \alpha < 1$. For $0 < \alpha < 1$, $(-\Delta_x)^\alpha$ is the convolution operator defined by

$$(-\Delta_x)^\alpha \psi(x, t) = -c_{n,\alpha} \lim_{\delta \rightarrow 0^+} \int_{|y-x|>\delta} (\psi(y, t) - \psi(x, t)) |y-x|^{-n-2\alpha} dy$$

for all $\psi \in C_c^\infty(H)$ and $(x, t) \in H$, where $c_{n,\alpha} = -4^\alpha \pi^{-n/2} \Gamma((n+2\alpha)/2) / \Gamma(-\alpha) > 0$. A function $u \in C(H)$ is said to be α -parabolic on H if u satisfies the following condition: for every $\psi \in C_c^\infty(H)$,

$$\int_H |u \cdot \tilde{L}^{(\alpha)} \psi| dV < \infty \quad \text{and} \quad \int_H u \cdot \tilde{L}^{(\alpha)} \psi dV = 0,$$

where $\tilde{L}^{(\alpha)} = -\partial_t + (-\Delta_x)^\alpha$ is the adjoint operator of $L^{(\alpha)}$.

We introduce the fundamental solution of $L^{(\alpha)}$. For $x \in \mathbb{R}^n$, the fundamental solution $W^{(\alpha)}$ of $L^{(\alpha)}$ is defined by

$$W^{(\alpha)}(x, t) = \begin{cases} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp\left(-t|\xi|^{2\alpha} + \sqrt{-1} x \cdot \xi\right) d\xi & t > 0 \\ 0 & t \leq 0, \end{cases}$$

where $x \cdot \xi$ denotes the inner product on $\mathbb{R}^n$. It is known that $W^{(\alpha)}$ is α -parabolic on H and $W^{(\alpha)} \in C^\infty(H)$.

We present definitions of fractional integral and differential operators. Let $C(\mathbb{R}_+)$ be the set of all continuous functions on $\mathbb{R}_+ = (0, \infty)$. For a positive real number κ , let $\mathcal{FC}^{-\kappa}$ be the set of all functions $\varphi \in C(\mathbb{R}_+)$ such that there exists a constant $\varepsilon > 0$ with $\varphi(t) = O(t^{-\kappa-\varepsilon})$ as $t \rightarrow \infty$. We remark that $\mathcal{FC}^{-\nu} \subset \mathcal{FC}^{-\kappa}$ if $0 < \kappa \leq \nu$. For $\varphi \in \mathcal{FC}^{-\kappa}$, we can define the fractional integral of φ with order κ by

$$\mathcal{D}_t^{-\kappa} \varphi(t) = \frac{1}{\Gamma(\kappa)} \int_0^\infty \tau^{\kappa-1} \varphi(t+\tau) d\tau, \quad t \in \mathbb{R}_+. \quad (2.1)$$

Furthermore, let $\mathcal{FC}^\kappa$ be the set of all functions $\varphi \in C^{[\kappa]}(\mathbb{R}_+)$ such that $\partial_t^{[\kappa]} \varphi \in \mathcal{FC}^{-(\lceil \kappa \rceil - \kappa)}$, where $\lceil \kappa \rceil$ is the smallest integer greater than or equal to κ . In particular, we will write $\mathcal{FC}^0 = C(\mathbb{R}_+)$. For $\varphi \in \mathcal{FC}^\kappa$, we can also define the fractional derivative of φ with order κ by

$$\mathcal{D}_t^\kappa \varphi(t) = \mathcal{D}_t^{-(\lceil \kappa \rceil - \kappa)} (-\partial_t)^{[\kappa]} \varphi(t), \quad t \in \mathbb{R}_+. \quad (2.2)$$

Also, we define $\mathcal{D}_t^0 \varphi = \varphi$. For a real number κ , we may often call $\mathcal{D}_t^\kappa$ the fractional derivative of φ with order κ . Moreover, we call $\mathcal{D}_t^\kappa$ the fractional differential operator with order κ .

We recall the definition of the parabolic Bergman space. For $1 \leq p < \infty$ and $\lambda > -1$, the parabolic Bergman space $\mathbf{b}_\alpha^p(\lambda)$ is the set of all α -parabolic functions u on H with

$$\|u\|_{L^p(\lambda)} := \left(\int_H |u(x, t)|^p t^\lambda dV(x, t) \right)^{\frac{1}{p}} < \infty.$$

We present basic properties with respect to fractional derivatives for parabolic Bergman functions, which are given in Proposition 4.1 of [1].

Lemma 2.1 (Proposition 4.1 of [1]) *Let $0 < \alpha \leq 1$, $1 \leq p < \infty$, $\lambda > -1$, $\beta \in \mathbb{N}_0^n$, and $\kappa > -(\frac{n}{2\alpha} + \lambda + 1)\frac{1}{p}$ be a real number. If $u \in \mathbf{b}_\alpha^p(\lambda)$, then the following statements hold.*

(1) *The derivatives $\partial_x^\beta \mathcal{D}_t^\kappa u$ and $\mathcal{D}_t^\kappa \partial_x^\beta u$ are well-defined, and*

$$\partial_x^\beta \mathcal{D}_t^\kappa u(x, t) = \mathcal{D}_t^\kappa \partial_x^\beta u(x, t)$$

for all $(x, t) \in H$. Moreover, there exists a constant $C > 0$ such that

$$|\partial_x^\beta \mathcal{D}_t^\kappa u(x, t)| \leq Ct^{-\frac{|\beta|}{2\alpha} - \kappa - (\frac{n}{2\alpha} + \lambda + 1)\frac{1}{p}} \|u\|_{L^p(\lambda)}$$

for all $(x, t) \in H$.

(2) Let v be a real number such that $\kappa + v > -(\frac{n}{2\alpha} + \lambda + 1)\frac{1}{p}$. Then,

$$\mathcal{D}_t^v \partial_x^\beta \mathcal{D}_t^\kappa u(x, t) = \partial_x^\beta \mathcal{D}_t^{\kappa+v} u(x, t)$$

for all $(x, t) \in H$.

(3) $\partial_x^\beta \mathcal{D}_t^\kappa u$ is α -parabolic on H .

We also present the following lemma, which is given in [3].

Lemma 2.2 (Lemma 5.1 of [3]) Let $0 < \alpha \leq 1$, $1 \leq p < \infty$, $\lambda > -1$, and $u \in \mathbf{b}_\alpha^p(\lambda)$. Then,

$$\left(\mathcal{D}_t^{\frac{1}{p}} + \Delta_x\right)u(x, t) = 0$$

for all $(x, t) \in H$.

In [1], we give the following reproducing formula on $\mathbf{b}_\alpha^p(\lambda)$, which plays an important role for our argument.

Lemma 2.3 (Theorem 5.2 of [1]) Let $0 < \alpha \leq 1$, $1 \leq p < \infty$, and $\lambda > -1$. Also, let $v > -\frac{\lambda+1}{p}$ and $\kappa > \frac{\lambda+1}{p}$ be real numbers. Then, the reproducing formula

$$u(x, t) = C_{v+\kappa} \int_H \mathcal{D}_t^v u(y, s) \mathcal{D}_t^\kappa W^{(\alpha)}(x - y, t + s) s^{v+\kappa-1} dV(y, s) \quad (2.3)$$

holds for all $u \in \mathbf{b}_\alpha^p(\lambda)$ and $(x, t) \in H$, where $C_\kappa = 2^\kappa / \Gamma(\kappa)$. Moreover, the reproducing formula (2.3) also holds when $p = 1$ and $\kappa = \lambda + 1$.

Let $\gamma \in \mathbb{N}_0^n$ and $\kappa, \rho \in \mathbb{R}$. For a function f on H , let

$$P_\alpha^{\gamma, \kappa, \rho} f(x, t) := \int_H f(y, s) \partial_x^\gamma \mathcal{D}_t^\kappa W^{(\alpha)}(x - y, t + s) s^\rho dV(y, s), \quad (x, t) \in H.$$

In Theorem 3.1 of [3], the authors show the following result.

Lemma 2.4 (Theorem 3.1 of [3]) Let $0 < \alpha \leq 1$, $1 \leq p < \infty$, and $\sigma \in \mathbb{R}$. Suppose that a multi-index $\gamma \in \mathbb{N}_0^n$ and $\kappa, \rho \in \mathbb{R}$ with $\kappa > -\frac{n}{2\alpha}$ satisfy

$$\sigma - \rho p < p - 1 < \left(\frac{|\gamma|}{2\alpha} + \kappa\right)p + \sigma - \rho p.$$

Then, for every $f \in L^p(\sigma)$, the following statements hold.

(1) There exists a constant $C > 0$ independent of f such that

$$\|P_\alpha^{\gamma, \kappa, \rho} f\|_{L^p(\eta)} \leq C \|f\|_{L^p(\sigma)},$$

where $\eta = (\frac{|\gamma|}{2\alpha} + \kappa - \rho - 1)p + \sigma$. Moreover, $P_\alpha^{\gamma, \kappa, \rho} f$ is α -parabolic on H .

(2) Furthermore, let $\beta \in \mathbb{N}_0^n$ be a multi-index and $v \in \mathbb{R}$. If v satisfies

$$v + \kappa > -\frac{n}{2\alpha} \quad \text{and} \quad p - 1 < \left(\frac{|\gamma|}{2\alpha} + v + \kappa \right) p + \sigma - \rho p,$$

then $\partial_x^\beta \mathcal{D}_t^v P_\alpha^{\gamma, \kappa, \rho} f(x, t)$ is well-defined for every $(x, t) \in H$ and $\partial_x^\beta \mathcal{D}_t^v P_\alpha^{\gamma, \kappa, \rho} f = P_\alpha^{\beta + \gamma, v + \kappa, \rho} f$, that is,

$$\partial_x^\beta \mathcal{D}_t^v P_\alpha^{\gamma, \kappa, \rho} f(x, t) = \int_H f(y, s) \partial_x^{\beta + \gamma} \mathcal{D}_t^{v + \kappa} W^{(\alpha)}(x - y, t + s) s^\rho dV(y, s).$$

Consequently, put $\eta = \left(\frac{|\beta| + |\gamma|}{2\alpha} + v + \kappa - \rho - 1 \right) p + \sigma$, then there exists a constant $C > 0$ independent of f such that

$$\left\| \partial_x^\beta \mathcal{D}_t^v P_\alpha^{\gamma, \kappa, \rho} f \right\|_{L^p(\eta)} \leq C \|f\|_{L^p(\sigma)}$$

and $\partial_x^\beta \mathcal{D}_t^v P_\alpha^{\gamma, \kappa, \rho} f \in \mathbf{b}_\alpha^p(\eta)$.

Finally, we present the following lemma, which gives estimates of derivative norms on parabolic Bergman spaces.

Lemma 2.5 (Corollary 3.2 of [3]) *Let $0 < \alpha \leq 1$, $1 \leq p < \infty$, and $\lambda > -1$. Then, for a real number $v > -\frac{\lambda+1}{p}$, there exists a constant $C = C(n, p, \alpha, \lambda, v) > 0$ such that*

$$C^{-1} \|u\|_{L^p(\lambda)} \leq \|t^v \mathcal{D}_t^v u\|_{L^p(\lambda)} \leq \sum_{|\gamma| < v + \frac{\lambda+1}{p}} \left\| t^{\frac{|\gamma|}{2\alpha} + v - |\gamma|} \partial_x^\gamma \mathcal{D}_t^{v - |\gamma|} u \right\|_{L^p(\lambda)} \leq C \|u\|_{L^p(\lambda)}$$

for all $u \in \mathbf{b}_\alpha^p(\lambda)$, where $\gamma \in \mathbb{N}_0^n$ denotes a multi-index.

3 $L^{(\alpha)}$ -Conjugates of $\mathbf{b}_\alpha^p(\lambda)$ -Functions

In this section, we give the proofs of Theorems 1 and 2. First, we prepare the following proposition, which is used in the later argument.

Proposition 3.1 *Let $0 < \alpha \leq 1$, $1 \leq p < \infty$, $\lambda > -1$, and $\gamma \in \mathbb{N}_0^n$ be a multi-index. Also, let v be a real number which satisfies $v > -\frac{|\gamma|}{2\alpha} - \frac{\lambda+1}{p}$, and put $\eta = \left(\frac{|\gamma|}{2\alpha} + v \right) p + \lambda > -1$. Then, for $u \in \mathbf{b}_\alpha^p(\lambda)$, the derivative $\mathcal{D}_t^\gamma \partial_x^\gamma u$ is well-defined and belongs to $\mathbf{b}_\alpha^p(\eta)$. Moreover, there exists a constant $C > 0$ independent of u such that*

$$\left\| \mathcal{D}_t^\gamma \partial_x^\gamma u \right\|_{L^p(\eta)} \leq C \|u\|_{L^p(\lambda)}. \quad (3.1)$$

Proof Let $u \in \mathbf{b}_\alpha^p(\lambda)$. Then Lemma 2.3 implies that

$$\begin{aligned} u(x, t) &= C_{\lambda+2} \int_H u(y, s) \mathcal{D}_t^{\lambda+2} W^{(\alpha)}(x - y, t + s) s^{\lambda+1} dV(y, s) \\ &= C_{\lambda+2} P_\alpha^{0, \lambda+2, \lambda+1} u(x, t). \end{aligned}$$

Let $\eta' = \frac{|\gamma|}{2\alpha} p + \lambda$. Then by (2) of Lemma 2.4, we have

$$\partial_x^\gamma u = \partial_x^\gamma \left(C_{\lambda+2} P_\alpha^{0, \lambda+2, \lambda+1} u \right) = C_{\lambda+2} P_\alpha^{\gamma, \lambda+2, \lambda+1} u$$

and

$$\|\partial_x^\gamma u\|_{L^p(\eta')} = C_{\lambda+2} \|P_\alpha^{\gamma, \lambda+2, \lambda+1} u\|_{L^p(\eta')} \leq C \|u\|_{L^p(\lambda)}.$$

Consequently, $\partial_x^\gamma u$ belongs to $\mathbf{b}_\alpha^p(\eta')$. Let ν be a real number which satisfies $\nu > -\frac{|\gamma|}{2\alpha} - \frac{\lambda+1}{p}$. Moreover, put $\eta = \left(\frac{|\gamma|}{2\alpha} + \nu\right)p + \lambda$. Then by (2) of Lemma 2.4, we have

$$\mathcal{D}_t^\nu \partial_x^\gamma u = \mathcal{D}_t^\nu (C_{\lambda+2} P_\alpha^{\gamma, \lambda+2, \lambda+1} u) = C_{\lambda+2} P_\alpha^{\gamma, \nu+\lambda+2, \lambda+1} u$$

and

$$\|\mathcal{D}_t^\nu \partial_x^\gamma u\|_{L^p(\eta)} = C_{\lambda+2} \|P_\alpha^{\gamma, \nu+\lambda+2, \lambda+1} u\|_{L^p(\eta)} \leq C \|u\|_{L^p(\lambda)}.$$

Hence, we obtain the inequality (3.1), and $\mathcal{D}_t^\nu \partial_x^\gamma u$ belongs to $\mathbf{b}_\alpha^p(\eta)$. This completes the proof. $\square$

Now, we give the proof of Theorem 1.

Proof of Theorem 1 Let $u \in \mathbf{b}_\alpha^p(\lambda)$ and $1 \leq j \leq n$. By Proposition 3.1, we can define a function v_j on H by

$$v_j(x, t) = -\mathcal{D}_t^{-\frac{1}{2\alpha}} \partial_j u(x, t), \quad (x, t) \in H \quad (3.2)$$

and there exists a constant $C > 0$ independent of u such that

$$\|v_j\|_{L^p(\lambda)} \leq C \|u\|_{L^p(\lambda)}. \quad (3.3)$$

Consequently, v_j belongs to $\mathbf{b}_\alpha^p(\lambda)$.

We show that the $(n+1)$ -tuple $(v_1, \dots, v_n, u)$ satisfies Eqs. N.1, N.2, and N.3. We note that $\partial_j u = \mathcal{D}_t^0 \partial_j u$ belongs to $\mathbf{b}_\alpha^p(\lambda + \frac{p}{2\alpha})$ by Proposition 3.1. Therefore by (2) of Lemma 2.1,

$$\partial_k v_j = -\partial_k \mathcal{D}_t^{-\frac{1}{2\alpha}} \partial_j u = -\mathcal{D}_t^{-\frac{1}{2\alpha}} \partial_j \partial_k u, \quad (1 \leq j, k \leq n). \quad (3.4)$$

Hence, we have $\partial_k v_j = \partial_j v_k$, that is, Eq. N.1 is satisfied. Also, (2) of Lemma 2.1 implies that

$$-\mathcal{D}_t^{\frac{1}{2\alpha}} v_j = \mathcal{D}_t^{\frac{1}{2\alpha}} \mathcal{D}_t^{-\frac{1}{2\alpha}} \partial_j u = \partial_j u,$$

that is, Eq. N.2 is satisfied. We show that Eq. N.3 is satisfied. By Eq. 3.4, we have

$$\sum_{j=1}^n \partial_j v_j = \sum_{j=1}^n \left(-\mathcal{D}_t^{-\frac{1}{2\alpha}} \partial_j^2 u \right) = \mathcal{D}_t^{-\frac{1}{2\alpha}} \left(-\sum_{j=1}^n \partial_j^2 u \right) = \mathcal{D}_t^{-\frac{1}{2\alpha}} (-\Delta_x u).$$

Hence, Lemma 2.2 and (2) of Lemma 2.1 imply that

$$\sum_{j=1}^n \partial_j v_j = \mathcal{D}_t^{-\frac{1}{2\alpha}} \mathcal{D}_t^{\frac{1}{\alpha}} u = \mathcal{D}_t^{\frac{1}{2\alpha}} u.$$

Thus, Eq. N.3 is satisfied.

We show the inequalities (1.4). Since the second inequality of Eq. 1.4 is already given by Eq. 3.3, it suffices to show the first inequality of Eq. 1.4. By Lemma 2.5 with $v = \frac{1}{2\alpha}$, Eq. N.3, and Proposition 3.1, we have the following estimates.

$$\begin{aligned} \|u\|_{L^p(\lambda)} &\leq C \left\| t^{\frac{1}{2\alpha}} \mathcal{D}_t^{\frac{1}{2\alpha}} u \right\|_{L^p(\lambda)} = C \left\| t^{\frac{1}{2\alpha}} \sum_{j=1}^n \partial_j v_j \right\|_{L^p(\lambda)} \\ &\leq C \sum_{j=1}^n \left\| t^{\frac{1}{2\alpha}} \partial_j v_j \right\|_{L^p(\lambda)} \leq C \sum_{j=1}^n \|v_j\|_{L^p(\lambda)}. \end{aligned}$$

Therefore, we obtain the first inequality of Eq. 1.4.

Suppose that $(u_1, \dots, u_n)$ be an $L^{(\alpha)}$ -conjugate of u with $u_j \in \mathbf{b}_\alpha^p(\lambda)$. We show that $v_j = u_j$. By Lemma 2.5 and Eq. N.2, we obtain

$$\|v_j - u_j\|_{L^p(\lambda)} \leq C \left\| t^{\frac{1}{2\alpha}} \mathcal{D}_t^{\frac{1}{2\alpha}} (v_j - u_j) \right\|_{L^p(\lambda)} = C \left\| t^{\frac{1}{2\alpha}} (\partial_j u - \partial_j u) \right\|_{L^p(\lambda)} = 0.$$

Hence, we have $v_j = u_j$. This completes the proof. $\square$

As an application of Theorem 3.2, we give the estimates of tangential derivative norms on parabolic Bergman spaces, that is, we give the proof of Theorem 2.

Proof of Theorem 2 Let $u \in \mathbf{b}_\alpha^p(\lambda)$. Since the second inequality of Eq. 1.5 is given by Lemma 2.5 with $v = m$, it suffices to show the first inequality of Eq. 1.5. We show the first inequality of Eq. 1.5 by the induction on m . Let $m = 1$. Then, Theorem 3.2 implies that there exists the $L^{(\alpha)}$ -conjugate $(v_1, \dots, v_n)$ of u with $v_j \in \mathbf{b}_\alpha^p(\lambda)$ satisfying Eq. 1.4. Hence, Eq. 1.4, Lemma 2.5, and Eq. N.2 imply that

$$\|u\|_{L^p(\lambda)} \leq C \sum_{j=1}^n \|v_j\|_{L^p(\lambda)} \leq C \sum_{j=1}^n \left\| t^{\frac{1}{2\alpha}} \mathcal{D}_t^{\frac{1}{2\alpha}} v_j \right\|_{L^p(\lambda)} = C \sum_{j=1}^n \left\| t^{\frac{1}{2\alpha}} \partial_j u \right\|_{L^p(\lambda)}. \quad (3.5)$$

Therefore, we obtain $\|u\|_{L^p(\lambda)} \leq \sum_{|\gamma|=1} \|t^{\frac{|\gamma|}{2\alpha}} \partial_x^\gamma u\|_{L^p(\lambda)}$. We assume that the first inequality of Eq. 1.5 holds for some m . We show that the first inequality of Eq. 1.5 also holds for $m + 1$. Let $\eta = \lambda + \frac{m}{2\alpha} p$. Then by the hypothesis, there exists a constant $C > 0$ such that

$$\|u\|_{L^p(\lambda)} \leq C \sum_{|\gamma|=m} \left\| t^{\frac{m}{2\alpha}} \partial_x^\gamma u \right\|_{L^p(\lambda)} = C \sum_{|\gamma|=m} \left\| \partial_x^\gamma u \right\|_{L^p(\eta)}. \quad (3.6)$$

Since $v = \partial_x^\gamma u \in \mathbf{b}_\alpha^p(\eta)$ by Proposition 3.1, the inequality (3.5) implies that

$$\left\| \partial_x^\gamma u \right\|_{L^p(\eta)} = \|v\|_{L^p(\eta)} \leq C \sum_{j=1}^n \left\| t^{\frac{1}{2\alpha}} \partial_j v \right\|_{L^p(\eta)} = C \sum_{j=1}^n \left\| t^{\frac{1}{2\alpha}} \partial_j \partial_x^\gamma u \right\|_{L^p(\eta)}. \quad (3.7)$$

Hence by Eqs. 3.6 and 3.7, we have

$$\begin{aligned}\|u\|_{L^p(\lambda)} &\leq C \sum_{|\gamma|=m} \|\partial_x^\gamma u\|_{L^p(\eta)} \leq C \sum_{|\gamma|=m} \sum_{j=1}^n \left\| t^{\frac{1}{2\alpha}} \partial_j \partial_x^\gamma u \right\|_{L^p(\eta)} \\ &= C \sum_{|\gamma|=m} \sum_{j=1}^n \left\| t^{\frac{m+1}{2\alpha}} \partial_j \partial_x^\gamma u \right\|_{L^p(\lambda)}.\end{aligned}$$

Therefore, we obtain the first inequality of Eq. 1.5. This completes the proof. $\square$

4 Inversion Theorem

In this section, we give the inversion theorem, that is, we give the proof of Theorem 3.

Proof of Theorem 3 Let $v_j \in \mathbf{b}_\alpha^p(\lambda)$ which satisfies Eq. N.1. By Proposition 3.1, we can define a function u on H by

$$u(x, t) = \sum_{j=1}^n \mathcal{D}_t^{-\frac{1}{2\alpha}} \partial_j v_j(x, t), \quad (x, t) \in H$$

and there exists a constant $C > 0$ independent of $v_1, \dots, v_n$ such that

$$\|u\|_{L^p(\lambda)} \leq C \sum_{j=1}^n \left\| \mathcal{D}_t^{-\frac{1}{2\alpha}} \partial_j v_j \right\|_{L^p(\lambda)} \leq C \sum_{j=1}^n \|v_j\|_{L^p(\lambda)}. \quad (4.1)$$

Consequently, u belongs to $\mathbf{b}_\alpha^p(\lambda)$.

We show that the $(n+1)$ -tuple $(v_1, \dots, v_n, u)$ satisfies Eqs. N.2 and N.3. We note that $\partial_j v_j$ belongs to $\mathbf{b}_\alpha^p(\lambda + \frac{p}{2\alpha})$ by Proposition 3.1. By (1) of Lemma 2.1 and Eq. N.1, we have

$$\partial_k u = \sum_{j=1}^n \partial_k \mathcal{D}_t^{-\frac{1}{2\alpha}} \partial_j v_j = \sum_{j=1}^n \mathcal{D}_t^{-\frac{1}{2\alpha}} \partial_j \partial_k v_j = \mathcal{D}_t^{-\frac{1}{2\alpha}} \left(\sum_{j=1}^n \partial_j^2 v_k \right) = \mathcal{D}_t^{-\frac{1}{2\alpha}} (\Delta_x v_k).$$

Hence, by Lemma 2.2 and (2) of Lemma 2.1, we obtain

$$\partial_k u = \mathcal{D}_t^{-\frac{1}{2\alpha}} (\Delta_x v_k) = \mathcal{D}_t^{-\frac{1}{2\alpha}} \left(-\mathcal{D}_t^{\frac{1}{\alpha}} v_k \right) = -\mathcal{D}_t^{\frac{1}{2\alpha}} v_k,$$

that is, Eq. N.2 is satisfied. Also, (2) of Lemma 2.1 implies that

$$\mathcal{D}_t^{\frac{1}{2\alpha}} u = \mathcal{D}_t^{\frac{1}{2\alpha}} \left(\sum_{j=1}^n \mathcal{D}_t^{-\frac{1}{2\alpha}} \partial_j v_j \right) = \sum_{j=1}^n \mathcal{D}_t^{\frac{1}{2\alpha}} \mathcal{D}_t^{-\frac{1}{2\alpha}} \partial_j v_j = \sum_{j=1}^n \partial_j v_j,$$

that is, Eq. N.3 is satisfied.

Suppose that $v \in \mathbf{b}_\alpha^p(\lambda)$ and $(v_1, \dots, v_n)$ is an $L^{(\alpha)}$ -conjugate of v . We show that $u = v$. By Lemma 2.5 and Eq. N.3, we obtain

$$\begin{aligned} \|u - v\|_{L^p(\lambda)} &\leq C \left\| t^{\frac{1}{2\alpha}} \mathcal{D}_t^{\frac{1}{2\alpha}}(u - v) \right\|_{L^p(\lambda)} = C \left\| t^{\frac{1}{2\alpha}} \left(\mathcal{D}_t^{\frac{1}{2\alpha}} u - \mathcal{D}_t^{\frac{1}{2\alpha}} v \right) \right\|_{L^p(\lambda)} \\ &= C \left\| t^{\frac{1}{2\alpha}} \left(\sum_{k=1}^n \partial_k v_k - \sum_{k=1}^n \partial_k v_k \right) \right\|_{L^p(\lambda)} = 0. \end{aligned}$$

Thus, we have $u = v$.

The estimates of Eq. 1.6 are given by Theorem 3.2, directly. Thus, this completes the proof. $\square$

5 A Generalization of $L^{(\alpha)}$ -Conjugates and their Applications

In this section, we generalize the definition of $L^{(\alpha)}$ -conjugates, and investigate properties of generalized $L^{(\alpha)}$ -conjugates on parabolic Bergman spaces. First, we introduce the following definition.

Definition 5.1 Let κ and ν be real numbers. Also, let u be a function on H . An n -tuple $(v_1, \dots, v_n)$ is called a (κ, ν) -conjugate of u if $u, v_j \in C^1(H)$, $v_j(x, \cdot) \in \mathcal{FC}^\kappa$, $u(x, \cdot) \in \mathcal{FC}^\nu$, and $(v_1, \dots, v_n, u)$ satisfies the following equations;

$$\partial_j v_k = \partial_k v_j \quad (1 \leq j, k \leq n), \quad (5.1)$$

$$\partial_j u = -\mathcal{D}_t^\kappa v_j \quad (1 \leq j \leq n), \quad (5.2)$$

and

$$\mathcal{D}_t^\nu u = \sum_{j=1}^n \partial_j v_j. \quad (5.3)$$

We remark that a (κ, ν) -conjugate with $\kappa = \nu = \frac{1}{2\alpha}$ coincides with an $L^{(\alpha)}$ -conjugate of Definition 1. Also, a (κ, ν) -conjugate with $\kappa = 1$ and $\nu = \frac{1}{\alpha} - 1$ coincides with an α -parabolic conjugate function of Definition B. Hence, the notion of (κ, ν) -conjugates contains that of $L^{(\alpha)}$ -conjugates and α -parabolic conjugate functions. Now, we show the main result of this section. In the following theorem, we investigate properties of (κ, ν) -conjugates on parabolic Bergman spaces.

Theorem 5.2 Let $0 < \alpha \leq 1$, $1 \leq p < \infty$, $\lambda > -1$, and $u \in \mathbf{b}_\alpha^p(\lambda)$. Furthermore, let κ and ν be real numbers such that $\kappa < \frac{1}{2\alpha} + \frac{\lambda+1}{p}$ and $\kappa + \nu = \frac{1}{\alpha}$. Put $\eta = \left(\frac{1}{2\alpha} - \kappa\right)p + \lambda$. Then, there exists a unique (κ, ν) -conjugate $(v_1, \dots, v_n)$ of u such that $v_j \in \mathbf{b}_\alpha^p(\eta)$. Moreover, there exists a constant $C > 0$ independent of u such that

$$C^{-1} \|u\|_{L^p(\lambda)} \leq \sum_{j=1}^n \|v_j\|_{L^p(\eta)} \leq C \|u\|_{L^p(\lambda)}. \quad (5.4)$$

Proof Let $u \in \mathbf{b}_\alpha^p(\lambda)$ and $1 \leq j \leq n$. By Proposition 3.1, we can define a function v_j on H by

$$v_j(x, t) = -\mathcal{D}_t^{-\kappa} \partial_j u(x, t), \quad (x, t) \in H$$

and there exists a constant $C > 0$ independent of u such that

$$\|v_j\|_{L^p(\eta)} \leq C \|u\|_{L^p(\lambda)}. \quad (5.5)$$

Consequently, v_j belongs to $\mathbf{b}_\alpha^p(\eta)$.

We show that the $(n+1)$ -tuple $(v_1, \dots, v_n, u)$ satisfies Eq. 5.1, 5.2, and 5.3. We note that $\partial_j u$ belongs to $\mathbf{b}_\alpha^p(\lambda + \frac{p}{2\alpha})$ by Proposition 3.1. Therefore, (2) of Lemma 2.1 implies that

$$\partial_k v_j = -\partial_k \mathcal{D}_t^{-\kappa} \partial_j u = -\mathcal{D}_t^{-\kappa} \partial_j \partial_k u. \quad (5.6)$$

Hence, we have $\partial_k v_j = \partial_j v_k$, that is, Eq. 5.1 is satisfied. Also by (2) of Lemma 2.1, we have

$$-\mathcal{D}_t^\kappa v_j = -\mathcal{D}_t^\kappa (-\mathcal{D}_t^{-\kappa} \partial_j u) = \partial_j u,$$

that is, Eq. 5.2 is satisfied. We show that Eq. 5.3 is satisfied. By Eq. 5.6, we have

$$\sum_{j=1}^n \partial_j v_j = \sum_{j=1}^n (-\mathcal{D}_t^{-\kappa} \partial_j \partial_j u) = \mathcal{D}_t^{-\kappa} \left(-\sum_{j=1}^n \partial_j^2 u \right) = \mathcal{D}_t^{-\kappa} (-\Delta_x u).$$

Thus, Lemma 2.2 and (2) of Lemma 2.1 imply that

$$\mathcal{D}_t^{-\kappa} (-\Delta_x u) = \mathcal{D}_t^{-\kappa} \mathcal{D}_t^{\frac{1}{\alpha}} u = \mathcal{D}_t^{-\kappa + \frac{1}{\alpha}} u = \mathcal{D}_t^\nu u,$$

because $\kappa + \nu = \frac{1}{\alpha}$. Therefore, Eq. 5.3 is satisfied.

We show the inequalities (5.4). Since the second inequality of Eq. 5.4 is already given by Eq. 5.5, it suffices to show the first inequality of Eq. 5.4. By Lemma 2.5, Eq. 5.3, and Proposition 3.1, there exists a constant $C > 0$ independent of u such that

$$\begin{aligned} \|u\|_{L^p(\lambda)} &\leq C \|t^\nu \mathcal{D}_t^\nu u\|_{L^p(\lambda)} = C \left\| t^{\frac{1}{\alpha} - \kappa} \sum_{j=1}^n \partial_j v_j \right\|_{L^p(\lambda)} = C \left\| t^{\frac{1}{2\alpha}} \sum_{j=1}^n \partial_j v_j \right\|_{L^p(\eta)} \\ &\leq C \sum_{j=1}^n \left\| t^{\frac{1}{2\alpha}} \partial_j v_j \right\|_{L^p(\eta)} \leq C \sum_{j=1}^n \|v_j\|_{L^p(\eta)}. \end{aligned}$$

Therefore, we obtain the first inequality of Eq. 5.4.

Suppose that $(u_1, \dots, u_n)$ is a (κ, ν) -conjugate of u with $u_j \in \mathbf{b}_\alpha^p(\eta)$. We show that $v_j = u_j$. By Lemma 2.5 and Eq. 5.2, we obtain

$$\|v_j - u_j\|_{L^p(\eta)} \leq C \|t^\kappa \mathcal{D}_t^\kappa (v_j - u_j)\|_{L^p(\eta)} = C \|t^\kappa (\partial_j u - \partial_j u)\|_{L^p(\eta)} = 0.$$

Therefore, we have $v_j = u_j$. This completes the proof. $\square$

We remark that Theorem 5.2 is a generalization of Theorems A and 1. Also, for $u \in \mathbf{b}_\alpha^p(\lambda)$ and $\eta > -1$, Theorem 5.2 implies that there exists a unique (κ, ν) -conjugate $(v_1, \dots, v_n)$ of u such that $v_j \in \mathbf{b}_\alpha^p(\eta)$, where $\kappa = \frac{1}{2\alpha} + \frac{\lambda - \eta}{p}$ and $\kappa + \nu = \frac{1}{\alpha}$.

As an application of Theorem 5.2, we study the isomorphism on parabolic Bergman spaces. We already gave the isomorphism on the parabolic Bergman space with $p = 1$ in Theorem 6.6 of [2].

Proposition 5.3 (Theorem 6.6 of [2]). *Let $0 < \alpha \leq 1$ and $\eta, \lambda > -1$. Then, $\mathbf{b}_\alpha^1(\eta) \cong \mathbf{b}_\alpha^1(\lambda)$ under the isomorphism $\pi(u) = \mathcal{D}_t^{\lambda-\eta}u$ for all $u \in \mathbf{b}_\alpha^1(\eta)$.*

In the following theorem, we show that Proposition 5.3 can be extended to the full range $1 \leq p < \infty$.

Theorem 5.4 *Let $0 < \alpha \leq 1$, $1 \leq p < \infty$, and $\eta, \lambda > -1$. Then, $\mathbf{b}_\alpha^p(\eta) \cong \mathbf{b}_\alpha^p(\lambda)$.*

Proof Let $u \in \mathbf{b}_\alpha^p(\lambda)$. Put $\kappa = \frac{1}{2\alpha} + \frac{\lambda-\eta}{p}$ and $v = \frac{1}{2\alpha} - \frac{\lambda-\eta}{p}$. Then by Theorem 5.2, there exists a unique (κ, v) -conjugate $(v_1, \dots, v_n)$ of u such that $v_j \in \mathbf{b}_\alpha^p(\eta)$. Also, Theorem 4.1 implies that there exists a unique function $v \in \mathbf{b}_\alpha^p(\eta)$ such that $(v_1, \dots, v_n)$ is the $L^{(\alpha)}$ -conjugate of v . Thus, this completes the proof. $\square$

Remark 5.5 In the proof of Theorem 5.4, we give the isomorphism $\pi : \mathbf{b}_\alpha^p(\eta) \rightarrow \mathbf{b}_\alpha^p(\lambda)$ by $\pi(v) = \mathcal{D}_t^{\frac{\lambda-\eta}{p}}v$. In fact, Theorems 4.1 and 5.2 imply that functions v, u , and v_j satisfy the following equations;

$$\mathcal{D}_t^\nu u = \sum_{j=1}^n \partial_j v_j \quad \text{and} \quad \mathcal{D}_t^{\frac{1}{2\alpha}} v = \sum_{j=1}^n \partial_j v_j.$$

Therefore by (2) of Lemma 2.1, we have $u = \mathcal{D}_t^{\frac{1}{2\alpha}-\nu}v = \mathcal{D}_t^{\frac{\lambda-\eta}{p}}v$.

Open Access This article is distributed under the terms of the Creative Commons Attribution License which permits any use, distribution, and reproduction in any medium, provided the original author(s) and the source are credited.

References

1. Hishikawa, Y.: Fractional calculus on parabolic Bergman spaces. *Hiroshima Math. J.* **38**, 471–488 (2008)
2. Hishikawa, Y.: The reproducing formula with fractional orders on the parabolic Bloch spaces. *J. Math. Soc. Japan* **62**, 1219–1255 (2010)
3. Hishikawa, Y., Nishio, M., Yamada, M.: A conjugate system and tangential derivative norms on parabolic Bergman space. *Hokkaido Math. J.* **39**, 85–114 (2010)
4. Kochnoff, E., Sagher, Y.: Conjugate temperatures. *J. Approx. Theory* **70**, 39–49 (1992)
5. Nishio, M., Shimomura, K., Suzuki, N.: α -parabolic Bergman spaces. *Osaka J. Math.* **42**, 133–162 (2005)
6. Stein, E.M., Weiss, G.: On the theory of harmonic functions of several variables I. The theory of H^p -spaces. *Acta Math.* **103**, 25–62 (1960)
7. Ramey, W., Yi, H.: Harmonic Bergman functions on half-spaces. *Trans. Amer. Math. Soc.* **348**, 633–660 (1996)