Equivalence of the Local Markov Inequality and a Kolmogorov Type Inequality in the Complex Plane

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Abstract We prove that a compact subset of the complex plane satisfies a local Markov inequality if and only if it satisfies a Kolmogorov type inequality. This result generalizes a theorem established by Bos and Milman in the real case. We also show that every set satisfying the local Markov inequality is a sum of Cantor type sets which are regular in the sense of the potential theory.

Keywords Markov inequality · Kolmogorov inequality · Green function · L-regularity of sets · Holomorphic functions · Cantor sets

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1 Introduction

Let $K \subset \mathbb{C}^N$ be a compact set. The pluricomplex Green function (with pole at infinity) of K can be defined by

$$V_K(z) := \sup \{u(z) : u \in \mathcal{L} \text{ and } u \leq 0 \text{ on } K\}, \quad z \in \mathbb{C}^N,$$

where $\mathcal{L}$ is the family of all plurisubharmonic functions in $\mathbb{C}^N$ of logarithmic growth at the infinity, i.e.

$$\mathcal{L} := \{u \text{ plurisubharmonic in } \mathbb{C}^N : u(z) - \log \|z\| \leq \mathcal{O}(1) \text{ as } \|z\| \rightarrow \infty\}.$$

In the one dimensional case V_K coincides with the Green function g_K of the unbounded component of $\hat{\mathbb{C}} \setminus K$ with logarithmic pole at infinity (as usual $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere).

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The (pluricomplex) Green function is closely related to polynomials in view of the formulas

$$V_K(z) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \Phi_n(K, z), \quad (1)$$

$$\Phi_n(K, z) = \sup \left\{ \frac{|P(z)|}{\|P\|_K} : P: \mathbb{C}^N \rightarrow \mathbb{C} \text{ polynomial of degree } n, P|_K \neq 0 \right\}$$

and $\|\cdot\|_K$ is the maximum norm on K (see [16] or [13, Theorem 5.1.7]).

A topic of great interest is the relationship between the regularity of the (pluricomplex) Green function and some Markov-type polynomial inequalities. These inequalities can be defined not only on subsets of $\mathbb{C}^N$ but also of $\mathbb{R}^N$ and in order to compare them with some properties of V_K , we consider $\mathbb{R}^N$ as a subspace of $\mathbb{C}^N$ in the usual way, i.e. $\mathbb{R}^N = (\operatorname{Re} \mathbb{C})^N \subset \mathbb{C}^N$, where $\operatorname{Re} z$ is the real part of z .

A compact set $E \subset \mathbb{K}^N$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) is said to have the *Global Markov Property* (GMP) if for every polynomial P of N variables

$$\|\nabla P\|_E \leq M(\deg P)^s \|P\|_E,$$

where $\nabla P := (\frac{\partial P}{\partial z_1}, \dots, \frac{\partial P}{\partial z_N})$, $|\nabla P| = (\sum_{j=1}^N |\frac{\partial P}{\partial z_j}|^2)^{1/2}$ and the positive constants M, s are independent of P .

Sets with this property play an important role in the constructive theory of functions, especially in problems of polynomial approximation and extension of $\mathcal{C}^\infty$ functions (see e.g. [1, 6, 8, 14, 21]). The GMP has found some important applications also in numerical analysis because of its connection with polynomial interpolation and approximation of $\mathcal{C}^\infty$ and holomorphic functions (e.g. [5, 9]).

It is worth noticing that by a result of Pleśniak [14], the global Markov property can be rewritten in the following form

$$\sup \left\{ \Phi_n(E, z) : \operatorname{dist}(z, E) \leq \frac{1}{n^s} \right\} \leq M,$$

where $M, s > 0$ are independent of n .

The last inequality and formula (1) suggest that the GMP of K may be closely related to the regularity of V_K . Indeed, the Hölder continuity of V_K implies the global Markov property of K (see [17, Lemma 3]) and every set with the GMP seems to be L -regular but that has been proved so far only for compact subsets of $\mathbb{R}$ (see [3]) (we say that K is *L-regular* if V_K is continuous). In the general (complex) case the question about the L -regularity of sets with the global Markov property remains an open problem posed explicitly in [14] and [19]. However, every compact set $K \subset \mathbb{C}$ with the GMP is not polar [2], which is a necessary condition for the continuity of g_K (see e.g. [15, Theorems 4.4.2, 3]).

On the other hand, it turns out that every compact subset of $\mathbb{C}$ with a local version of the Markov property (LMP, see Definition 1.4) is L -regular (see [3] or Section 4 in this paper). The implication $\text{LMP} \Rightarrow \text{GMP}$ is immediate but any attempt to prove the converse poses considerable difficulties.

Bos and Milman [7, 8] Bos and Milman (The equivalence of Markov and Sobolev type inequalities on compact sets in $\mathbb{R}^n$ (preprint)), Bos and Milman (A Geometric Interpretation and the Equality of Exponents in Markov and Gagliardo-Nirenberg (Sobolev) Type Inequalities for Singular Compact Domains (preprint)) have proved

that a local Markov inequality (Definition 1.1) is equivalent to the GMP in the real case. The proof is hard and requires demonstration of the equivalence of the local Markov inequality, a Kolmogorov type inequality (Definition 1.2) and an extension property for $C^\infty(K)$ functions. We were intrigued to obtain a corresponding result for sets in the complex plane because of the intricate interconnectedness of multiple distinct global and local properties: Markov inequalities, Kolmogorov type inequalities, polynomial approximation, extension operators, geometric properties and, ultimately, the behavior of the Green function, i.e. L-regularity, Hölder continuity and the Łojasiewicz-Siciak property. However, a simple adaptation to the complex case of the proof given by Bos and Milman does not seem possible. In this paper we show that the LMP is equivalent to a Kolmogorov type property for any compact subset of $\mathbb{C}$. This is the first step to show the equivalence (under an additional but necessary assumption) of the LMP and the GMP in the complex plane and, in consequence, to prove the L-regularity of sets with the global Markov property in $\mathbb{C}$. The next steps of the demonstration of this equivalence will be the subject of two papers that are now in their final stage of preparation (see Bialas-Ciez and Eggink (Łojasiewicz-Siciak inequality of Green's function and a version of Jackson's theorem in the complex plane (Preprint)), Bialas-Ciez and Eggink (Equivalence of local and global Markov inequality in the complex plane (Preprint))).

We first recall the definitions of the local Markov inequality and the Kolmogorov type inequality given by Bos and Milman.

We denote by $\mathcal{P}_n(\mathbb{R}^N)$ the space of all polynomials of N real variables with real coefficients of degree at most n . Let $\mathbb{N}_0 := \{0, 1, 2, \dots\}$. For $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}_0^N$ write $|\alpha| := \alpha_1 + \dots + \alpha_N$ and $D^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_N^{\alpha_N}}$.

Definition 1.1 ([Bos and Milman (A Geometric Interpretation and the Equality of Exponents in Markov and Gagliardo-Nirenberg (Sobolev) Type Inequalities for Singular Compact Domains (preprint)) Definition 2.3] or Bos and Milman (The equivalence of Markov and Sobolev type inequalities on compact sets in $\mathbb{R}^n$ (preprint)) [7]) Let $m \geq 1$. A compact set $K \subset \mathbb{R}^N$ admits the *Local Markov Inequality* (LMI(m)) in short) if there exist $c_1, \kappa \geq 1$ such that for every $n \in \mathbb{N}$, $x_0 \in K, r \in (0, 1], P \in \mathcal{P}_n(\mathbb{R}^N)$ and $\alpha \in \mathbb{N}_0^N$

$$|D^\alpha P(x_0)| \leq \left(\frac{c_1 n^\kappa}{r^m} \right)^{|\alpha|} \|P\|_{K \cap B(x_0, r)}, \quad (2)$$

where $B(x_0, r) := \{x : |x - x_0| \leq r\}$ and $\|g\|_F = \sup\{|g(x)| : x \in F\}$.

Note that here and below we do not assume $m \geq 1$ to be an integer.

For $f \in C^\infty(\mathbb{R}^N)$, $j \in \mathbb{N}$ we put

$$|f|_{K,j} := \sum_{|\alpha|=j} \|D^\alpha f\|_K, \quad \|f\|_{K,j} := \|f\|_K + |f|_{K,j}, \quad \|f\|_{K,0} := \|f\|_K.$$

Consider also the quotient norms

$$\|f\|_{K,j} := \inf \left\{ \|g\|_{\text{conv}(K),j} : g \in C^\infty(\mathbb{R}^N), \quad g|_K \equiv f|_K \right\}, \quad j \in \mathbb{N}_0,$$

where $\text{conv}(K)$ is the convex hull of K .

Definition 1.2 ([Bos and Milman (A Geometric Interpretation and the Equality of Exponents in Markov and Gagliardo-Nirenberg (Sobolev) Type Inequalities for Singular Compact Domains (preprint)), Definition 2.15] or Bos and Milman (The equivalence of Markov and Sobolev type inequalities on compact sets in $\mathbb{R}^n$ (preprint)) [7]) A compact set $K \subset \mathbb{R}^N$ admits the *Kolmogorov Inequality in Quotient Norms* ($KIQ(m)$, $m \geq 1$) if there are constants $c_1, \kappa \geq 1$ such that for all $f \in C^\infty(\mathbb{R}^N)$

$$\|f\|_{K,j} \leq (c_1 l^\kappa)^j \|f\|_K^{1-\frac{mj}{l}} \|f\|_{K,l}^{\frac{mj}{l}} \quad \text{for all } j, l \in \mathbb{N} \text{ such that } mj \leq l. \quad (3)$$

This property has been called by Bos and Milman a *Sobolev inequality in the quotient topology*. However, M. Baran noted that condition (3) is more closely related to Kolmogorov (or Landau) type inequalities.

Theorem 1.3 ([Bos and Milman (A Geometric Interpretation and the Equality of Exponents in Markov and Gagliardo-Nirenberg (Sobolev) Type Inequalities for Singular Compact Domains (preprint)), Theorem A] or [7, Theorem 3.9]) *If K is a compact subset of $\mathbb{R}^N$ and $m \geq 1$, then*

- (a) $KIQ(m) \Rightarrow LMI(m')$ for any $m' > m$,
 $KIQ(1) \Rightarrow LMI(1)$,
- (b) $LMI(m) \Rightarrow KIQ(m)$.

Our first attempts to prove an analogous theorem in the complex plane failed owing to several obstacles. The difficulty was that we did not know how to replace the space of C^∞ functions by another class of functions that would be well adapted to the Kolmogorov inequality in view of a proof of the local Markov inequality. An idea to use the functions of class $\mathcal{A}^\infty$ appeared first in [11]. Another obstacle was the fact that a cutoff function multiplied by a complex polynomial may not be of class $\mathcal{A}^\infty$ on a fixed compact set. We got over this problem by a sequence of lemmas.

Let E be a compact set in the complex plane. From now on, we will use the following notation.

For $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$, $f \in C^\infty(\mathbb{C})$ let $z^\alpha := z^{\alpha_1} \bar{z}^{\alpha_2}$ and $D^\alpha f := \frac{\partial^{|\alpha|}}{\partial z^{\alpha_1} \partial \bar{z}^{\alpha_2}} f$. As usual, $\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \binom{\alpha_2}{\beta_2}$ and $\beta \leq \alpha$ iff $\beta_1 \leq \alpha_1$ and $\beta_2 \leq \alpha_2$ for $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$, $\beta = (\beta_1, \beta_2) \in \mathbb{N}_0^2$. Of course if f is a holomorphic function at z_0 then $D^\alpha f(z_0) = 0$ whenever $\alpha_2 > 0$. Denote by $\mathcal{P}_n(\mathbb{C})$ the space of all holomorphic polynomials with complex coefficients of degree at most n .

Definition 1.4 Let $m, \kappa \geq 1$. A compact set $E \subset \mathbb{C}$ is said to admit the *Local Markov Property* $LMP(m, \kappa)$, if for all $n \in \mathbb{N}$, $z_0 \in E$, $r \in (0, 1]$, $P \in \mathcal{P}_n(\mathbb{C})$ and $j \in \mathbb{N}$

$$|P^{(j)}(z_0)| \leq \left(\frac{c_1 n^\kappa}{r^m} \right)^j \|P\|_{E \cap B(z_0, r)} \quad (4)$$

with $c_1 > 0$ depending only on E . We will write that the set E admits $LMP(m)$ if it admits $LMP(m, \kappa)$ for some $\kappa \geq 1$.

Evidently, if $E \subset \mathbb{R}$ then $LMP(m)$ is equivalent to $LMI(m)$ with $N = 1$.

A smooth function f is *flat* at $z_0 \in \mathbb{C}$ if $D^\alpha f(z_0) = 0$ for all $\alpha \in \mathbb{N}_0^2$. We call $f \in \mathcal{C}^\infty(\mathbb{C})$ $\bar{\partial}$ -flat on E if $\frac{\partial f}{\partial \bar{z}}$ is flat at every point of E . We will denote by $\mathcal{A}^\infty(E)$ the set of all smooth functions $\bar{\partial}$ -flat on E

$$\mathcal{A}^\infty(E) := \{f \in \mathcal{C}^\infty(\mathbb{C}) : f \text{ is } \bar{\partial}\text{-flat on } E\}$$

(compare with [21, Section 5] and [18]). For $f \in \mathcal{C}^\infty(\mathbb{C})$ and $j \in \mathbb{N}$ let

$$|f|_{E,j} := \sum_{\alpha \in \mathbb{N}_0^2, |\alpha|=j} \|D^\alpha f\|_E, \quad |f|_{E,0} := \|f\|_E$$

$$\|f\|_{E,j} := \|f\|_E + |f|_{E,j}, \quad \|f\|_{E,0} := \|f\|_E$$

and for $f \in \mathcal{A}^\infty(E)$ let

$$\|f\|_{E,j} := \inf \{ \|g\|_{\text{conv}(E),j} : g \in \mathcal{A}^\infty(E), g|_E \equiv f|_E \}, \quad \|f\|_{E,0} := \|f\|_E.$$

Definition 1.5 A compact set $E \subset \mathbb{C}$ admits the *Kolmogorov Property in Quotient Norms* $\text{KPQ}(m, \kappa)$, where $m, \kappa \geq 1$, if there exists a constant $c_1 \geq 1$ such that for all $f \in \mathcal{A}^\infty(E)$

$$|f|_{E,j} \leq (c_1 l^\kappa)^j \|f\|_E^{1-\frac{mj}{l}} \|f\|_{E,l}^{\frac{mj}{l}} \quad \text{for all } j, l \in \mathbb{N} \text{ such that } mj \leq l.$$

We will write that the set E admits $\text{KPQ}(m)$ if it admits $\text{KPQ}(m, \kappa)$ for some $\kappa \geq 1$.

It is worth noticing that for $E \subset \mathbb{R}$ we have $\mathcal{A}^\infty(E)/_{\mathbb{R}} = \mathcal{C}^\infty(\mathbb{R})$ (see [18]) and the $\text{KPQ}(m)$ is equivalent to the $\text{KIQ}(m)$ with $N = 1$.

We are now ready to state an analogue of Theorem 1.3.

Theorem 1.6 *If E is a compact set in $\mathbb{C}$ and $m \geq 1$, then*

- (a) $\text{KPQ}(m) \Rightarrow \text{LMP}(m')$ for any $m' > m$,
 $\text{KPQ}(1) \Rightarrow \text{LMP}(1)$,
- (b) $\text{LMP}(m) \Rightarrow \text{KPQ}(m)$.

This theorem is an easy consequence of a more general result. In order to formulate it, we need some additional definitions of Kolmogorov type inequalities. These properties will be necessary in the next stages of the proof of the equivalence of the local and global Markov inequalities (see Białas-Ciez and Eggink (Lojasiewicz-Siciak inequality of Green's function and a version of Jackson's theorem in the complex plane (Preprint)), Białas-Ciez and Eggink (Equivalence of local and global Markov inequality in the complex plane (Preprint))).

Let $F \subset \mathbb{C}$ be a compact set. Denote by $\mathcal{H}^\infty(F)$ the family of smooth functions that are holomorphic in some open neighbourhood of the set F , i.e.

$$\mathcal{H}^\infty(F) := \left\{ f \in \mathcal{C}^\infty(\mathbb{C}) : \frac{\partial f}{\partial \bar{z}} \equiv 0 \text{ in some open neighbourhood of } F \right\}.$$

Of course $\mathcal{H}^\infty(F) \subset \mathcal{A}^\infty(F)$. For $f \in \mathcal{H}^\infty(F)$ and $j \in \mathbb{N}$ let

$$\langle\langle f \rangle\rangle_{F,j} := \inf \{ \|g\|_{\text{conv}(F),j} : g \in \mathcal{H}^\infty(F), g|_F \equiv f|_F \}, \quad \langle\langle f \rangle\rangle_{F,0} := \|f\|_F.$$

Clearly, if $f \in \mathcal{H}^\infty(F)$ we have $\|f\|_{F,j} \leq \langle f \rangle_{F,j}$. For $\delta \in (0, 1]$ we define F_δ to be the set $\{z \in \mathbb{C} : \exists w \in F \ |z - w| \leq \delta\}$.

The next definition is a variant of the Kolmogorov Property in Quotient Norms KPQ adapted to holomorphic functions from $\mathcal{H}^\infty(E_\delta)$. We additionally require the constants to be dependent on δ in a specific way.

Definition 1.7 A compact set $E \subset \mathbb{C}$ admits the *Kolmogorov Property in Quotient Norms for Holomorphic Functions* KPQH(m, s, κ), where $m, \kappa \geq 1, s \geq 0$, if there exists a constant $c_1 \geq 1$ such that for all $\delta \in (0, 1], f \in \mathcal{H}^\infty(E_\delta)$

$$|f|_{E,j} \leq \left(\frac{c_1 l^\kappa}{\delta^s} \right)^j \|f\|_E^{1-\frac{mj}{l}} \langle f \rangle_{E_\delta,l}^{\frac{mj}{l}} \quad \text{for all } j, l \in \mathbb{N} \text{ such that } mj \leq l. \quad (5)$$

We give also a weaker version of the KPQH. Namely, we replace the norms $\langle \cdot \rangle_{E_\delta,l}$ by $\| \cdot \|_{E_\delta}$. We increase the first exponent by a factor c_o only in order to have better control over the constants (see Biała-Cież and Eggink (Equivalence of local and global Markov inequality in the complex plane (Preprint))).

Definition 1.8 A compact set $E \subset \mathbb{C}$ admits the *Kolmogorov Property for Holomorphic Functions* KPH(m, s, κ), where $m, s, \kappa \geq 1$, if there exist constants $c_o \geq 0, c_1 \geq 1$ such that for all $\delta \in (0, 1], f \in \mathcal{H}^\infty(E_\delta)$ and

$$|f|_{E,j} \leq \left(\frac{c_1 l^\kappa}{\delta^s} \right)^{j+c_o} \|f\|_E^{1-\frac{mj}{l}} \|f\|_{E_\delta}^{\frac{mj}{l}} \quad \text{for all } j, l \in \mathbb{N} \text{ such that } mj \leq l. \quad (6)$$

The statement given below is a refinement of Theorem 1.6 and expresses the equivalence not only of the LMP and the KPQ but also of the KPQH and KPH.

Theorem 1.9 If E is a compact set in $\mathbb{C}$ and $m, \kappa \geq 1$, then

- (a) $KPH(m, s, \kappa) \Rightarrow LMP(m', \kappa')$ for all $s \geq 1$ and $m' > s, \kappa' > \kappa + 3s$,
 $KPH(m, 1, \kappa)$ with $c_o = 0 \Rightarrow LMP(1, \kappa')$ for any $\kappa' > \kappa + 3$,
- (b) $LMP(m, \kappa) \Rightarrow KPQ(m, \kappa)$,
- (c) $KPQ(m, \kappa) \Rightarrow KPQH(m, s, \kappa)$ for all $s \geq 0$,
- (d) $KPQH(m, s, \kappa) \Rightarrow KPH(m, m + s, \kappa + 5m)$ with $c_o = 0$ for all $s \geq 0$.

In particular $KPQH(1, 0, \kappa) \Rightarrow KPH(1, 1, \kappa + 5)$ with $c_o = 0 \Rightarrow LMP(1, \kappa') \Rightarrow KPQ(1, \kappa') \Rightarrow KPQH(1, s, \kappa')$ for all $s \geq 0$ and for any $\kappa' > \kappa + 8$.

In Sections 2, 3 and 4 we prove the statements a, b and d of Theorem 1.9 respectively. The Proof of Theorem 1.9.c is immediate because $\mathcal{H}^\infty(E_\delta) \subset \mathcal{H}^\infty(E) \subset \mathcal{A}^\infty(E)$ and consequently for all $l \in \mathbb{N}, \delta \in (0, 1]$ and $f \in \mathcal{H}^\infty(E_\delta)$ we have $\|f\|_{E,l} \leq \langle f \rangle_{E,l} \leq \langle f \rangle_{E_\delta,l}$.

Section 5 is devoted to a geometric property of sets with the LMP(m). We can see at once that if $E \subset \mathbb{C}$ admits the LMP(m) then it is an m -perfect set, i.e. there exists $c > 0$ such that for all $z_0 \in E$ and $r \in (0, 1]$

$$\left\{ z \in E : \frac{r^m}{c} \leq |z - z_0| \leq r \right\} \neq \emptyset.$$

m -Perfectness seems to be an interesting property of E but it is not closely connected with the $\text{LMP}(m)$ in view of L -regularity. Namely, the $\text{LMP}(m)$ implies L -regularity (see [3, Theorem 2.2]) and there exist sets which are m -perfect for some m without being L -regular (see [19]). In Section 5, we introduce the notion of (m, ∞) -perfectness, which is more closely related with the $\text{LMP}(m)$ than m -perfectness. The L -regularity of (m, ∞) -perfect sets is a consequence of [3, Proposition 3.6]. In Section 5 we prove that all sets with the $\text{LMP}(m)$ are (m, ∞) -perfect.

2 The KPH Implies the LMP

We start with the construction of Fekete extremal points which are a useful tool in proving Theorem 1.9.a. Let $n \in \mathbb{N}$ and $E \subset \mathbb{C}$ be a compact set containing at least $n + 1$ points. Put

$$V(x_0, x_1, \dots, x_n) := \prod_{0 \leq \mu < \nu \leq n} (x_\nu - x_\mu).$$

Points $z_0, z_1, \dots, z_n \in E$ such that

$$|V(z_0, z_1, \dots, z_n)| = \max\{|V(x_0, x_1, \dots, x_n)| : x_0, x_1, \dots, x_n \in E\}$$

are called *Fekete extremal points of order n for E* . We denote by L_μ , $\mu = 0, \dots, n$, the Lagrange polynomials associated to the Fekete extremal points $z_0, \dots, z_n$, i.e.

$$L_\mu(z) := \frac{V(z_0, \dots, z_{\mu-1}, z, z_{\mu+1}, \dots, z_n)}{V(z_0, \dots, z_n)} \prod_{v=0, v \neq \mu}^n \frac{z - z_v}{z_\mu - z_v}.$$

It is easy to see that $\|L_\mu\|_E = 1$ for every $\mu = 0, 1, \dots, n$.

In order to prove the implication $\text{KPH} \Rightarrow \text{LMP}$, we need to do some preparations. The first proposition says that one can deduce a certain pointwise Markov inequality for the set $E \subset \mathbb{C}$ from a polynomial Markov-type estimate for a special circular projection of E .

Proposition 2.1 *Let $E \subset \mathbb{C}$ be a compact set and $z_0 \in E$, $r > 0$, $n \in \mathbb{N}$ be fixed. Put*

$$T = T(z_0, r) := \{t \in [0, r] : \exists z \in E : |z - z_0| = t\}.$$

Assume that T contains at least $n + 1$ different points and $t_0, \dots, t_n$ are Fekete extremal points of order n for T , $t_0 = \min\{t_0, \dots, t_n\}$. If there exists a constant $C_n > 0$ such that for all Lagrange polynomials $L_1, \dots, L_n$ associated to $t_0, \dots, t_n$ we have

$$|L'_\mu(0)| \leq C_n \|L_\mu\|_T, \quad \text{as } \mu = 1, \dots, n, \quad (7)$$

then for every polynomial $P \in \mathcal{P}_n(\mathbb{C})$ it follows that

$$|P'(z_0)| \leq 2n C_n \|P\|_{E \cap B(z_0, r)}. \quad (8)$$

Proof We first observe that $t_0 = 0$. Choose $z_1, \dots, z_n \in E \cap B(z_0, r)$ such that $|z_v - z_0| = t_v$ for $v = 1, \dots, n$. Fix $P \in \mathcal{P}_n(\mathbb{C})$. It is easy to check that

$$P'(z_0) = \sum_{\mu=1}^n \left[\frac{P(z_\mu) - P(z_0)}{z_\mu - z_0} \prod_{v=1, v \neq \mu}^n \frac{z_0 - z_v}{z_\mu - z_v} \right].$$

Hence

$$|P'(z_0)| \leq 2 \|P\|_{E \cap B(z_0, r)} \sum_{\mu=1}^n \frac{\prod_{v=1, v \neq \mu}^n |z_0 - z_v|}{\prod_{v=0, v \neq \mu}^n |z_\mu - z_v|}.$$

Since $|z_\mu - z_v| \geq |t_\mu - t_v|$ for each μ and v , by inequality (7), we have

$$\begin{aligned} |P'(z_0)| &\leq 2 \|P\|_{E \cap B(z_0, r)} \sum_{\mu=1}^n \frac{\prod_{v=1, v \neq \mu}^n t_v}{\prod_{v=0, v \neq \mu}^n |t_\mu - t_v|} = 2 \|P\|_{E \cap B(z_0, r)} \sum_{\mu=1}^n |L'_\mu(0)| \\ &\leq 2 \|P\|_{E \cap B(z_0, r)} \sum_{\mu=1}^n C_n \|L_\mu\|_T = 2n C_n \|P\|_{E \cap B(z_0, r)}. \end{aligned}$$

□

Proposition 2.1 and its generalizations ([4, Theorems 1.1 and 1.2]) seem to be a result of independent interest (see [4, Section 3]).

Note that inequality (8) does not imply property (7). It is sufficient to consider $E := \{0\} \cup \{z \in \mathbb{C} : |z| = r\}$ and $z_0 = 0$. By Cauchy's inequality, condition (8) is satisfied but $T = \{0, r\}$ does not admit any Markov inequality.

The next proposition shows that in the definition of the local Markov property we can replace $P^{(j)}(z_0)$ by $P'(z_0)$.

Proposition 2.2 *If $E \subset \mathbb{C}$ is a compact set, $m, \kappa \geq 1$ and there exists $c_1 > 0$ such that for every $z_0 \in E$, $r \in (0, 1]$, $P \in \mathcal{P}_n(\mathbb{C})$*

$$|P'(z_0)| \leq \frac{c_1 n^\kappa}{r^m} \|P\|_{E \cap B(z_0, r)}, \quad (9)$$

then the set E admits the LMP($m, \kappa + m$).

Proof Fix arbitrary $n \in \mathbb{N}$, z_0 , r , P and $j \in \{2, \dots, n\}$. By applying the assumption to the derivative $P^{(j-1)}$ and radius r/n , we see that

$$|P^{(j)}(z_0)| \leq \frac{c_1 n^{m+\kappa}}{r^m} \|P^{(j-1)}\|_{E \cap B(z_0, r/n)}.$$

Let z_1 be a point of $E \cap B(z_0, \frac{r}{n})$ such that $\|P^{(j-1)}\|_{E \cap B(z_0, r/n)} = |P^{(j-1)}(z_1)|$. Next we obtain similarly

$$|P^{(j)}(z_0)| \leq \frac{c_1 n^{m+\kappa}}{r^m} |P^{(j-1)}(z_1)| \leq \left(\frac{c_1 n^{m+\kappa}}{r^m} \right)^2 \|P^{(j-2)}\|_{E \cap B(z_1, r/n)}.$$

We continue in this fashion to obtain points $z_1, \dots, z_{j-1}$ such that for all $v = 1, \dots, j-1$ we have $z_v \in E \cap B(z_{v-1}, \frac{r}{n})$ and $\|P^{(j-v)}\|_{E \cap B(z_{v-1}, \frac{r}{n})} = |P^{(j-v)}(z_v)|$. This way we conclude that

$$\begin{aligned} |P^{(j)}(z_0)| &\leq \left(\frac{c_1 n^{m+\kappa}}{r^m} \right)^j \|P\|_{E \cap B(z_{j-1}, r/n)} \\ &\leq \left(\frac{c_1 n^{m+\kappa}}{r^m} \right)^j \|P\|_{E \cap B(z_0, jr/n)} \leq \left(\frac{c_1 n^{m+\kappa}}{r^m} \right)^j \|P\|_{E \cap B(z_0, r)}. \end{aligned}$$

□

Lemma 2.3 *If $P \in \mathcal{P}_n$ and $r > 0$, then there exists an interval $I \subset [0, r]$ of length at least $\frac{r}{4n^2}$ such that*

$$\|P\|_{[0,r]} \leq 2|P(x)| \quad \text{for all } x \in I.$$

Proof Let x_0 be a point of $[0, r]$ such that $|P(x_0)| = \|P\|_{[0,r]}$. Put $I := [x_0 - \frac{r}{4n^2}, x_0 + \frac{r}{4n^2}] \cap [0, r]$ and consider an arbitrary point $x \in I$. The mean value theorem leads to

$$|P(x_0) - P(x)| \leq \|P'\|_I \cdot |x_0 - x| \leq \frac{r}{4n^2} \|P'\|_{[0,r]}.$$

The interval $[0, r]$ admits the classical Markov inequality (see e.g. [10, Ch.4, Theorem 1.4]), hence

$$\|P'\|_{[0,r]} \leq \frac{2n^2}{r} \|P\|_{[0,r]} = \frac{2n^2}{r} |P(x_0)|.$$

From the above it follows that $|P(x_0)| - |P(x)| \leq \frac{1}{2} |P(x_0)|$, and finally $|P(x_0)| \leq 2|P(x)|$. □

Proof of Theorem 1.9.a Fix an arbitrary integer $a \geq m+1$ and put $m_a = \frac{s(a+c_0)-m}{a-m} \geq 1$ and $\kappa_a = \frac{(\kappa+2s)(a+c_0)}{a-m} > 3$. We will use Proposition 2.2 to prove $\text{LMP}(m_a, \kappa_a + m_a)$, which for sufficiently large a leads to the assertion of the theorem, because $\lim_{a \rightarrow \infty} m_a = s$ and $\lim_{a \rightarrow \infty} (\kappa_a + m_a) = \kappa + 3s$.

Hence we need to prove the assumption of Proposition 2.2. For that purpose, fix arbitrarily $n \in \mathbb{N}$, $z_0 \in E$ and $r \in (0, 1]$. Define $T = T(z_0, r)$, $t_0, \dots, t_n$ and L_μ for $\mu = 1, \dots, n$ as in Proposition 2.1. Note that the set $T(z_0, r)$ contains at least $n+1$ different points, because every compact admitting KPH is a perfect set and thus z_0 is not an isolated point of E and $t_0 = \min\{t_0, \dots, t_n\} = 0$ is not an isolated point of $T(z_0, r)$. Let I_μ be an interval of length at least $\frac{r}{4n^2}$ constructed for the polynomial L_μ as in Lemma 2.3.

If for every $\mu = 1, \dots, n$ there exists $z_\mu \in E$ such that $|z_\mu - z_0| \in I_\mu$, then we use Proposition 2.1. Specifically, in this case for $\mu = 1, \dots, n$, the interval I_μ intersects the set $T(z_0, r)$, say at the point τ_μ . By applying the classic Markov inequality for the interval $[0, r]$ and Lemma 2.3, we obtain for $\mu = 1, \dots, n$

$$|L'_\mu(0)| \leq \frac{2n^2}{r} \|L_\mu\|_{[0,r]} \leq \frac{4n^2}{r} |L_\mu(\tau_\mu)| \leq \frac{4n^2}{r} \|L_\mu\|_{T(z_0, r)}.$$

Hence by Proposition 2.1, we have

$$|P'(z_0)| \leq \frac{8n^3}{r} \|P\|_{E \cap B(z_0, r)} \leq \frac{8n^{\kappa_a}}{r^{m_a}} \|P\|_{E \cap B(z_0, r)}$$

for all polynomials $P \in \mathcal{P}_n$ as required in the assumption of Proposition 2.2.

We now turn to the case where $I_\mu \cap T(z_0, r) = \emptyset$ for some $\mu \in \{1, \dots, n\}$, which implies that there is an empty annulus around z_0 of a certain minimum size. We shall have established the first implication in Theorem 1.9.a if we prove that in this case we have for all $P \in \mathcal{P}_n$

$$|P'(z_0)| \leq C n^{\kappa \frac{a+c_0}{a-m}} \left(\frac{r^m}{l_\mu^{s(a+c_0)}} \right)^{\frac{1}{a-m}} \|P\|_{E \cap B(z_0, r)}, \quad (10)$$

where C is a constant independent of P, n, r, z_0 and l_μ is the length of I_μ . Indeed, by Lemma 2.3, $l_\mu \geq \frac{r}{4n^2}$ and applying this to inequality (10) we obtain

$$|P'(z_0)| \leq C n^{\kappa \frac{a+c_0}{a-m}} \frac{(4n^2)^{\frac{s(a+c_0)}{a-m}}}{r^{\frac{s(a+c_0)-m}{a-m}}} \|P\|_{E \cap B(z_0, r)} = 4^{s \frac{a+c_0}{a-m}} C \frac{n^{\kappa_a}}{r^{m_a}} \|P\|_{E \cap B(z_0, r)},$$

as required.

It remains to show inequality (10) for all $P \in \mathcal{P}_n$. For this purpose, we write $[\rho_0, \rho_1]$ for the interval I_μ . Of course, $0 < \rho_0 < \rho_1 \leq r$ and $\rho_1 - \rho_0 = l_\mu$.

The rest of the proof is adapted from [Bos and Milman (A Geometric Interpretation and the Equality of Exponents in Markov and Gagliardo-Nirenberg (Sobolev) Type Inequalities for Singular Compact Domains (preprint)), Theorem A]. Let $\varepsilon \in C^\infty(\mathbb{R})$ be any cutoff function with the following properties:

- (a) $0 \leq \varepsilon(x) \leq 1$ for all $x \in \mathbb{R}$,
- (b) $\varepsilon(x) = 1$ for $x \leq \frac{1}{3}$,
- (c) $\varepsilon(x) = 0$ for $x \geq \frac{2}{3}$

and put $h(z) := \varepsilon\left(\frac{|z-z_0|-\rho_0}{l_\mu}\right)$ so that $h \in C^\infty(\mathbb{C})$.

Now fix arbitrarily $P \in \mathcal{P}_n$ and let $Q(z) := (P(z) - P(z_0))^a$, $Q \in \mathcal{P}_{an}$, and $f(z) := h(z)Q(z)$. Choose any $l \in \mathbb{N}$ such that $a^2n \leq l \leq 2a^2n$. Since $T(z_0, r) \cap [\rho_0, \rho_0 + l_\mu] = \emptyset$, we have $f \in \mathcal{H}^\infty(E_\delta)$, where $\delta := \frac{l_\mu}{3} \leq \frac{r}{3}$.

We see that

$$\|f\|_E = \|f\|_{E \cap B(z_0, \rho_1)} \leq \|Q\|_{E \cap B(z_0, \rho_1)} \leq \|Q\|_{E \cap B(z_0, r)}, \quad (11)$$

$$\|f\|_{E_\delta} = \|f\|_{E_\delta \cap B(z_0, \rho_1)} \leq \|f\|_{B(z_0, \rho_1)} \leq \|Q\|_{B(z_0, \rho_1)} \leq \|Q\|_{B(z_0, r)}, \quad (12)$$

and by the KPH(m, s, κ), we have for arbitrary $j \in \mathbb{N}$ such that $j \leq an \leq \frac{l}{a} < \frac{l}{m}$

$$|Q^{(j)}(z_0)| \leq \left\| \frac{\partial^j f}{\partial z^j} \right\|_E = \|f\|_{E, j} \leq \left(\frac{c_1 l^\kappa}{\delta^s} \right)^{j+c_0} \|f\|_E^{1-\frac{mj}{T}} \|f\|_{E_\delta}^{\frac{mj}{T}}. \quad (13)$$

From inequalities (11–13) and the choice of δ , it follows that

$$|Q^{(j)}(z_0)| \leq \left(\frac{3^s c_1 l^\kappa}{l_\mu^s} \right)^{j+c_0} \|Q\|_{E \cap B(z_0, r)}^{1-\frac{mj}{T}} \|Q\|_{B(z_0, r)}^{\frac{mj}{T}}. \quad (14)$$

Our next objective is to estimate

$$\lambda := \left(\frac{\|Q\|_{B(z_0, r)}}{\|Q\|_{E \cap B(z_0, r)}} \right)^{\frac{1}{an}} \geq 1.$$

By inequality (14), we have for any $j = 1, \dots, an$

$$|Q^{(j)}(z_0)| \leq \left(\frac{3^s c_1 l^\kappa}{l_\mu^s} \right)^{j+c_o} \lambda^{\frac{mj}{a}} \|Q\|_{E \cap B(z_0, r)} \quad (15)$$

and for $j = 0$ this is trivially true too. From this, applying Taylor's formula and the fact that $l_\mu \leq r \leq 1$, we get

$$\begin{aligned} \|Q\|_{B(z_0, r)} &\leq \sum_{j=0}^{an} \frac{r^j |Q^{(j)}(z_0)|}{j!} \leq \sum_{j=0}^{an} (3^s c_1 l^\kappa)^{j+c_o} \left(\frac{r}{l_\mu^s} \right)^j \lambda^{\frac{mj}{a}} \|Q\|_{E \cap B(z_0, r)} \\ &\leq (2 \cdot 3^s c_1 l^\kappa)^{an+c_o} \left(\frac{r}{l_\mu^s} \right)^{an} \frac{\lambda^{mn}}{l_\mu^{sc_o}} \|Q\|_{E \cap B(z_0, r)}. \end{aligned}$$

It follows that

$$\lambda^{an} = \frac{\|Q\|_{B(z_0, r)}}{\|Q\|_{E \cap B(z_0, r)}} \leq (2 \cdot 3^s c_1 l^\kappa)^{an+c_o} \left(\frac{r}{l_\mu^s} \right)^{an} \frac{\lambda^{mn}}{l_\mu^{sc_o}},$$

and consequently

$$\lambda \leq (2 \cdot 3^s c_1 l^\kappa)^{\frac{a+c_o/n}{a-m}} \left(\frac{r}{l_\mu^s} \right)^{\frac{a}{a-m}} \cdot \left(\frac{1}{l_\mu^{sc_o}} \right)^{\frac{1}{(a-m)n}}.$$

By combining this estimate with inequality (15), we can assert that

$$\begin{aligned} |Q^{(j)}(z_0)| &\leq (2 \cdot 3^s c_1 l^\kappa)^{j+c_o + \frac{mj(a+\frac{c_o}{n})}{a(a-m)}} \left(\frac{1}{l_\mu} \right)^{s(j+c_o) + \frac{jms}{a-m} + \frac{sc_o mj}{an(a-m)}} r^{\frac{jm}{a-m}} \|Q\|_{E \cap B(z_0, r)} \\ &\leq (2 \cdot 3^s c_1 l^\kappa)^{\frac{a}{a-m}(j+c_o)} \left(\frac{1}{l_\mu} \right)^{\frac{as}{a-m}(j+c_o)} r^{\frac{jm}{a-m}} \|Q\|_{E \cap B(z_0, r)}, \end{aligned}$$

because $\frac{mj}{a}(a + \frac{c_o}{n}) = m(j + \frac{j}{an}c_o) \leq m(j + c_o)$ and $(a-m)s(j+c_o) + jms + sc_o m \frac{j}{an} \leq (a-m)s(j+c_o) + jms + sc_o m = as(j+c_o)$. Specifically, for $j = a$ we have

$$a! |P'(z_0)|^a = |Q^{(a)}(z_0)| \leq (2 \cdot 3^s c_1 l^\kappa)^{a \frac{a+c_o}{a-m}} \left(\frac{r^m}{l_\mu^{s(a+c_o)}} \right)^{\frac{a}{a-m}} \|Q\|_{E \cap B(z_0, r)}.$$

By the definition of Q and since $l \leq 2a^2n$, we finally get

$$|P'(z_0)| \leq 2 (2^{1+\kappa} 3^s a^{2\kappa} c_1 n^\kappa)^{\frac{a+c_o}{a-m}} \left(\frac{r^m}{l_\mu^{s(a+c_o)}} \right)^{\frac{1}{a-m}} \|P\|_{E \cap B(z_0, r)},$$

which implies inequality (10).

The second statement of Theorem 1.9.a is easy to show because $m_a = 1$ for $s = 1$, $c_o = 0$ and arbitrary m . $\square$

3 The LMP Implies the KPQ

Lemma 3.1 *If $E \subset \mathbb{C}$ is a compact set with the LMP(m, κ), then there exist $c_1 \geq 1$ such that for every $z_0 \in E$ and $l \in \mathbb{N}$*

$$\left| \frac{\partial^j g}{\partial z^j}(z_0) \right| \leq (c_1 l^\kappa)^j \|g\|_E^{1-\frac{mj}{l}} \|g\|_{\text{conv}(E), l}^{\frac{mj}{l}}, \quad j = 1, \dots, l$$

for any function $g \in C^\infty(\mathbb{C})$ that is $\bar{\partial}$ -flat at z_0 and $g|_E \neq 0$.

Proof Our reasoning is a modification of that given by Bos and Milman (A Geometric Interpretation and the Equality of Exponents in Markov and Gagliardo-Nirenberg (Sobolev) Type Inequalities for Singular Compact Domains (preprint)). Fix $z_0 \in E$, $l \in \mathbb{N}$ and $g \in C^\infty(\mathbb{C})$ satisfying the assumptions of the lemma. Let $T_{z_0}^l g$ be the Taylor polynomial defined by $T_{z_0}^l g(z) := \sum_{|\alpha| \leq l-1} \frac{1}{\alpha!} D^\alpha g(z_0)(z - z_0)^\alpha = \sum_{j=0}^{l-1} \frac{1}{j!} \frac{\partial^j g}{\partial z^j}(z_0) (z - z_0)^j$, because g is $\bar{\partial}$ -flat in z_0 . By the local Markov property, it follows that

$$\left| \frac{\partial^j g}{\partial z^j}(z_0) \right| = \left| \frac{d^j}{dz^j} T_{z_0}^l g(z_0) \right| \leq \left[\frac{c_1 (l-1)^\kappa}{r^m} \right]^j \|T_{z_0}^l g\|_{E \cap B(z_0, r)}$$

for every $r \in (0, 1]$ and $j = 1, \dots, l-1$. If $R_{z_0}^l g := g - T_{z_0}^l g$ is the Taylor remainder, then we have

$$\begin{aligned} \left| \frac{\partial^j g}{\partial z^j}(z_0) \right| &\leq \left(\frac{c_1 l^\kappa}{r^m} \right)^j (\|g\|_{E \cap B(z_0, r)} + \|R_{z_0}^l g\|_{E \cap B(z_0, r)}) \\ &\leq \left(\frac{c_1 l^\kappa}{r^m} \right)^j (\|g\|_E + r^l |g|_{\text{conv}(E), l}) \end{aligned}$$

for $j = 1, \dots, l-1$ and this goes also for $j=l$, which is easy to check because $\left| \frac{\partial^l g}{\partial z^l}(z_0) \right| \leq |g|_{\text{conv}(E), l}$. Put

$$r = \left(\frac{\|g\|_E}{A} \right)^{\frac{1}{l}}, \quad \text{where} \quad A := \|g\|_E + |g|_{\text{conv}(E), l}.$$

Of course, $r \in (0, 1]$. From the above it follows that

$$\begin{aligned} \left| \frac{\partial^j g}{\partial z^j}(z_0) \right| &\leq \left(\frac{c_1 l^\kappa}{\|g\|_E^{m/l}} A^{\frac{m}{l}} \right)^j \left(\|g\|_E + \frac{\|g\|_E}{A} |g|_{\text{conv}(E), l} \right) \\ &= (c_1 l^\kappa)^j A^{\frac{mj}{l}} \|g\|_E^{1-\frac{mj}{l}} \left(1 + \frac{1}{A} |g|_{\text{conv}(E), l} \right). \end{aligned}$$

Since $|g|_{\text{conv}(E), l} < A \leq \|g\|_{\text{conv}(E), l}$, we conclude that

$$\begin{aligned} \left| \frac{\partial^j g}{\partial z^j}(z_0) \right| &\leq (c_1 l^\kappa)^j \|g\|_{\text{conv}(E), l}^{\frac{mj}{l}} \|g\|_E^{1-\frac{mj}{l}} \cdot 2 \\ &\leq (2c_1 l^\kappa)^j \|g\|_E^{1-\frac{mj}{l}} \|g\|_{\text{conv}(E), l}^{\frac{mj}{l}}, \quad j \in \{1, \dots, l\}, \end{aligned}$$

and this is precisely the assertion of the lemma. $\square$

Proof of Theorem 1.9.b We first observe that E is perfect and thus $\mathcal{A}^\infty$ -determining, i.e., if $f \in \mathcal{A}^\infty(E)$ and $f|_E \equiv 0$ then $(D^\alpha f)|_E \equiv 0$ for all $\alpha \in \mathbb{N}_0^2$.

Fix $f \in \mathcal{A}^\infty(E)$. We can certainly assume that $f|_E \not\equiv 0$, since otherwise the inequality in KPQ would be fulfilled trivially. Consider an arbitrary function $g \in C^\infty(\mathbb{C})$ that is $\bar{\partial}$ -flat on E and $g|_E \equiv f|_E$. Lemma 3.1 now gives

$$\left\| \frac{\partial^j g}{\partial \bar{z}^j} \right\|_E \leq (c_1 l^k)^j \|g\|_E^{1-\frac{mj}{7}} \|g\|_{\text{conv}(E), l}^{\frac{mj}{7}} \quad \text{as } 1 \leq j \leq l.$$

We have $\frac{\partial^j g}{\partial \bar{z}^j}|_E \equiv \frac{\partial^j f}{\partial \bar{z}^j}|_E$, because $g - f \in \mathcal{A}^\infty(E)$, $(g - f)|_E \equiv 0$ and E is $\mathcal{A}^\infty$ -determining. Consequently, for $1 \leq j \leq l$ we get

$$\left\| \frac{\partial^j f}{\partial \bar{z}^j} \right\|_E \leq (c_1 l^k)^j \|f\|_E^{1-\frac{mj}{7}} \|f\|_{E, l}^{\frac{mj}{7}}$$

which completes the proof. $\square$

4 The KPQH Implies the KPH

First we give a lemma on cutoff functions, which allows us to estimate the holomorphic quotient norms. The lemma is an adaptation to the complex case of a result given e.g. in [Bos and Milman (A Geometric Interpretation and the Equality of Exponents in Markov and Gagliardo-Nirenberg (Sobolev) Type Inequalities for Singular Compact Domains (preprint)), Lemma 4.12].

Lemma 4.1 *For any compact set $E \subset \mathbb{C}$ and radius $\varepsilon \in (0, 1]$ there exists a cutoff function $u \in C^\infty(\mathbb{C})$ such that*

- (a) $0 \leq u(z) \leq 1$ for all $z \in \mathbb{C}$,
- (b) $u(z) = 1$ as $\text{dist}(z, E) \leq \frac{\varepsilon}{8}$,
- (c) $u(z) = 0$ as $\text{dist}(z, E) \geq \varepsilon$,
- (d) $\|D^\alpha u\|_{\mathbb{C}} \leq \frac{C_{|\alpha|}}{\varepsilon^{|\alpha|}}$ for all $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$,

where $C_j = d j^4$ for $j \in \mathbb{N}$, $C_0 = 1$ and $d \geq 1$ is some absolute constant.

Proposition 4.2 *For any compact set $E \subset \mathbb{C}$, $\delta \in (0, 1]$, $f \in \mathcal{H}^\infty(E_\delta)$ and $l \in \mathbb{N}$ we have*

$$\langle\langle f \rangle\rangle_{E_{\delta/17}, l} \leq \frac{(2dl)^{5l}}{\delta^l} \|f\|_{E_\delta},$$

where $d \geq 1$ is the absolute constant from Lemma 4.1.

Proof Let $u \in C^\infty(\mathbb{C})$ be the cutoff function given in Lemma 4.1 for the set E and radius $\varepsilon = \frac{\delta}{2}$. We put $\tilde{f} = uf$ and see that $\tilde{f} \in \mathcal{H}^\infty(E_{\frac{\delta}{17}})$ because $\tilde{f} \equiv f$ on $E_{\frac{\delta}{16}}$, which contains an open neighbourhood of the set $E_{\frac{\delta}{17}}$. In $\mathbb{C} \setminus E_{\frac{\delta}{2}}$ we have $u \equiv 0$ and hence $D^\alpha \tilde{f} \equiv 0$ for all $\alpha \in \mathbb{N}_0^2$. Consequently, by the definition of the holomorphic quotient

norms, we have $\langle\langle f \rangle\rangle_{E_{\delta/17}, l} \leq \|\tilde{f}\|_{\text{conv}(E_{\delta/17}), l} \leq \|\tilde{f}\|_{C, l} = \|\tilde{f}\|_{E_{\delta/2}, l}$. By the Leibniz rule, we obtain for every $z \in E_{\delta/2}$ and $\alpha \in \mathbb{N}_0^2$

$$D^\alpha \tilde{f}(z) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta u(z) D^{\alpha-\beta} f(z) = \sum_{\beta_1 \leq \alpha_1} \binom{\alpha_1}{\beta_1} D^{(\beta_1, \alpha_2)} u(z) f^{(\alpha_1-\beta_1)}(z)$$

because f is holomorphic in E_δ . As usual, $\beta \leq \alpha$ means that $\beta_1 \leq \alpha_1$, $\beta_2 \leq \alpha_2$ and $\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \binom{\alpha_2}{\beta_2}$. The properties of the cutoff function and Cauchy's integral formula lead us to

$$\begin{aligned} \|D^\alpha \tilde{f}\|_{E_{\frac{\delta}{2}}} &\leq \sum_{\beta_1 \leq \alpha_1} \binom{\alpha_1}{\beta_1} \frac{C_{\beta_1+\alpha_2}}{(\delta/2)^{\beta_1+\alpha_2}} \frac{(\alpha_1-\beta_1)!}{(\delta/2)^{\alpha_1-\beta_1}} \|f\|_{E_\delta} \\ &= \sum_{\beta_1 \leq \alpha_1} \frac{\alpha_1!}{\beta_1!} \frac{C_{\beta_1+\alpha_2}}{(\delta/2)^{|\alpha|}} \|f\|_{E_\delta} \leq \frac{e! C_l}{(\delta/2)^l} \|f\|_{E_\delta} \leq \frac{2e! C_l}{\delta^l} \|f\|_{E_\delta} = \frac{2edl^{5l}}{\delta^l} \|f\|_{E_\delta} \end{aligned}$$

for $|\alpha| = l$. Finally, we see that

$$\langle\langle f \rangle\rangle_{E_{\frac{\delta}{17}}, l} \leq \|\tilde{f}\|_{E_{\frac{\delta}{2}}} + \sum_{|\alpha|=l} \|D^\alpha \tilde{f}\|_{E_{\frac{\delta}{2}}} \leq (l+2) \frac{2edl^{5l}}{\delta^l} \|f\|_{E_\delta} \leq \frac{(2dl)^{5l}}{\delta^l} \|f\|_{E_\delta},$$

because $(l+2)2e \leq 2^{5l}$. $\square$

Proof of Theorem 1.9.d Fix $j, l \in \mathbb{N}$ such that $l \geq mj$, $\delta \in (0, 1]$ and $f \in \mathcal{H}^\infty(E_\delta)$. We combine KPQH with Proposition 4.2 to obtain

$$\begin{aligned} \|f\|_{E, j} &\leq \left(\frac{c_1 l^k}{(\delta/17)^s} \right)^j \|f\|_E^{1-\frac{mj}{l}} \langle\langle f \rangle\rangle_{E_{\delta/17}, l}^{\frac{mj}{l}} \leq \left(\frac{17^s c_1 l^k}{\delta^s} \right)^j \|f\|_E^{1-\frac{mj}{l}} \frac{(2dl)^{5mj}}{\delta^{mj}} \|f\|_{E_\delta}^{\frac{mj}{l}} \\ &= \left(\frac{17^s c_1 l^k (2dl)^{5m}}{\delta^{m+s}} \right)^j \|f\|_E^{1-\frac{mj}{l}} \|f\|_{E_\delta}^{\frac{mj}{l}} = \left(\tilde{c}_1 l^{k+5m} \right)^j \|f\|_E^{1-\frac{mj}{l}} \|f\|_{E_\delta}^{\frac{mj}{l}}, \end{aligned}$$

where $\tilde{c}_1 = 17^s c_1 (2d)^{5m}$. This proves Theorem 1.9.d. $\square$

It is worth noticing that if we replace in Definitions 1.4, 1.5, 1.7 and 1.8 $c_1 n^k$ or $c_1 l^k$ by c_n or c_l respectively, then Theorem 1.9. remains true. Bos and Milman labeled such properties of Markov or Sobolev type in the real case as *weak* inequalities.

5 The LMP(m) Implies (m, ∞) -perfectness

Definition 5.1 (see [3, Definition 3.1] for more details) Let $m \geq 1$ and $k \in \{2, 3, 4, \dots\}$. A compact set $E \subset \mathbb{C}$ is (m, k) -perfect if there exist constants $a, b > 0$ such that for every $z \in E$ we can construct a family of closed discs $\{B\} \cup \{B_{i_1, \dots, i_n}\}_{n \in \mathbb{N}, i_1, \dots, i_n \in \{1, \dots, k\}}$ satisfying the following conditions

- (i) $B_{i_1, \dots, i_{n+1}} \subset B_{i_1, \dots, i_n}$ for $n \in \mathbb{N}$, $i_1, \dots, i_{n+1} = 1, 2, \dots, k$,
 $B_{i_1} \subset B$ for $i_1 = 1, 2, \dots, k$,

- (ii) $\text{diam } B_{i_1, \dots, i_{n+1}} = a (\text{diam } B_{i_1, \dots, i_n})^m$ for $i_1, \dots, i_{n+1} = 1, 2, \dots, k$,
 $\text{diam } B_{i_1} = a (\text{diam } B)^m = a$ for $i_1 = 1, 2, \dots, k$,
- (iii) $\text{dist}(B_{i_1, \dots, i_n, p}, B_{i_1, \dots, i_n, q}) \geq b (\text{diam } B_{i_1, \dots, i_n})^m$ for $n \in \mathbb{N}$, $i_1, \dots, i_n, p, q = 1, 2, \dots, k$, $p \neq q$,
 $\text{dist}(B_p, B_q) \geq b (\text{diam } B)^m = b$ for $p, q = 1, 2, \dots, k$, $p \neq q$,
- (iv) $z \in \bigcap_{n=1}^{\infty} \bigcup_{i_n=1}^k B_{i_1, \dots, i_n} \subset E$,

where $\text{diam } D$ is the diameter of a set D and $\text{dist}(D_1, D_2)$ is the distance between sets D_1, D_2 .

We have shown in [3, Theorem 3.4] that if E is an m -perfect set then it is $(m, 2)$ -perfect and if E is (m, k) -perfect for some $k \in \{2, 3, \dots\}$ then it is m^2 -perfect. Furthermore, an (m, k) -perfect set E with $m < k$ satisfies the following estimate of the logarithmic capacity: there exists a constant $c > 0$ such that

$$\text{cap}(E \cap B(z, r)) \geq cr^{\frac{k-1}{k-m}m^2} \quad (16)$$

for every $z \in E$ and $r \in (0, 1]$ (see [3, Proposition 3.6]).

Definition 5.2 We call a compact set (m, ∞) -perfect if it is (m, k) -perfect for every $k \in \{2, 3, \dots\}$.

According to the above, by the Wiener criterion (see e.g. [15, Theorem 5.4.1]), every (m, k) -perfect set with $m < k$ is L-regular. Therefore, (m, ∞) -perfectness implies L-regularity.

Theorem 5.3 If $m \geq 1$ and E is a compact subset of the complex plane satisfying the following Markov property: for each $n \in \mathbb{N}$ there exists a constant $c_n \geq 1$ such that

$$|P'(z_0)| \leq \frac{c_n}{r^m} \|P\|_{E \cap B(z_0, r)} \quad (17)$$

for all $z_0 \in E$, $r \in (0, 1]$ and $P \in \mathcal{P}_n(\mathbb{C})$, then E is an (m, ∞) -perfect set.

Proof Fix $z_0 \in E$, $r \in (0, 1]$ and $k \in \{3, 4, \dots\}$. We will show that there exists a positive constant a_k independent of z_0, r and there exist points $z_1, \dots, z_{k-1} \in E \cap B(z_0, r)$ such that

$$|z_i - z_j| \geq a_k r^m \quad \text{for each } i, j \in \{0, 1, \dots, k-1\}, i \neq j,$$

which, according to [3, Lemma 3.2], is sufficient for (m, k) -perfectness. Note that $(m, 3)$ -perfectness implies $(m, 2)$ -perfectness.

Put $a_k := [2(2e)^m(k-1)^m c_{k-1}]^{-1}$. We construct the points $\{z_j\}_{j=1, \dots, k-1}$ as follows. Let $\mu = 1$. Put

$$r_\mu := r \left(\frac{k-2}{k-1} \right)^{\mu-1}. \quad (18)$$

Now find Fekete extremal points of order $k - 1$ (see the beginning of Section 2) for the intersection $E \cap B(z_0, r_\mu)$ and denote them by $y_1^{(\mu)}, \dots, y_k^{(\mu)}$. Let $L_i^{(\mu)}$ be the Lagrange polynomials associated with $y_1^{(\mu)}, \dots, y_k^{(\mu)}$

$$L_i^{(\mu)}(z) = \frac{\prod_{l=1, l \neq i}^k (z - y_l^{(\mu)})}{\prod_{l=1, l \neq i}^k (y_i^{(\mu)} - y_l^{(\mu)})}, \quad i = 1, \dots, k.$$

Observe that for all $1 \leq i < j \leq k$

$$\frac{1}{|y_j^{(\mu)} - y_i^{(\mu)}|^2} = \left| \frac{d}{dz} L_i^{(\mu)}(y_j^{(\mu)}) \right| \cdot \left| \frac{d}{dz} L_j^{(\mu)}(y_i^{(\mu)}) \right|.$$

Thus, by inequality (17), for every $\varrho \in (0, 1]$

$$\frac{1}{|y_j^{(\mu)} - y_i^{(\mu)}|^2} \leq \left(\frac{c_{k-1}}{\varrho^m} \right)^2 \|L_i^{(\mu)}\|_{E \cap B(y_j^{(\mu)}, \varrho)} \|L_j^{(\mu)}\|_{E \cap B(y_i^{(\mu)}, \varrho)}.$$

Assume that for each $i = 1, \dots, k$ we have

$$|y_i^{(\mu)} - z_0| \leq \frac{k - \frac{3}{2}}{k - 1} r_\mu. \quad (19)$$

In that case we put $\varrho = \frac{1/2}{k-1} r_\mu$ and in consequence we conclude

$$E \cap B(y_i^{(\mu)}, \varrho) \subset E \cap B(z_0, r_\mu) \quad \text{for every } i = 1, \dots, k$$

and

$$\begin{aligned} \frac{1}{|y_j^{(\mu)} - y_i^{(\mu)}|^2} &\leq \left[\frac{c_{k-1}}{(\frac{1/2}{k-1} r_\mu)^m} \right]^2 \|L_i^{(\mu)}\|_{E \cap B(z_0, r_\mu)} \|L_j^{(\mu)}\|_{E \cap B(z_0, r_\mu)} \\ &= \left[\frac{2^m (k-1)^m c_{k-1}}{r_\mu^m} \right]^2 \quad \text{for all } 1 \leq i < j \leq k, \end{aligned}$$

because the norm of Lagrange polynomials on the set $E \cap B(z_0, r_\mu)$ equals 1. Finally

$$|y_j^{(\mu)} - y_i^{(\mu)}| \geq \frac{r_\mu^m}{2^m (k-1)^m c_{k-1}} = \frac{[r(\frac{k-2}{k-1})^{\mu-1}]^m}{2^m (k-1)^m c_{k-1}}.$$

Note that for all $x > 0$ we have $(\frac{x+1}{x})^x = (1 + \frac{1}{x})^x < e$ and in particular, putting $x := k - 2$, we obtain $(\frac{k-2}{k-1})^{k-2} > \frac{1}{e}$. From this it follows that for all $1 \leq i < j \leq k$

$$|y_j^{(\mu)} - y_i^{(\mu)}| > \frac{r_\mu^m}{(2e)^m (k-1)^m c_{k-1}} = 2a_k r_\mu^m, \quad \text{as long as } \mu \leq k - 1.$$

Thus at most one point of the set $\{y_i^{(\mu)}\}_{i=1, \dots, k}$ can be included in the interior of $B(z_0, a_k r_\mu^m)$. After removing from $\{y_i^{(\mu)}\}_{i=1, \dots, k}$ that one point (or any arbitrary point

if none belongs to the interior of $B(z_0, a_k r^m)$ we are left with $k - 1$ points that meet the requirements of Lemma 3.2 in [3].

If assumption (19) is not met then we conclude that for a certain $i \in \{1, \dots, k\}$ we have $|y_i^{(\mu)} - z_0| > \frac{k-3/2}{k-1} r_\mu$. We put $z_\mu := y_i^{(\mu)}$ after which we increase μ by 1 and we return to inequality (18).

This way, either for a certain $\mu \in \{1, \dots, k-1\}$ condition (19) will be satisfied and then the problem will be solved or we end up with a set of points $\{z_\mu\}_{\mu=1, \dots, k-1} \subset E$ with the following property

$$r \left(\frac{k-2}{k-1} \right)^{\mu-1} \frac{k-\frac{3}{2}}{k-1} = \frac{k-\frac{3}{2}}{k-1} r_\mu < |z_\mu - z_0| \leq r_\mu = r \left(\frac{k-2}{k-1} \right)^{\mu-1}$$

for each $\mu \in \{1, \dots, k-1\}$. From this it is easy to see that for all $1 \leq \mu < \nu \leq k-1$

$$\begin{aligned} |z_\mu - z_\nu| &\geq |z_\mu - z_0| - |z_\nu - z_0| \geq r \left(\frac{k-2}{k-1} \right)^{\mu-1} \frac{k-\frac{3}{2}}{k-1} - r \left(\frac{k-2}{k-1} \right)^{\nu-1} \\ &\geq r \left(\frac{k-2}{k-1} \right)^{\mu-1} \left(\frac{k-\frac{3}{2}}{k-1} - \frac{k-2}{k-1} \right) \end{aligned}$$

hence

$$|z_\mu - z_\nu| \geq \frac{r}{2e(k-1)} \geq a_k r^m$$

since $m \geq 1$ and $c_k \geq 1$. But also

$$|z_\mu - z_0| \geq r \left(\frac{k-2}{k-1} \right)^{\mu-1} \frac{k-\frac{3}{2}}{k-1} \geq \frac{r}{2e(k-1)} \geq a_k r^m$$

for each $\mu = 1, \dots, k-1$, which finishes the proof. $\square$

The Markov property defined by inequality (17) can be found in [3, 8, 20] Bos and Milman (A Geometric Interpretation and the Equality of Exponents in Markov and Gagliardo-Nirenberg (Sobolev) Type Inequalities for Singular Compact Domains (preprint)). If a set $E \subset \mathbb{C}$ has the LMP(m, κ) with constant c_1 , then c_n in inequality (17) equals $c_1 n^\kappa$ and in the proof of Theorem 4.3 we can put $a_k = \frac{c'}{k^{m+\kappa}}$ with some constant c' independent of $z_0 \in E$, $r \in (0, 1]$ and $k \in \{2, 3, \dots\}$.

Remark 5.4 By Theorem 5.3, every compact set $E \subset \mathbb{C}$ with the LMP(m) is (m, ∞) -perfect and, by consequence, it is L-regular. In this fashion, the L-regularity of sets with the LMP is proved in a different way than in [3] (compare estimate (16) with [3, Proposition 2.1]).

Remark 5.5 Let $E \subset \mathbb{C}$ be an (m, ∞) -perfect set. From what has already been described at the beginning of this section, we see that E is L-regular, i.e. the Green function g_E of the unbounded component of $\widehat{\mathbb{C}} \setminus E$ with the pole at ∞ is continuous. One may ask whether the function g_E has the following Hölder continuity property

$$g_E(z) \leq M [\text{dist}(z, E)]^\delta, \quad \text{as } \text{dist}(z, E) \leq 1,$$

the constants $M, s > 0$ being independent of z . The answer is negative. Indeed, consider the set

$$E := \{0\} \cup \bigcup_{l=2}^{\infty} [a_l, b_l], \quad \text{where } b_l := \frac{1}{l!}, \quad a_l := \frac{l}{(l+1)!} \text{ for } l = 2, 3, \dots$$

(compare with [12]).

Proposition 5.6 *The set E constructed above is an (m, ∞) -perfect set for any $m > 1$ and E does not have the Hölder continuity property.*

Proof The lack of the Hölder continuity property of E is due to Goncharov and Uzun (Markov's property of compact sets in $\mathbb{R}$ (preprint)). Let us only outline the proof of the (m, ∞) -perfectness of E . By [3, Lemma 3.2], it is sufficient to find $a > 0$ for fixed $m > 0$ and $k \in \mathbb{N}$ such that for every $l \in \mathbb{N}$

$$b_{l+1} - a_{l+1} > a a_l^m k.$$

Thus we only need to show that there exists $a > 0$ such that

$$0 < a < \frac{(l+1)!^m}{l^m(l+2)!k}.$$

Since $m > 1$, the assertion follows. $\square$

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