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The Łojasiewicz–Siciak Condition of the Pluricomplex Green Function

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Abstract The aim of this paper is to address a problem raised originally by L. Gendre, later by W. Pleśniak and recently by L. Białas–Cież and M. Kosek. This problem concerns the pluricomplex Green function and consists in finding new examples of sets with so–called Łojasiewicz–Siciak ((ŁS) for short) property. So far, the known examples of such sets are rather of particular nature. We prove that each compact subset of $\mathbb{R}^N$, treated as a subset of $\mathbb{C}^N$, satisfies the Łojasiewicz–Siciak condition. We also give a sufficient geometric criterion for a semialgebraic set in $\mathbb{R}^2$, but treated as a subset of $\mathbb{C}$, to satisfy this condition. This criterion applies more generally to a set in $\mathbb{C}$ definable in a polynomially bounded o–minimal structure.

Keywords Siciak's extremal function · (HCP) property · Semialgebraic set · O–minimal structure

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1 Introduction

In $\mathbb{C}^N$ we consider the Euclidean norm: $|z| := \sqrt{|z_1|^2 + \ldots + |z_N|^2}$, for $z = (z_1, \ldots, z_N)$. If $b \in \mathbb{C}^N$ and $r > 0$, then $K(b, r) := \{z \in \mathbb{C}^N : |z - b| < r\}$. For a nonempty set $A \subset \mathbb{C}^N$ and $h : A \longrightarrow \mathbb{C}^{N'}$, we put $\|h\|_A := \sup_{z \in A} |h(z)|$. Moreover, $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, $\mathbb{N} := \{1, 2, 3, \ldots\}$.

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Our paper is devoted to Siciak's extremal function. Recall that the extremal function, associated with a (nonempty) compact set $K \subset \mathbb{C}^N$ and introduced by J. Siciak in [26], is defined by the formula

$$\Phi_K(z) := \sup\{|p(z)|^{1/\deg p} : \ p \in \mathbb{C}[Z] \text{ is nonconstant and } \|p\|_K \leq 1\},$$

for $z \in \mathbb{C}^N$ (cf. [11, 22, 26, 27]). It is a deep result that $\log \Phi_K = V_K$, where

$$V_K(z) := \sup\{u(z) : \ u \in \mathcal{L}(\mathbb{C}^N), \ u \leq 0 \text{ on } K\}$$

and $\mathcal{L}(\mathbb{C}^N)$ denotes the class of plurisubharmonic functions u in $\mathbb{C}^N$ satisfying the condition: $\sup_{z\in\mathbb{C}^N} \left[u(z) - \log(1 + |z|)\right] < \infty$ (cf. [27, 29]). The extremal function is a powerful tool in real and complex analysis (for example, in the theory of holomorphic functions, in approximation theory, as well as in potential and pluripotential theory–for the latter two see [2, 8, 24]). A spectacular example of usefulness of the extremal function is the Siciak's extension of the Bernstein–Walsh theorem to the case of several variables (cf. [26]).

If $N = 1$ and K is of positive logarithmic capacity, the upper semicontinuous regularization V_K^* of V_K is the Green function of the unbounded component of $\mathbb{C} \setminus K$ (with pole at infinity). If $N > 1$, V_K^* is therefore sometimes called the pluricomplex Green function.

It is particularly important to recognize, given a point $a \in K$, whether Φ_K is continuous at a (if so, then we say that K is L–regular at a). This problem was studied among others in [1, 15, 17–21, 25, 27].

While the problem of finding new L–regular sets is rather difficult, the construction of sets with so–called (HCP) property, which is a stronger condition than L–regularity, is significantly harder. We say that a compact set $K \subset \mathbb{C}^N$ has the Hölder continuity property (HCP) if there exist constants $\varpi > 0$, $\mu > 0$ such that

$$\Phi_K(z) \leq 1 + \varpi\,(\mathrm{dist}(z;\ K))^{\mu} \quad \text{as } \mathrm{dist}(z;\ K) \leq 1$$

($z \in \mathbb{C}^N$). The (HCP) property finds applications in the theory of polynomial inequalities (for example, Markov's inequality) and was investigated in [9, 13, 14, 16, 28].

Recently, L. Gendre introduced another condition, called the Łojasiewicz–Siciak condition, or (ŁS) for short (cf. [7]). We say that a compact set $K \subset \mathbb{C}^N$ satisfies the (ŁS) condition if it is polynomially convex[1] and there exist constants $\eta > 0$, $\kappa > 0$ such that

$$\Phi_K(z) \geq 1 + \eta\,(\mathrm{dist}(z;\ K))^{\kappa} \quad \text{as } \mathrm{dist}(z;\ K) \leq 1$$

($z \in \mathbb{C}^N$). This condition is useful in approximation theory. For example, it was used by L. Gendre to prove a result on approximation of functions in holomorphic Carleman classes. In a planned sequel to the present article, we shall discuss other motivations for studying this condition.

L. Białas–Cież and M. Kosek claim in [3] that so far very few examples of sets with the (ŁS) property are known. Their paper is devoted to the problem

[1]We call a compact set $K \subset \mathbb{C}^N$ polynomially convex if $K = \hat{K} := \{z \in \mathbb{C}^N : \ |P(z)| \leq \|P\|_K$ for all polynomials $P \in \mathbb{C}[Z]\}$. One can prove that each compact subset of $\mathbb{R}^N$, treated as a subset of $\mathbb{C}^N$, is polynomially convex. Moreover, a compact set $K \subset \mathbb{C}$ is polynomially convex if and only if $\mathbb{C} \setminus K$ is connected.

of delivering some new examples of such sets (which are connected with iterated function systems). The main thrust of our paper is to prove Theorems 1.1 and 1.2.

Theorem 1.1 *Each (nonempty) compact subset of $\mathbb{R}^N$, treated as a subset of $\mathbb{C}^N$, satisfies the (ŁS) condition with $\kappa = 1$.*

For a compact set $K \subset \mathbb{R}^2$ treated as a subset of $\mathbb{C}$, we have a much different result.

We say that $b \in A \subset \mathbb{R}^N$ is a regular point of A, if A in some neighbourhood of b is a $\mathcal{C}^1$ submanifold of $\mathbb{R}^N$. The set of regular points is denoted by $\operatorname{Reg} A$. Moreover, we put $\operatorname{Sing} A := A \setminus \operatorname{Reg} A$.

Recall that a subset of $\mathbb{R}^N$ is semialgebraic if it is a finite union of sets of the form

$$\{x \in \mathbb{R}^N :\ f(x) = 0,\ g_1(x) > 0,\ \dots,\ g_k(x) > 0\},$$

where $f, g_1, \dots, g_k \in \mathbb{R}[X_1, \dots, X_N]$ (cf. [4]). A map is said to be semialgebraic if its graph is semialgebraic. We define the dimension $\dim A$ of a nonempty semialgebraic set $A \subset \mathbb{R}^N$ to be the maximum of all $d \in \mathbb{N} \cup \{0\}$ such that A contains a d–dimensional $\mathcal{C}^1$ submanifold of $\mathbb{R}^N$.

In the next theorem the following assumption is made: all the interior angles of the set $\mathbb{R}^2 \setminus K$ at b are greater than 0. We will explain in the Section 3 (cf. Definition 3.6) what this precisely means. Roughly speaking, we require that $\mathbb{R}^2 \setminus K$ have no cusps.

Theorem 1.2 *Assume that a (nonempty) set $K \subset \mathbb{R}^2$ is compact, connected, semialgebraic and such that $\mathbb{R}^2 \setminus K$ is connected. Suppose additionally that, for each $b \in \partial K := K \setminus \operatorname{Int} K$, all the interior angles of the set $\mathbb{R}^2 \setminus K$ at b are greater than 0. Then K, treated as a subset of $\mathbb{C}$, satisfies the (ŁS) condition.*

Moreover, the exponent κ can be given effectively as follows. For each $b \in \operatorname{Sing} \partial K$, let $\theta_1(b), \dots, \theta_{p(b)}(b) \in (0, 2]$ be such that the interior angles of $\mathbb{R}^2 \setminus K$ at b are equal $\pi\theta_1(b), \dots, \pi\theta_{p(b)}(b)$. Put

$$\sigma := \inf\{\theta_\nu(b) :\ \nu \le p(b), b \in \operatorname{Sing} \partial K\}.$$

(If $\operatorname{Sing} \partial K = \emptyset$, we let $\sigma := 1$). Then $\sigma > 0$ and (ŁS) holds with $\kappa := \max\{1, \sigma^{-1}\}$.

Remark 1.3 To make the paper as accessible as possible we decided to avoid, with the exception of Section 6, the o–minimal setting. However, the statement of Theorem 1.2 remains valid if the assumption that K is semialgebraic is replaced by the assumption that K is definable in some polynomially bounded o–minimal structure (see Section 6). The latter case is much more general. Nevertheless, the way we prove Theorem 1.2 is such that it works for sets definable in polynomially bounded o–minimal structures (and of course satisfying the remaining assumptions of the theorem).

2 The Real Case

Theorem 1.1 completely solves the problem of the Łojasiewicz–Siciak condition for compact subsets of $\mathbb{R}^N$, treated as subsets of $\mathbb{C}^N$. It is really surprising, because as we will see in the next sections, the situation becomes completely different, when we consider a set in $\mathbb{R}^2$ not as a subset of $\mathbb{C}^2$, but as a subset of $\mathbb{C}$. Namely, in the latter case

- there are (very simple—even semialgebraic) polynomially convex sets which do not satisfy the (ŁS) condition;
- there is no universal exponent $\kappa > 0$ for sets satisfying the (ŁS) condition, while in the real case we can always take $\kappa = 1$.

Theorem 1.1 follows immediately from Proposition 2.1 below. Before we state it recall that:

- $\Phi_{[-1,\,1]}(z) = |z + \sqrt{z^2 - 1}|$ for $z \in \mathbb{C}$, where the square root is so chosen that $\Phi_{[-1,\,1]} \geq 1$;
- If $K \subset E \subset \mathbb{C}^N$, then $\Phi_K \geq \Phi_E$;
- If $F : \mathbb{C}^N \longrightarrow \mathbb{C}^N$ is an affine isomorphism, then $\Phi_{F(K)} = \Phi_K \circ F^{-1}$ $(K \subset \mathbb{C}^N)$.

Proposition 2.1 (*R. Pierzchała, Approximation of holomorphic functions on compact subsets of* $\mathbb{R}^N$ *(in preparation)*). *Assume that* $K \subset \mathbb{R}^N$ *is a compact set containing at least two distinct points, treated as a subset of* $\mathbb{C}^N$. *Then for each* $z \in \mathbb{C}^N$,

$$\Phi_K(z) \geq 1 + \varpi(z)\,\mathrm{dist}(z;\ K)\,,$$

where $\varpi(z) := \sqrt{(\mathrm{diam} K)^{-1}\,(\mathrm{diam} K + 2\,\mathrm{dist}(\mathrm{Re}(z);\ K))^{-1}}$.

The proof of this proposition is given in (R. Pierzchała, Approximation of holomorphic functions on compact subsets of $\mathbb{R}^N$ (in preparation)). However, for the sake of completeness, we include the proof.

Proof Fix $a \in \mathbb{C}^N$. Put $b := \mathrm{Re}(a)$, $\delta := (\mathrm{dist}(b;\ K))^2$, $\delta' := (\mathrm{dist}(a;\ K))^2$ and $R := \mathrm{diam} K \left(\mathrm{diam} K + 2\sqrt{\delta}\right)$. Define

$$K_a := \left\{x \in \mathbb{R}^N :\ \sqrt{\delta} \leq |x - b| \leq \sqrt{R + \delta}\right\}$$

and consider the polynomial

$$\Upsilon : \mathbb{C}^N \ni z \longmapsto R + \delta - \sum (z_\nu - b_\nu)^2 \in \mathbb{C}\,.$$

Since $\Upsilon(K_a) = [0,\ R]$ and $\Upsilon(a) = R + \delta'$, it follows that

$$\Phi_{\Upsilon(K_a)}(\Upsilon(a)) = \Phi_{[0,\,R]}(\Upsilon(a)) = \Phi_{[-1,\,1]}\left(\frac{2\Upsilon(a)}{R} - 1\right)$$
$$= \Phi_{[-1,\,1]}\left(\frac{2(R + \delta')}{R} - 1\right) = \frac{\left(\sqrt{R + \delta'} + \sqrt{\delta'}\right)^2}{R}\,.$$

As $K \subset K_a$ we obtain easily the following estimates

$$\Phi_K(a) \geq \Phi_{K_a}(a) \geq \sqrt{\Phi_{\Upsilon(K_a)}(\Upsilon(a))}$$

and therefore

$$\Phi_K(a) \geq \sqrt{\Phi_{\Upsilon(K_a)}(\Upsilon(a))} = \frac{\sqrt{R+\delta'}+\sqrt{\delta'}}{\sqrt{R}} \geq 1+\sqrt{\frac{\delta'}{R}} = 1+\frac{1}{\sqrt{R}}\,\mathrm{dist}(a;\ K),$$

which is the desired estimate. □

Remark 2.2 Theorem 1.1 along with the earlier results of W. Pawłucki and W. Pleśniak (cf. [13]), as well as the results obtained by the author (cf. [14, 16]), enables to give a rich family of natural sets satisfying both (HCP) and (ŁS) condition. All compact, fat and semialgebraic sets constitute a part of this family. (Recall that a set B is said to be fat if $\overline{B} = \overline{\mathrm{Int}B}$.)

Remark 2.3 L. Białas–Cież informed me that Theorem 1.1 in the special case $N = 1$ was obtained independently by L. Białas–Cież and R. Eggink.

3 The Complex Case Semialgebraic Sets

Definition 3.1 (see [23]) Assume that $\Gamma \subset \mathbb{R}^2 = \mathbb{C}$. We say that Γ is a Dini–smooth arc if there exists $h : [\alpha,\ \beta] \longrightarrow \mathbb{R}^2$ of class $\mathcal{C}^1$ $(\alpha < \beta)$ such that

(1) $\Gamma = h([\alpha,\ \beta])$;
(2) h is injective and $h'(t) \neq (0, 0)$ for $t \in [\alpha,\ \beta]$;
(3) For some (weakly) increasing function $\omega : [0,\ \beta-\alpha] \longrightarrow [0, +\infty)$ the following conditions hold:
- $\displaystyle\int_0^{\beta-\alpha} \frac{\omega(x)}{x}\,\mathrm{d}x < \infty$,
- $|h'(u) - h'(v)| \leq \omega(|u - v|)$, for $u, v \in [\alpha,\ \beta]$.

Definition 3.2 Suppose that $b \in \mathbb{R}^2 = \mathbb{C}$, $r > 0$ and $\theta \in [0,\ 2]$. A set $\Omega \subset \mathbb{R}^2 = \mathbb{C}$ is said to be a (θ, b, r)–set if there exist Dini–smooth arcs $\Gamma_1, \Gamma_2 \subset \mathbb{R}^2$ with endpoints a_1, b and a_2, b respectively such that

- $a_1, a_2 \in \partial K(b, r)$,
- $\Gamma_1 \setminus \{a_1\} \subset K(b, r)$, $\Gamma_2 \setminus \{a_2\} \subset K(b, r)$,
- $\Gamma_1 \cap \Gamma_2 = \{b\}$ (in particular, $a_1 \neq a_2$),
- Ω is one of the two connected components of the set $K(b, r) \setminus (\Gamma_1 \cup \Gamma_2)$,
- the interior angle of Ω at b is equal $\pi\theta$. (The arcs Γ_1, Γ_2 have the tangent lines at b. These lines make the interior angle of Ω at b. For example, if $\theta = 0$, then Ω has a cusp at b, and if $\theta = 2$, then $\mathbb{R}^2 \setminus \Omega$ has a cusp at b.)

By l_Ω we will denote the open (i.e. without endpoints) subarc of $\partial K(b, r)$ connecting the points a_1, a_2 and contained in $\overline{\Omega}$.

Definition 3.3 A set $\Gamma \subset \mathbb{R}^2$ is called a simple semialgebraic arc if there exists a semialgebraic function $\xi : [\alpha,\ \beta] \longrightarrow \mathbb{R}$ of class $\mathcal{C}^1$ $(\alpha < \beta)$ such that

$$\Gamma = \{(x, \xi(x)) : x \in [\alpha,\ \beta]\} \qquad \text{or} \qquad \Gamma = \{(\xi(x), x) : x \in [\alpha,\ \beta]\}.$$

Lemma 3.4 *Each simple semialgebraic arc is a Dini–smooth arc.*

Proof Let $\xi : [\alpha, \beta] \longrightarrow \mathbb{R}$ $(\alpha < \beta)$ be of class $\mathcal{C}^1$ and semialgebraic. We will show that the set $\Gamma := \{(x, \xi(x)) : x \in [\alpha, \beta]\}$ is a Dini–smooth arc. By the Łojasiewicz inequality (cf. [12]), there exist $\Theta, \mu > 0$ such that

$$|\xi'(u) - \xi'(v)| \leq \Theta|u - v|^{\mu},$$

for $u, v \in [0, 1]$. We see that Γ satisfies the definition of a Dini–smooth arc with $h(t) := (t, \xi(t))$, $t \in [\alpha, \beta]$, and $\omega(x) = \Theta x^{\mu}$, $x \in [0, \beta - \alpha]$. In the same way we show that $\{(\xi(x), x) : x \in [\alpha, \beta]\}$ is a Dini–smooth arc. □

The following lemma is a special case of Lemma 6.1.

Lemma 3.5 *Let $E \subset \mathbb{R}^2$ be a closed semialgebraic set. If $b \in \partial E$ is not an isolated point of E, then one of the following two conditions holds:*

(1) *There exist $p = p(b) \in \mathbb{N}$ and $\theta_1 = \theta_1(b), \ldots, \theta_p = \theta_p(b) \in [0, 2]$ such that, for each sufficiently small $r > 0$,*
- *$K(b, r) \setminus E = \Omega_1 \cup \ldots \cup \Omega_p$,*
- *Ω_ν $(\nu = 1, \ldots, p)$ are certain pairwise disjoint (θ_ν, b, r)–sets,*
- *$l_{\Omega_\nu} \subset \mathbb{R}^2 \setminus E$ $(\nu = 1, \ldots, p)$.*

(2) *For each sufficiently small $r > 0$,*
- *$E \cap \overline{K(b, r)} = \Gamma$,*
- *$\Gamma \subset \mathbb{R}^2$ is a simple semialgebraic arc with endpoints a, b,*
- *$a \in \partial K(b, r)$, $\Gamma \setminus \{a\} \subset K(b, r)$.*

Definition 3.6 We keep the notation of the above lemma and let $b \in \partial E$.

- If the condition (1) holds, then the collection $\{\pi\theta_1(b), \ldots, \pi\theta_{p(b)}(b)\}$ will be called the interior angles of the set $\mathbb{R}^2 \setminus E$ at b;
- If the condition (2) holds or b is an isolated point of E, then we will say that 2π is the interior angle of the set $\mathbb{R}^2 \setminus E$ at b.

4 The Complex Case Proof of Theorem 1.2

In this section we will derive Theorem 1.2 from the following result.

Proposition 4.1 *Let $K \subset \mathbb{C}$ be a compact, connected set such that $\#K \geq 2$ and $\mathbb{C} \setminus K$ is connected. Suppose that there exists $\delta > 0$ such that each point $b \in \partial K$ is one of the following types:*

Type I*:* *There exist $p = p(b) \in \mathbb{N}$, $\theta_1 = \theta_1(b), \ldots, \theta_p = \theta_p(b) \in [0, 2]$ and $r = r(b) > 0$ such that*
- *$K(b, r) \setminus K = \Omega_1 \cup \ldots \cup \Omega_p$,*
- *Ω_ν $(\nu = 1, \ldots, p)$ are certain pairwise disjoint (θ_ν, b, r)–sets and $\theta_\nu \geq \delta$,*
- *$l_{\Omega_\nu} \subset \mathbb{C} \setminus K$ $(\nu = 1, \ldots, p)$.*

Type II*: There exists* $r = r(b) > 0$ *such that*

- $K \cap \overline{K(b, r)} = \Gamma$*,*
- $\Gamma \subset \mathbb{C}$ *is a Dini–smooth arc with endpoints* a, b*,*
- $a \in \partial K(b, r)$*,* $\Gamma \setminus \{a\} \subset K(b, r)$*.*

Then there exists $\eta > 0$ *such that*

$$\Phi_K(z) \geq 1 + \eta\,(\mathrm{dist}(z;\ K))^{\kappa} \quad \text{as } \mathrm{dist}(z;\ K) \leq 1$$

$(z \in \mathbb{C})$*, where* $\kappa := \max\{1, \delta^{-1}\}$*.*

Proof Note first that

- $\Phi_K \geq 1$ in $\mathbb{C}$;
- $\Phi_K(z) = 1 \Longleftrightarrow z \in K$ (because $K = \hat{K}$);
- Φ_K is continuous in $\mathbb{C}$ (cf. [10]).

By the Riemann Mapping Theorem, there exists a conformal map $\varphi : \overline{\mathbb{C}} \setminus K \longrightarrow K(0, 1)$ such that $\varphi(\infty) = 0$. The function $z \longmapsto \log \dfrac{1}{|\varphi(z)|}$ is the Green function of $\mathbb{C} \setminus K$ (with pole at infinity). Therefore

$$V_K(z) = \log \frac{1}{|\varphi(z)|} \qquad \text{for } z \in \mathbb{C} \setminus K,$$

or equivalently

$$\Phi_K(z) = \frac{1}{|\varphi(z)|} \qquad \text{for } z \in \mathbb{C} \setminus K$$

(see the Introduction).

A standard argument via the Koebe One–Quarter Theorem shows that

$$|\varphi'(z)| \leq 4\frac{1 - |\varphi(z)|}{\mathrm{dist}(z;\ K)},$$

for $z \in \mathbb{C} \setminus K$. Therefore

$$|g'(w)| \geq \frac{1}{4}\frac{\mathrm{dist}\,(g(w);\ K)}{1 - |w|}, \tag{1}$$

for $w \in K(0, 1) \setminus \{0\}$, where $g := \varphi^{-1} : K(0, 1) \longrightarrow \overline{\mathbb{C}} \setminus K$.

Suppose that, contrary to our claim, the statement of Proposition 4.1 is not valid. Then we will find sequences

- $z_j \in \mathbb{C}$ $(j \in \mathbb{N})$, $\mathrm{dist}(z_j;\ K) \leq 1$,
- $\eta_j > 0$ $(j \in \mathbb{N})$, $\eta_j \to 0$

such that

$$\Phi_K(z_j) < 1 + \eta_j\left(\mathrm{dist}(z_j;\ K)\right)^{\kappa}. \tag{2}$$

(Recall that $\kappa := \max\{1, \delta^{-1}\}$.) Clearly, $z_j \in \mathbb{C} \setminus K$. Put $w_j := \varphi(z_j) = g^{-1}(z_j)$.

Passing to a subsequence if necessary, we can assume that $z_j \to b \in \mathbb{C} \setminus \mathrm{Int}K$. Note that

$$\Phi_K(b) = \lim_{j\to\infty} \Phi_K(z_j) \leq \lim_{j\to\infty} \left[1 + \eta_j \left(\mathrm{dist}(z_j;\ K)\right)^{\kappa}\right] = 1$$

and therefore $b \in K$. Consequently, $b \in K \setminus \mathrm{Int}K = \partial K$.

Case 1 b is of Type I. Take $p = p(b) \in \mathbb{N}, r = r(b) > 0$ and the sets $\Omega_1, \ldots, \Omega_p$ as in definition of Type I. Passing to a subsequence if necessary, we can assume that $z_j \in \Omega_{\nu_0}$ $(j \in \mathbb{N})$ for some $\nu_0 \leq p$. Put $\Omega := \Omega_{\nu_0}$. Recall that Ω is a (θ, b, r)–set with some $\theta \geq \delta$, and take $a_1, a_2, \Gamma_1, \Gamma_2$ and l_Ω of Definition 3.2. Since $\Gamma_1 \setminus \{a_1\}, \Gamma_2 \setminus \{a_2\} \subset K$, it follows that $\Gamma_1 \cup \Gamma_2 \subset K$.

Take injective and continuous $\gamma : [0,\ 1] \longrightarrow \partial K(b, r)$ such that

- $\gamma(0) = a_1, \gamma(1) = a_2$,
- $\gamma\,((0,\ 1)) = l_\Omega$.

By Theorem 2.3 in [23],

- the limits

$$a'_1 := \lim_{t\to 0^+} \varphi(\gamma(t))\ , \qquad a'_2 := \lim_{t\to 1^-} \varphi(\gamma(t))$$

  exist,
- $a'_1 \neq a'_2$ and $a'_1, a'_2 \in \partial K(0, 1)$.

Therefore $l'_\Omega \cup \{a'_1, a'_2\}$, where $l'_\Omega := \varphi(l_\Omega)$, is a Jordan arc (that is, a homeomorphic image of $[0,\ 1]$) with endpoints a'_1, a'_2. Note that

$$\overline{\Omega} \cap [(\overline{\mathbb{C}} \setminus K) \setminus l_\Omega] = [\Omega \cup \Gamma_1 \cup \Gamma_2 \cup l_\Omega] \cap [(\overline{\mathbb{C}} \setminus K) \setminus l_\Omega] = \Omega\,.$$

Hence Ω is closed in $(\overline{\mathbb{C}} \setminus K) \setminus l_\Omega$. It is also open. Consequently, $\varphi(\Omega)$ is open and closed in $K(0, 1) \setminus l'_\Omega$ and thus it must be one of the two connected components of this set. Put $\Omega' := \varphi(\Omega)$.

The map $\Omega' \ni w \longmapsto g(w) \in \Omega$ is a conformal map between Jordan domains. Therefore it can be extended to a homeomorphism $G : \overline{\Omega'} \longrightarrow \overline{\Omega}$ (cf. [23], Theorem 2.1). Note that

$$\partial\Omega' = G^{-1}(\partial\Omega) = l'_\Omega \cup G^{-1}(\Gamma_1 \cup \Gamma_2)\,.$$

It is clear that

- $G^{-1}(\Gamma_1 \cup \Gamma_2)$ is a (closed) subarc of $\partial K(0, 1)$ connecting the points a'_1, a'_2;
- $G^{-1}(\Gamma_1) \cap G^{-1}(\Gamma_2) = G^{-1}(\Gamma_1 \cap \Gamma_2) = \{G^{-1}(b)\}$;
- G maps the arcs $G^{-1}(\Gamma_1)$, $G^{-1}(\Gamma_2)$ respectively onto Dini–smooth arcs Γ_1, Γ_2.

By Theorem 3.7 in [23], the map

$$w \longmapsto \frac{g'(w)}{\left(w - G^{-1}(b)\right)^{\theta-1}}$$

is bounded near $G^{-1}(b)$. Therefore for some $M \in (0, +\infty)$,

$$|g'(w_j)| \leq M|G^{-1}(b) - w_j|^{\theta-1}\,.$$

Combining this with Eq. 1 we obtain

$$\frac{1}{4M}\mathrm{dist}(z_j;\ K) \leq (1 - |w_j|)|G^{-1}(b) - w_j|^{\theta-1}\,. \tag{3}$$

Note that, for $j \in \mathbb{N}$ large enough, $|G^{-1}(b) - w_j| \leq 1$ and therefore

$$\begin{aligned}|G^{-1}(b) - w_j|^{\theta-1} &\leq |G^{-1}(b) - w_j|^{\min\{0,\theta-1\}} \leq \left(|G^{-1}(b)| - |w_j|\right)^{\min\{0,\theta-1\}}\\ &= \left(1 - |w_j|\right)^{\min\{0,\theta-1\}} \leq \left(1 - |w_j|\right)^{\min\{0,\delta-1\}}\,.\end{aligned}$$

The above estimates and Eq. 3 imply that, for $j \in \mathbb{N}$ large enough,

$$\begin{aligned}\frac{1}{4M}\mathrm{dist}(z_j;\ K) &\leq (1 - |w_j|)(1 - |w_j|)^{\min\{0,\delta-1\}} = (1 - |w_j|)^{\frac{1}{\kappa}}\\ &= \left(1 - \frac{1}{\Phi_K(z_j)}\right)^{\frac{1}{\kappa}} \leq \left(\Phi_K(z_j) - 1\right)^{\frac{1}{\kappa}}\end{aligned}$$

and thus

$$\Phi_K(z_j) \geq 1 + \frac{1}{(4M)^\kappa}\left(\mathrm{dist}(z_j;\ K)\right)^\kappa\,.$$

This contradicts the fact that Eq. 2 holds and $\eta_j \to 0$.

Case 2 b is of Type II. Take $r = r(b) > 0$, $a \in \partial K(b, r)$ and Γ as in definition of Type II. Clearly, there exists a Jordan arc L with endpoints $c \in \partial K(b, r)$ and b such that $L \cap \Gamma = \{b\}$ and $L \setminus \{c\} \subset K(b, r)$. Let l_1, l_2 denote the two (open) subarcs of $\partial K(b, r)$ connecting the points a and c. Each of the two curves $\Gamma \cup L \cup l_1$, $\Gamma \cup L \cup l_2$ bounds a Jordan domain contained in $K(b, r)$. These domains will be denoted Ω_1, Ω_2 respectively.

Note that Ω_1 is closed in $(\overline{\mathbb{C}} \setminus K) \setminus (L \cup l_1) = (\overline{\mathbb{C}} \setminus K) \setminus [(L \setminus \{b\}) \cup l_1]$, because

$$\overline{\Omega_1} \cap [(\overline{\mathbb{C}} \setminus K) \setminus (L \cup l_1)] = [\Omega_1 \cup \Gamma \cup L \cup l_1] \cap [(\overline{\mathbb{C}} \setminus K) \setminus (L \cup l_1)] = \Omega_1\,.$$

Consequently, $\varphi(\Omega_1)$ is open and closed in $K(0, 1) \setminus [\varphi(L \setminus \{b\}) \cup \varphi(l_1)]$ and thus it must be one of the two connected components of this set. (Note that these components are Jordan domains–compare the argument given in Case 1.) Put $\Omega'_1 := \varphi(\Omega_1)$. Similarly, we define $\Omega'_2 := \varphi(\Omega_2)$.

The following maps

$$\Omega_1' \ni w \longmapsto g(w) \in \Omega_1 \,, \qquad \Omega_2' \ni w \longmapsto g(w) \in \Omega_2$$

are conformal maps between Jordan domains. Therefore they can be extended to homeomorphisms

$$G_1 : \overline{\Omega_1'} \longrightarrow \overline{\Omega_1} \,, \qquad G_2 : \overline{\Omega_2'} \longrightarrow \overline{\Omega_2}$$

(cf. [23], Theorem 2.1). Note that

$$\partial\Omega_1' = G_1^{-1}(\partial\Omega_1) = \varphi(L \setminus \{b\}) \cup \varphi(l_1) \cup G_1^{-1}(\Gamma) \,,$$
$$\partial\Omega_2' = G_2^{-1}(\partial\Omega_2) = \varphi(L \setminus \{b\}) \cup \varphi(l_2) \cup G_2^{-1}(\Gamma) \,.$$

It is clear that

- $G_1^{-1}(\Gamma)$, $G_2^{-1}(\Gamma)$ are closed subarcs of $\partial K(0,1)$ connecting the points a_1', b' and a_2', b' respectively, where $a_1' := G_1^{-1}(a)$, $a_2' := G_2^{-1}(a)$ and $b' := G_1^{-1}(b) = G_2^{-1}(b)$;
- $\{b'\} \subset G_1^{-1}(\Gamma) \cap G_2^{-1}(\Gamma) \subset \{b', a_1'\}$;
- g (more precisely, the extension of g to $\overline{K(0,1)}$) maps each of the arcs $G_1^{-1}(\Gamma)$, $G_2^{-1}(\Gamma)$ onto Γ.

By Theorem 3.7 in [23], the map

$$w \longmapsto \frac{g'(w)}{w - b'}$$

is bounded near b'. We easily check that $w_j \to b'$ and therefore, for some $\tilde{M} \in (0, +\infty)$,

$$|g'(w_j)| \le \tilde{M}|b' - w_j| \,.$$

Combining this with Eq. 1 we obtain for $j \in \mathbb{N}$ large enough

$$\frac{1}{4\tilde{M}}\operatorname{dist}(z_j;\ K) \le 1 - |w_j| = 1 - \frac{1}{\Phi_K(z_j)} \le \Phi_K(z_j) - 1$$

and thus

$$\Phi_K(z_j) \ge 1 + \frac{1}{4\tilde{M}}\operatorname{dist}(z_j;\ K) \ge 1 + \frac{1}{4\tilde{M}}\left(\operatorname{dist}(z_j;\ K)\right)^{\kappa} \,.$$

The above estimates contradict the fact that Eq. 2 holds and $\eta_j \to 0$. □

Proof that Proposition 4.1 $\Longrightarrow$ Theorem 1.2 We keep the notation of Theorem 1.2. We may assume that $\#K \ge 2$, because otherwise $\Phi_K = +\infty$ in $\mathbb{C} \setminus K$. Therefore K is without isolated points. Recall that

$$\sigma := \begin{cases} \inf\{\theta_\nu(b) : \nu \le p(b), b \in \operatorname{Sing}\partial K\} & \text{if } \operatorname{Sing}\partial K \neq \emptyset \\ 1 & \text{otherwise}. \end{cases}$$

Since $\dim \operatorname{Sing} \partial K < \dim \partial K$, it follows that $\operatorname{Sing} \partial K$ is a finite set (cf. [5]). In particular, $\sigma > 0$. Put $\delta := \min\{1, \sigma\}$. Fix $b \in \partial K$. We apply Lemma 3.5 to $E := K$ and consider the following three cases.

Case 1: $b \in \operatorname{Sing} \partial K$ and the condition (1) of Lemma 3.5 holds. Note that $\theta_1(b), \ldots, \theta_{p(b)}(b) \geq \delta$. Therefore the point b is of Type I (cf. Proposition 4.1).

Case 2: $b \in \operatorname{Reg} \partial K$. Then the condition (1) of Lemma 3.5 holds with $p = p(b) = 1$ and $\theta_1 = \theta_1(b) = 1$ or with $p = p(b) = 2$ and $\theta_1 = \theta_1(b) = 1, \theta_2 = \theta_2(b) = 1$. Clearly, $\theta_\nu(b) \geq \delta$. Therefore the point b is of Type I.

Case 3: the condition (2) of Lemma 3.5 holds. By Lemma 3.4, the point b is of Type II.

We have checked that the assumptions of Proposition 4.1 are satisfied. Hence there exists $\eta > 0$ such that

$$\Phi_K(z) \geq 1 + \eta\,(\operatorname{dist}(z;\ K))^\kappa \quad \text{as } \operatorname{dist}(z;\ K) \leq 1$$

($z \in \mathbb{C}$), where $\kappa := \max\{1, \delta^{-1}\} = \delta^{-1} = \max\{1, \sigma^{-1}\}$. This finishes the proof. □

5 Examples

In the first example we give a family of compact semialgebraic subsets of $\mathbb{R}^2$, treated as subsets of $\mathbb{C}$, such that:

- For each set of this family, the exponent κ obtained via Theorem 1.2 is optimal;
- There is no universal upper bound for these exponents. This reveals the first major difference between the complex and the real case. Recall that in the latter one we have the universal exponent $\kappa = 1$ (cf. Theorem 1.1).

Example 1 Let $a > 0, r > 0, R := \sqrt{a^2 + r^2}$. Put $K := K_1 \cup K_2$, where $K_1 := \overline{K(a, R)}$, $K_2 := \overline{K(-a, R)}$. Take $\delta > 0$ such that $\pi\delta$ is the interior angle of the set $\mathbb{C} \setminus K$ at ir. Clearly, $\delta \in (0,\ 1)$. Put

$$\phi : \overline{\mathbb{C}} \ni z \longmapsto \frac{z - ir}{z + ir} \in \overline{\mathbb{C}}\,.$$

It is easy to check that

$$\phi(\partial K_1) = \mathbb{R} \cdot (a - ir) \cup \{\infty\}\ , \qquad \phi(\partial K_2) = \mathbb{R} \cdot (a + ir) \cup \{\infty\}\,.$$

It follows that $\phi(\overline{\mathbb{C}} \setminus K) = \overline{\mathbb{C}} \setminus (\phi(K_1) \cup \phi(K_2)) = D$, where

$$D := \left\{ w = w_1 + i w_2 \in \mathbb{C} :\ w_1 > 0,\ \ |w_2| < \frac{r}{a} w_1 \right\}\,.$$

Consider the following conformal maps

$$\phi_1 : D \ni w \longmapsto w^{\frac{1}{\delta}} \in \{u \in \mathbb{C} :\ \operatorname{Re} u > 0\}\,,$$

$$\phi_2 : \{u \in \mathbb{C} :\ \operatorname{Re} u > 0\} \ni u \longmapsto \frac{u + 1}{u - 1} \in \overline{\mathbb{C}} \setminus \overline{K(0, 1)}\,.$$

Since $\phi(\infty)=1, \phi_1(1)=1, \phi_2(1)=\infty$, it follows that

$$V_K(z)=V_{\overline{K(0,1)}}((\phi_2\circ\phi_1\circ\phi)(z))=\log|(\phi_2\circ\phi_1\circ\phi)(z)|=\log\left|\frac{\phi(z)^{\frac{1}{\delta}}+1}{\phi(z)^{\frac{1}{\delta}}-1}\right|,$$

for $z\in\mathbb{C}\setminus K$.

By Theorem 1.2, there exists $\eta>0$ such that, for each $z\in\mathbb{C}$ with $\operatorname{dist}(z;\ K)\le 1$,

$$\Phi_K(z)\ge 1+\eta\,(\operatorname{dist}(z;\ K))^{\frac{1}{\delta}}\ . \tag{$\star$}$$

Note that, for $t\in(r,+\infty)$,

$$\Phi_K(it)=\exp V_K(it)=\left|\frac{\phi(it)^{\frac{1}{\delta}}+1}{\phi(it)^{\frac{1}{\delta}}-1}\right|=1+\frac{2(t-r)^{\frac{1}{\delta}}}{(t+r)^{\frac{1}{\delta}}-(t-r)^{\frac{1}{\delta}}}$$

and

$$\operatorname{dist}(it;\ K)=\sqrt{a^2+t^2}-R=\frac{t+r}{\sqrt{a^2+t^2}+R}(t-r)\,.$$

It follows that in ($\star$) the exponent $\frac{1}{\delta}$ is optimal. That is, it cannot be replaced by a smaller one.

The next example shows the second major difference between the complex and the real case. Namely, in $\mathbb{C}$ there are very simple (even semialgebraic) polynomially convex sets which do not satisfy the (ŁS) condition.

Example 2 Let us see what happens in the previous example if a is fixed, say $a=1$, and $r\to 0$. In the limit we obtain the set $K:=\overline{K(1,1)}\cup\overline{K(-1,1)}$. Consider the following conformal maps

$$\xi_1:\overline{\mathbb{C}}\setminus K\ni z\longmapsto\frac{1}{z}\in\{w\in\mathbb{C}:\ 2|\operatorname{Re} w|<1\}\,,$$

$$\xi_2:\{w\in\mathbb{C}:\ 2|\operatorname{Re} w|<1\}\ni w\longmapsto e^{i\pi w}\in\{u\in\mathbb{C}:\ \operatorname{Re} u>0\}\,,$$

$$\xi_3:\{u\in\mathbb{C}:\ \operatorname{Re} u>0\}\ni u\longmapsto\frac{u+1}{u-1}\in\overline{\mathbb{C}}\setminus\overline{K(0,1)}\,.$$

Since $\xi_1(\infty)=0, \xi_2(0)=1, \xi_3(1)=\infty$, it follows that

$$V_K(z)=V_{\overline{K(0,1)}}((\xi_3\circ\xi_2\circ\xi_1)(z))=\log|(\xi_3\circ\xi_2\circ\xi_1)(z)|=\log\left|\frac{e^{\frac{i\pi}{z}}+1}{e^{\frac{i\pi}{z}}-1}\right|,$$

for $z\in\mathbb{C}\setminus K$. Therefore for $t>0$,

$$\Phi_K(it)=\exp V_K(it)=\left|\frac{e^{\frac{\pi}{t}}+1}{e^{\frac{\pi}{t}}-1}\right|=1+\frac{2}{e^{\frac{\pi}{t}}-1}\,.$$

It is straightforward now to check that K does not satisfy the (ŁS) condition. (Note that quite similar example is due to Siciak and given in [3].)

Remark 5.1 Let K be as in Example 2. Note that the interior angles of the set $\mathbb{R}^2 \setminus K$ at 0 are equal 0. Consequently, in Theorem 1.2 the assumption concerning the interior angles of $\mathbb{R}^2 \setminus K$ cannot be removed.

6 O–minimal Version of Theorem 1.2

In this section, we will describe how we can obtain a generalization of Theorem 1.2 to the case of sets definable in polynomially bounded o–minimal structures (see Remark 1.3).

The theory of o–minimal structures is a far–reaching extension of the theory of semialgebraic sets (cf. [5, 6]). Let $\mathcal{M}$ be an o–minimal structure. A typical example is $\mathcal{M} = (\mathcal{M}_n)_{n\in\mathbb{N}}$, where $\mathcal{M}_n$ consists of all semialgebraic subsets of $\mathbb{R}^n$. A set $E \subset \mathbb{R}^n$ is called definable (in $\mathcal{M}$) if $E \in \mathcal{M}_n$. A map is said to be definable (in $\mathcal{M}$) if its graph is definable. An o–minimal structure is polynomially bounded if for every definable $f : \mathbb{R} \longrightarrow \mathbb{R}$ there exists some $m \in \mathbb{N}$ such that $f(x) = O(x^m)$ as $x \to +\infty$.

Similarly to simple semialgebraic arcs we can define simple $\mathcal{M}$–arcs (just replace in Definition 3.3 "semialgebraic" by "definable in $\mathcal{M}$"). If $\mathcal{M}$ is additionally polynomially bounded, then each simple $\mathcal{M}$–arc is a Dini–smooth arc. The proof is the same as the proof of Lemma 3.4 (one needs to apply the o–minimal version of the Łojasiewicz inequality).

The method of the proof of Theorem 1.2 carries over to the polynomially bounded o–minimal setting, because we have at our disposal the above mentioned generalization of Lemma 3.4 and the following generalization of Lemma 3.5.

Lemma 6.1 *Let $E \subset \mathbb{R}^2$ be a closed set definable in a polynomially bounded o–minimal structure $\mathcal{M}$. If $b \in \partial E$ is not an isolated point of E, then one of the following two conditions holds:*

(1) *There exist $p = p(b) \in \mathbb{N}$ and $\theta_1 = \theta_1(b), \ldots, \theta_p = \theta_p(b) \in [0, 2]$ such that, for each sufficiently small $r > 0$,*
   - $K(b, r) \setminus E = \Omega_1 \cup \ldots \cup \Omega_p$,
   - Ω_ν *($\nu = 1, \ldots, p$) are certain pairwise disjoint (θ_ν, b, r)–sets,*
   - $l_{\Omega_\nu} \subset \mathbb{R}^2 \setminus E$ *($\nu = 1, \ldots, p$).*

(2) *For each sufficiently small $r > 0$,*
   - $E \cap \overline{K(b, r)} = \Gamma$,
   - $\Gamma \subset \mathbb{R}^2$ *is a simple $\mathcal{M}$–arc with endpoints a, b,*
   - $a \in \partial K(b, r)$, $\Gamma \setminus \{a\} \subset K(b, r)$.

Proof Fix $b \in \partial E$ which is not an isolated point of E. Without loss of generality we can assume that $b = 0$. Take a cell decomposition $\mathcal{C}$ of $\mathbb{R}^2$ partitioning the sets: ∂E, $[-1, 1]^2$, $\{0\}$ (cf. [5]). Put

$$\mathcal{D} := \{D \in \mathcal{C} : D \subset \partial E, \ 0 \in \overline{D}\}.$$

Clearly, $\partial E = \bigcup_{D\in\mathcal{C},\, D\subset\partial E} D$ and therefore, for some $\epsilon_0 > 0$,

$$\partial E \cap \overline{K(0,\epsilon_0)} = \Big(\bigcup_{D\in\mathcal{D}} D\Big) \cap \overline{K(0,\epsilon_0)}\,. \tag{4}$$

Since $\dim \partial E < 2$, it follows that the family $\mathcal{D}$ consists of the set $\{0\}$ and of a finite number of cells of dimension 1. More precisely, $\mathcal{D} = \{\{0\}\} \cup \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3 \cup \mathcal{D}_4$, where

(i) $\mathcal{D}_1 = \emptyset$ or there exist: $\epsilon_1 > 0$, $m_1 \in \mathbb{N}$ and definable $f_1, \ldots, f_{m_1} : (0, \epsilon_1) \longrightarrow \mathbb{R}$ such that

- $f_1 < \ldots < f_{m_1}$,
- $\mathcal{D}_1 = \{\mathrm{graph}(f_1), \ldots, \mathrm{graph}(f_{m_1})\}$;

(ii) $\mathcal{D}_2 = \emptyset$ or there exists $\rho_1 > 0$ such that $\mathcal{D}_2 = \{\{0\} \times (0, \rho_1)\}$;

(iii) $\mathcal{D}_3 = \emptyset$ or there exist: $\epsilon_2 > 0$, $m_2 \in \mathbb{N}$ and definable $g_1, \ldots, g_{m_2} : (-\epsilon_2, 0) \longrightarrow \mathbb{R}$ such that

- $g_1 > \ldots > g_{m_2}$,
- $\mathcal{D}_3 = \{\mathrm{graph}(g_1), \ldots, \mathrm{graph}(g_{m_2})\}$;

(iv) $\mathcal{D}_4 = \emptyset$ or there exists $\rho_2 > 0$ such that $\mathcal{D}_4 = \{\{0\} \times (-\rho_2, 0)\}$.

We arrange the elements of the set $\mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3 \cup \mathcal{D}_4$ into a sequence $D_1, \ldots, D_k$. Note that $k \geq 1$, because $0 \in \partial E$ and 0 is not an isolated point of E.

Take $r_0 > 0$ such that

- $r_0 \leq \min\{\epsilon_0, \epsilon_1, \epsilon_2, \rho_1, \rho_2\}$;
- If $\mathcal{D}_1 \neq \emptyset$, then $f_j|_{(0,r_0)}$ is of class $\mathcal{C}^1$ ($j = 1, \ldots, m_1$) and
$$f_j|_{(0,r_0)} \equiv 0 \qquad \text{or} \qquad f_j'(x) \neq 0 \text{ for } x \in (0, r_0)\,;$$
- If $\mathcal{D}_3 \neq \emptyset$, then $g_j|_{(-r_0,0)}$ is of class $\mathcal{C}^1$ ($j = 1, \ldots, m_2$) and
$$g_j|_{(-r_0,0)} \equiv 0 \qquad \text{or} \qquad g_j'(x) \neq 0 \text{ for } x \in (-r_0, 0)\,.$$

Fix $r \in (0, r_0)$. By Eq. 4,

$$\partial E \cap \overline{K(0,r)} = \Gamma_1 \cup \ldots \cup \Gamma_k\,, \tag{5}$$

where $\Gamma_j := \{0\} \cup [D_j \cap \overline{K(0,r)}]$. It is easy to check that

(a) For each $t \in (0, r]$, the set $\Gamma_j \cap \partial K(0,t)$ is a singleton. Denote its only element by $a_j(t)$;

(b) The points $a_1(t), \ldots, a_k(t)$ are distinct, for $t \in (0, r]$;

(c) $\Gamma_j = \{0\} \cup \{a_j(t) : t \in (0, r]\}$;

(d) Γ_j is a simple $\mathcal{M}$–arc[2] with endpoints $0, a_j(r)$ and

- $\Gamma_j \setminus \{a_j(r)\} \subset K(0,r)$,
- Γ_j is a Dini–smooth arc.[3]

[2]Use the fact that the following limits exist (finite or not): $\lim_{x\to 0^+} f_j'(x)$, $\lim_{x\to 0^-} g_j'(x)$.

[3]We need here the extension of Lemma 3.4 to polynomially bounded o–minimal structures.

Case 1 $k \geq 2$. Note first that the set $K(0,r) \setminus (\Gamma_1 \cup \ldots \cup \Gamma_k) = K(0,r) \setminus \partial E$ (cf. (5)) has exactly k connected components. Denote them $W_1, \ldots, W_k$. Clearly,

$$W_j \subset [K(0,r) \setminus E] \cup [\mathrm{Int} E \cap K(0,r)]$$

and therefore

$$W_j \subset K(0,r) \setminus E \qquad \text{or} \qquad W_j \subset \mathrm{Int} E \cap K(0,r)$$

($j = 1, \ldots, k$). It follows that, for each $j = 1, \ldots, k$,

$$W_j \subset K(0,r) \setminus E \qquad \text{or} \qquad W_j \cap [K(0,r) \setminus E] = \emptyset\,.$$

Consequently, the elements of the family

$$\mathcal{A} := \left\{ W_j : \ W_j \subset K(0,r) \setminus E \right\}$$

are exactly the connected components of the set $K(0,r) \setminus E$. Denote these elements by $\Omega_1, \ldots, \Omega_p$. It is straightforward to verify that:

- $p \geq 1$;
- Ω_ν is a $(\theta_\nu, b\,, r)$–set for some $\theta_\nu \in [0,\ 2]$ ($\nu = 1, \ldots, p$);
- $\Omega_\nu \cup l_{\Omega_\nu} \subset \mathbb{R}^2 \setminus \partial E = (\mathbb{R}^2 \setminus E) \cup \mathrm{Int} E$. The set $\Omega_\nu \cup l_{\Omega_\nu}$ is connected[4] and $\Omega_\nu \subset \mathbb{R}^2 \setminus E$, hence $l_{\Omega_\nu} \subset \mathbb{R}^2 \setminus E$ ($\nu = 1, \ldots, p$).

Therefore the condition (1) of our lemma holds.

Case 2 $k = 1$. We know already that Γ_1 is a simple $\mathcal{M}$–arc with endpoints $0, a_1(r)$, where $a_1(r) \in \partial K(0,r)$, and $\Gamma_1 \setminus \{a_1(r)\} \subset K(0,r)$. By Eq. 5,

$$\overline{K(0,r)} \setminus \Gamma_1 = \overline{K(0,r)} \setminus \partial E \subset \mathbb{R}^2 \setminus \partial E = (\mathbb{R}^2 \setminus E) \cup \mathrm{Int} E\,.$$

Since $\overline{K(0,r)} \setminus \Gamma_1$ is connected, it follows that

$$\overline{K(0,r)} \setminus \Gamma_1 \subset \mathrm{Int} E \qquad \text{or} \qquad \mathrm{Int} E \cap [\overline{K(0,r)} \setminus \Gamma_1] = \emptyset\,.$$

Equivalently,

$$\overline{K(0,r)} \setminus \Gamma_1 \subset \mathrm{Int} E \qquad \text{or} \qquad \mathrm{Int} E \cap \overline{K(0,r)} = \emptyset\,.$$

Suppose that $\overline{K(0,r)} \setminus \Gamma_1 \subset \mathrm{Int} E$. Then

$$\overline{K(0,r)} = [\overline{K(0,r)} \setminus \Gamma_1] \cup \Gamma_1 \subset \mathrm{Int} E \cup \partial E = E\,,$$

which is impossible, because $0 \in \partial E$. Therefore $\mathrm{Int} E \cap \overline{K(0,r)} = \emptyset$ and via Eq. 5

$$\Gamma_1 = \partial E \cap \overline{K(0,r)} = E \cap \overline{K(0,r)}\,.$$

This means that the condition (2) of our lemma holds. □

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[4]because $\Omega_\nu \subset \Omega_\nu \cup l_{\Omega_\nu} \subset \overline{\Omega_\nu}$.

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