An Upper Bound for the Poisson Kernel on Higher Rank NA Groups

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Abstract Let S be a semi direct product $S = N \rtimes A$ where N is a connected and simply connected nilpotent Lie group and A is isomorphic with $\mathbb{R}^k$, $k > 1$. We obtain an upper bound for the Poisson kernel for the class of second order left-invariant differential operators on S .

Keywords Poisson kernel · Harmonic functions · Solvable Lie groups · Higher rank NA groups · Homogeneous group · Left invariant operators · Brownian motion

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1 Statement of the Result

Let S be a semi direct product $S = N \rtimes A$ where N is a connected and simply connected nilpotent Lie group and A is isomorphic with $\mathbb{R}^k$. For $g \in S$ we let $n(g) = n$ and $a(g) = a$ denote the components of g in this product so that $g = (n, a)$.

In what follows we identify the group A , its Lie algebra $\mathfrak{a}$, and $\mathfrak{a}^*$, the space of linear forms on $\mathfrak{a}$, with the Euclidean space $\mathbb{R}^k$ endowed with the usual scalar product

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$\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$. For the vectors $x, y \in \mathbb{R}^k$ we write $x \cdot y = \langle x, y \rangle = \sum_{i=1}^k x_i y_i$ and $x^2 = x \cdot x = \langle x, x \rangle$.

We assume that there is a basis $X_1, \dots, X_m$ for $\mathfrak{n}$ that diagonalizes the A -action. Let $\xi_1, \dots, \xi_m \in \mathfrak{a}^* = \mathbb{R}^k$ be the corresponding roots, i.e., for every $H \in \mathfrak{a}$, $[H, X_j] = \xi_j(H)X_j$, $j = 1, \dots, m$. As in [2] we assume that there is an element $H_o \in \mathbb{R}^k$ such that

$$\xi_j(H_o) > 0 \text{ for } 1 \leq j \leq m.$$

Let, for $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{R}^k$ and real d_j 's,

$$\mathcal{L}_\alpha = \sum_{j=1}^r \left(e^{2\xi_j(\alpha)} X_j^2 + d_j e^{\xi_j(\alpha)} X_j \right) + \Delta_\alpha, \quad (1.1)$$

where

$$\Delta_\alpha = \sum_{j=1}^k \partial_{a_j}^2 - 2\alpha_j \partial_{a_j},$$

and $X_1, \dots, X_r$ satisfy the Hörmander condition, i.e., they generate the Lie algebra $\mathfrak{n}$ of N .

Define

$$\rho_0 = \sum_{j=1}^m \xi_j.$$

Let

$$\chi(g) = \det(\text{Ad}(g)) = e^{\rho_0 \cdot a},$$

where

$$\text{Ad}(g)x = gxg^{-1}, x \in S.$$

Let dx be left-invariant Haar measure on S . We have

$$\int_S f(xg)dx = \chi(g) \int_S f(x)dx.$$

We set

$$A^+ = \text{Int}\{a \in \mathbb{R}^k : \xi_j(a) \geq 0 \text{ for } 1 \leq j \leq r\}.$$

Remark It is clear that we could have used all of the roots in defining A^+ since from the Hörmander condition the span over $\mathbb{N}$ of ξ_j , $1 \leq j \leq r$ contains all of the roots.

If $\alpha \in A^+$ then there exists a *Poisson kernel* v for $\mathcal{L}_\alpha$, [2]. That is, there is a C^∞ function v on N such that every bounded $\mathcal{L}_\alpha$ -harmonic function F on S may be written as a Poisson integral against a bounded function f on $S/A = N$,

$$F(g) = \int_{S/A} f(gx)v(x)dx = \int_N f(x)\check{v}^a(x^{-1}n)dx,$$

where

$$\check{v}^a(x) = v(a^{-1}x^{-1}a)\chi(a)^{-1}.$$

Conversely the Poisson integral of any $f \in L^\infty(N)$ is a bounded $\mathcal{L}_\alpha$ -harmonic function.

For $t \in \mathbb{R}^+$ and $\rho \in A^+$, let

$$\delta_t^\rho = \text{Ad}((\log t)\rho)|_N.$$

Then $t \mapsto \delta_t^\rho$ is a one parameter group of automorphisms (*dilations*) of N for which the corresponding eigenvalues on $\mathfrak{n}$ are all positive. It is known [8] that then N has δ_t^ρ -homogeneous norm: a non-negative continuous function $|\cdot|_\rho$ on N such that $|n|_\rho = 0$ if and only if $n = e$ and

$$|\delta_t^\rho n|_\rho = t|n|_\rho.$$

The main result of this paper is the following.

Theorem 1.1 *Assume that the rank (dimension of A) is $k > 1$. Let v be the Poisson kernel for the operator $\mathcal{L}_\alpha$ defined in Eq. 1.1 with $\alpha \in A^+$. Then for all $q > 1$ there exists a constant $C_q = C_{q,\alpha} > 0$ such that for all $x \in N$, the following estimate holds*

$$v(x) \leq C_q(1 + |x|_\alpha)^{-\frac{2}{q} \min_{1 \leq j \leq r} (\xi_j \alpha)^2 / \xi_j^2}.$$

For the estimates for the Poisson kernel and its derivatives on rank-one NA groups, i.e., $\dim A = 1$, see [1, 4, 5, 16].

Example Consider the operator $\mathcal{L}_\alpha$ with $\xi_1 = (1, 0)$, $\xi_2 = (0, 1)$ and $\alpha = (1/2, 1/2)$, i.e.,

$$\mathcal{L}_{(1/2, 1/2)} = e^{2a_1} \partial_x^2 + e^{2a_2} \partial_y^2 + \partial_{a_1}^2 - \partial_{a_1} + \partial_{a_2}^2 - \partial_{a_2}.$$

This operator is the Laplace-Beltrami operator for the product of two half-planes where each half-plane H^+ is identified with $\mathbb{R}^2$ using the map of $\mathbb{R}^2$ onto H^+ defined by $(x, t) \mapsto x + ie^t$. The Poisson kernel for this operator is

$$v(x, y) = \frac{\text{const}}{(1 + x^2)(1 + y^2)}.$$

In fact, in order to check that v is the Poisson kernel we note that in the multiplicative notation

$$\begin{aligned} \mathcal{L}_{(1/2, 1/2)} &= a^2 \partial_x^2 + b^2 \partial_y^2 + (a \partial_a)^2 - a \partial_a + (b \partial_b)^2 - b \partial_b \\ &= a^2 \partial_x^2 + b^2 \partial_y^2 + a^2 \partial_a^2 + b^2 \partial_b^2. \end{aligned}$$

Now it is enough to check that $LF = 0$, where

$$F(x, y, a, b) = a^{-1}b^{-1}\nu(a^{-1}x, b^{-1}y).$$

By the result obtained in [13] we have the following upper bound for ν . There exists a constant $C > 0$ such that, for every $(x, y) \in \mathbb{R}^2$,

$$\nu(x, y) \leq C(1 + |(x, y)|_\alpha)^{-1/12} = C(1 + |x|^2 + |y|^2)^{-1/12}. \quad (1.2)$$

Our Theorem 1.1 gives

$$\nu(x, y) \leq C_q(1 + |x|^2 + |y|^2)^{-\frac{1}{2q}}, \quad (1.3)$$

for every $q > 1$. If $q < 6$ this estimate is much better than Eq. 1.2.

The outline of the rest of the paper is as follows. In Section 2 we split the diffusion on S generated by $\mathcal{L}_\alpha$ into a skew-product of diffusions on A and N , and we state the estimate for the “horizontal component” of the diffusion on N . In Section 3 we recall the construction of the Poisson kernel ν on N and its extension $\nu^a(x)$ to S . In Section 4 we study some exponential functionals of Brownian motion. In Section 5 we prove the main estimate for $\nu^a(x)$. Combining this with the material from previous sections we get required estimate for the Poisson kernel in Section 6.

2 Disintegration of the Diffusion into Vertical and Horizontal Components

Let

$$\mathcal{L}_\alpha = \sum_{i=1}^k (\partial_{a_i}^2 - 2\alpha_i \partial_{a_i}) + \sum_{j=1}^r (e^{2\xi_j(a)} X_j^2 + d_j e^{\xi_j(a)} X_j) = \Delta_\alpha + L_\alpha. \quad (2.1)$$

2.1 Vertical Component

The k -dimensional Brownian motion σ_t (with drift -2α) generated by the operator

$$\Delta_\alpha = \sum_{i=1}^k \partial_{a_i}^2 - 2\alpha_i \partial_{a_i}$$

is called the *vertical component* of the diffusion generated by $\mathcal{L}_\alpha$.

2.2 Horizontal Component

Let $C_\infty(N)$ be the space of continuous functions f on N for which $\lim_{x \rightarrow \infty} f(x)$ exists. For $X \in \mathfrak{n}$, we let $\tilde{X}$ denote the corresponding right-invariant vector field. For a multi-index $I = (i_1, \dots, i_m)$, $i_j \in \mathbb{Z}^+$ and a basis $X_1, \dots, X_m$ of the Lie algebra $\mathfrak{n}$ we write $X^I = X_1^{i_1} \dots X_m^{i_m}$. For $k, l = 0, 1, 2, \dots, \infty$ we define

$$C^{(k,l)}(N) = \{f : \tilde{X}^I X^J f \in C_\infty(N) \text{ for every } |I| < k+1 \text{ and } |J| < l+1\}$$

and

$$\|f\|_{(k,l)}^0 = \sup_{|I|=k, |J|=l} \|\tilde{X}^I X^J f\|_\infty,$$

$$\|f\|_{(k,l)} = \sup_{|I|\leq k, |J|\leq l} \|\tilde{X}^I X^J f\|_\infty.$$

In particular $C^{(0,k)}(N)$ is a Banach space with the norm $\|f\|_{(0,2)}$.

For $a \in \mathbb{R}^k$, let

$$L_a = \sum_{j=1}^r (e^{2\xi_j(a)} X_j^2 + d_j e^{\xi_j(a)} X_j).$$

For a continuous function $\sigma : [0, \infty) \rightarrow \mathbb{R}^k$, we consider the following operator

$$L_{\sigma_t} = \sum_{j=1}^r (e^{2\xi_j(\sigma_t)} X_j^2 + d_j e^{\xi_j(\sigma_t)} X_j). \quad (2.2)$$

Let $\{U^\sigma(s, t) : 0 \leq s \leq t\}$ be the (unique) family of bounded operators on C_∞ which satisfies

- (i) $U^\sigma(s, s) = \text{Id}$, for all $s \geq 0$,
- (ii) $\lim_{h \rightarrow 0} U^\sigma(s, s+h)f = f$ in $C_\infty(N)$,
- (iii) $U^\sigma(s, r)U^\sigma(r, t) = U^\sigma(s, t)$, $0 \leq s \leq r \leq t$,
- (iv) $\partial_s U^\sigma(s, t)f = -L_{\sigma_s} U^\sigma(s, t)f$ for every $f \in C^{(0,2)}(N)$,
- (v) $\partial_t U^\sigma(s, t)f = U^\sigma(s, t)L_{\sigma_t}f$ for every $f \in C^{(0,2)}(N)$,
- (vi) $U^\sigma(s, t) : C^{(0,2)}(N) \rightarrow C^{(0,2)}(N)$ for all $s \leq t$.

The operator $U^\sigma(s, t)$ is a convolution operator with a probability measure with a smooth density, i.e., $U^\sigma(s, t)f = f * p^\sigma(t, s)$. In particular, $U^\sigma(s, t)$ is left-invariant. By iii), $p^\sigma(t, r) * p^\sigma(r, s) = p^\sigma(t, s)$ for $t \geq r \geq s$. Existence of $U^\sigma(s, t)$ follows from [14]. Notice that from above properties it follows that

- (vii) $U^{\sigma \circ \theta_u}(s, t) = U^\sigma(s+u, t+u)$, where $\sigma \circ \theta_u(s) = \sigma_{s+u}$ is the shift operator.

In fact $V(s, t) := U^\sigma(s+u, t+u)$ satisfies (i)–(vi) with the operator $L_{\sigma_{t+u}}$. Hence, the result follows from the uniqueness of $U^{\sigma_{t+u}}(s, t)$.

For a function f on N and $a \in \mathbb{R}^k$, let

$$\text{Ad}_*(a)f = f \circ \text{Ad}(a)$$

and

$$\chi(a) = \det(\text{Ad}(a)).$$

Then for $X \in \mathfrak{n}$,

$$X(\text{Ad}_*(a)f)(x) = \text{Ad}_*(a)(\text{Ad}(a)Xf)(x).$$

Thus, since $\text{Ad}(a)^{-1} = \text{Ad}(-a)$,

$$\text{Ad}_*(-a)X\text{Ad}_*(a) = \text{Ad}(a)X,$$

and

$$L_a = \text{Ad}(a)L_0 = \text{Ad}_*(-a)L_0\text{Ad}_*(a). \quad (2.3)$$

We will need the following property of the family $U^\sigma(s, t)$.

Lemma 2.1 *We have the following equality on $C_\infty(N)$,*

$$\text{Ad}_*(-a)U^\sigma(s, t)\text{Ad}_*(a) = U^{\sigma+a}(s, t). \quad (2.4)$$

Hence,

$$\chi(a)^{-1}\text{Ad}_*(-a)p^\sigma(t, s) = p^{\sigma+a}(t, s).$$

Proof It is easy to check, using Eq. 2.3, that the family

$$\tilde{U}^\sigma(s, t) := \text{Ad}_*(-a)U^\sigma(s, t)\text{Ad}_*(a)$$

satisfies properties (i)–(vi) with the operator L_{σ_t+a} . Hence, the first claim follows from the uniqueness of $\tilde{U}^\sigma(s, t)$. For the second one we use Eq. 2.4 and we write for $f \in C_c(N)$,

$$\begin{aligned} \int_N f(y)p^{\sigma+a}(t, s)(y^{-1}x)dy &= \int_N f(\text{Ad}(a)y)p^\sigma(t, s)(y^{-1}\text{Ad}(-a)x)dy \\ &= \chi(a)^{-1} \int_N f(y)p^\sigma(t, s)(\text{Ad}(-a)y^{-1}\text{Ad}(-a)x)dy \\ &= \chi(a)^{-1} \int_N f(y)\text{Ad}_*(-a)p^\sigma(t, s)(y^{-1}x)dy. \end{aligned}$$

Thus the second equality follows. $\square$

A stochastic process (evolution) in N corresponding to transition probabilities $p^\sigma(t, s)$ is called a *horizontal component* of the diffusion generated by $\mathcal{L}_\alpha$.

2.3 A Skew-product Formula for the Heat Semigroup on $N \rtimes \mathbb{R}^k$

Consider the operators $\mathcal{L}_\alpha$, Δ_α and L_a defined in Eq. 2.1. In this section we split the diffusion on S generated by $\mathcal{L}_\alpha$ into a skew-product of diffusions on A and N , generated by Δ_α and L_{σ_t} , respectively.

Let $U^\sigma(s, t)$ be the evolution generated by the operator L_{σ_t} defined in Eq. 2.2. For $f \in C_c(N \times \mathbb{R}^n)$ and $t \geq 0$, we put

$$T_t f(x, a) = \mathbf{E}_a U^\sigma(0, t) f(x, \sigma_t) = \mathbf{E}_a p^\sigma(t, 0) *_N f(x, \sigma_t), \quad (2.5)$$

where the expectation is taken with respect to the distribution of the process σ_t (Brownian motion with drift) in $\mathbb{R}^k$ with generator Δ_α . The operator $U^\sigma(0, t)$ acts on the first variable of the function f (as a convolution operator).

We have the following

Theorem 2.2 *The family T_t defined in Eq. 2.5 is the semigroup of operators generated by $\mathcal{L}_\alpha$. That is*

$$\partial_t T_t f = \mathcal{L}_\alpha T_t f \text{ and } \lim_{t \rightarrow 0} T_t f = f.$$

We refer to formula Eq. 2.5 as the *skew-product formula*. By now the proof of the above statement is standard and it goes along the lines of [5] with obvious changes.

Notice that Theorem 2.2 is a special case of the following general situation. Consider the product of two manifolds $M \times N$. Let L_1 be a differential operator acting on M and let, for every $a \in M$, $L_2(a)$ be an operator acting on N . Let L be a *skew-product* of L_1 and $L_2(a)$. That is L is the operator, on the product $N \times M$, of the form

$$L = L_1 + L_2(a).$$

Then it is natural to expect that the semigroup T_t generated by L is given by Eq. 2.5, where the expectation is taken with respect to the diffusion σ_t on M generated by L_1 and $U^\sigma(0, t)$ is the evolution generated by $L_2(\sigma_t)$. The idea of such a decomposition of the diffusion on $N \times M$ goes back to [10, 11] (see also [15]).

2.4 Estimate for the Evolution Kernel $p^\sigma(t, 0)$

Let $\sigma : [0, \infty) \rightarrow \mathbb{R}^k$ be a continuous function. We define the functional

$$A^\sigma(s, t) = \int_s^t \max_{\substack{j=1, \dots, r \\ d=1, 2}} e^{d\xi_j(\sigma_u)} du. \quad (2.6)$$

The following theorem is a generalization of the rank-one result [5, Theorem 4.1] to the higher rank setting. A detailed proof is given in [13]. Here τ is a subadditive, δ_t^ρ -homogeneous norm on N , $\rho \in A^+$, which is smooth on $N \setminus \{e\}$. The existence of such a subadditive norm is proved in [9].

Theorem 2.3 *Let $K \subset N$ be closed and $e \notin K$. Then there exist positive constants C_1, C_2 and ϑ such that for every $x \in K$ and for every t ,*

$$p^\sigma(t, 0)(x) \leq C_1 \left(\int_0^t \chi(\sigma_u)^{2/\vartheta} du \right)^{-\vartheta/2} \exp \left(\frac{\tau(x)}{4} - \frac{\tau(x)^2}{C_2 A^\sigma(0, t)} \right).$$

Remark The constant ϑ is an arbitrary positive number such that $d \leq \vartheta \leq D$, where d and D denote the local dimension and the dimension at infinity of a group N (for the definition of d and D see [18]).

3 The Poisson Kernel ν

Let μ_t be the semigroup of probability measures on $S = N \rtimes A$ generated by $\mathcal{L}_\alpha$. It is known [3, 4] that

$$\lim_{t \rightarrow \infty} (\pi_N(\check{\mu}_t), f) = (\nu, f),$$

where π_N denotes the projection from S onto N , and $(\check{\mu}, f) = (\mu, \check{f})$, $\check{f}(x) = f(x^{-1})$. Let $a \in \mathbb{R}^k$ and let μ be a measure on N . We define

$$(\mu^a, f) = (\mu, f \circ \text{Ad}(a)).$$

For $a \in \mathbb{R}^k$ we define

$$\nu^a(x) = \nu(a^{-1}xa)\chi(a)^{-1},$$

where $\chi(b) = e^{\rho_0 \cdot b}$, $\rho_0 = \sum_{j=1}^m \xi_j$. It is an easy calculation to check that

$$\lim_{t \rightarrow \infty} (\pi_N(\check{\mu}_t)^a, f) = (\nu^a, f). \quad (3.1)$$

Lemma 3.1 *We have*

$$(\pi_N(\check{\mu}_t)^a, f) = (\mathbf{E}_a \check{p}^\sigma(t, 0), f).$$

Proof Let T_t be the semigroup of operators generated by $\mathcal{L}_\alpha$, i.e.,

$$T_t f(x, a) = f * \mu_t(x, a) = \int_S p_t(x, a; y, b) f(y, b) \chi(b)^{-1} dy db.$$

By Theorem 2.2,

$$T_t f(x, a) = \mathbf{E}_a \int_N f(xy^{-1}, \sigma_t) p^\sigma(t, 0)(y) dy.$$

Define, for a given function $f : N \rightarrow \mathbb{R}$,

$$\tilde{f} : N \rtimes A \rightarrow \mathbb{R}, \quad \tilde{f}(x, a) = f(x).$$

Now we can write,

$$\begin{aligned} (\pi_N(\check{\mu}_t)^a, f) &= (\pi_N(\check{\mu}_t), f \circ \text{Ad}(a)) = (\check{\mu}_t, \tilde{f} \circ \text{Ad}(a)) \\ &= T_t(\tilde{f} \circ \text{Ad}(a))(e, 0) \\ &= \mathbf{E}_0 U^\sigma(0, t)(\tilde{f} \circ \text{Ad}(a))(e, \sigma_t) \\ &= \mathbf{E}_0 p^\sigma(t, 0) * (f \circ \text{Ad}(a))(e) \\ &= \mathbf{E}_0 \int_N f(\text{Ad}(a)y^{-1}) p^\sigma(t, 0)(y) dy \\ &= \mathbf{E}_0 \int_N f(\text{Ad}(a)y) \check{p}^\sigma(t, 0)(y) dy \\ &= \mathbf{E}_0 \int_N f(x) \check{p}^\sigma(t, 0)(\text{Ad}(-a)x) \chi(a)^{-1} dx. \end{aligned}$$

By Lemma 2.1, the last expected value is equal to

$$\mathbf{E}_0 \int_N f(x) \check{p}^{\sigma+a}(t, 0)(x) dx = \mathbf{E}_a \int_N f(x) \check{p}^{\sigma}(t, 0)(x) dx = (\mathbf{E}_a \check{p}^{\sigma}(t, 0), f).$$

Thus the proof is complete. $\square$

By Eq. 3.1 and Lemma 3.1 it follows that

$$(v^a, f) = \lim_{t \rightarrow \infty} (\pi_N(\check{\mu}_t)^a, f) = \lim_{t \rightarrow \infty} (\mathbf{E}_a \check{p}^{\sigma}(t, 0), f). \quad (3.2)$$

4 Exponential Functionals of Brownian Motion

Let w_s , $s \geq 0$, be the Brownian motion on $\mathbb{R}$ starting from $a \in \mathbb{R}$ and normalized so that

$$\mathbf{E}_a f(w_s) = \int_{\mathbb{R}} f(x+a) \frac{1}{\sqrt{4\pi s}} e^{-x^2/4s} dx. \quad (4.1)$$

Hence $\mathbf{E}w_s = a$ and $\text{Var}w_s = 2s$.

Remark Our normalization of the Brownian motion w_s is different than that typically used by probabilists who tend to assume that $\text{Var}w_s = s$.

For $d > 0$ and $\mu > 0$ we define the functional

$$I_{d,\mu} = \int_0^\infty e^{d(w_s - \mu s)} ds. \quad (4.2)$$

Such functionals are called *perpetual functionals* in financial mathematics where they play an important role (see e.g. [7, 17]).

The following theorem gives the distribution of $I_{2,\mu}$.

Theorem 4.1 (Dufresne [7]) *Let $w_0 = 0$. Then the functional $I_{2,\mu}$ is distributed as $(4\gamma_{\mu/2})^{-1}$, where $\gamma_{\mu/2}$ denotes a gamma random variable with parameter $\mu/2$, i.e., $\mathbf{P}(\gamma_{\mu/2} \in dx) = (1/\Gamma(\mu/2))x^{\frac{\mu}{2}-1}e^{-x}\mathbf{1}_{[0,+\infty)}(x)dx$, where Γ is the gamma function.*

Many authors have been interested in this functional and the proof can be found in many places. See for example [4, 6] or the survey paper [12] and the references therein.

As a corollary from Theorem 4.1, by scaling the Brownian motion and changing the variable, we get the following

Theorem 4.2 *Let $w_0 = a$. Then*

$$\mathbf{E}_a f(I_{d,\mu}) = c_{d,\mu} e^{\mu a} \int_0^\infty f(x) x^{-\mu/d} \exp\left(-\frac{e^{da}}{d^2 x}\right) \frac{dx}{x}.$$

The *inverse gamma density* (with respect to dx) is defined by

$$h_{\mu,\gamma} = C_{\mu,\gamma} x^{-\mu-1} e^{-\frac{\gamma}{x}} \mathbf{1}_{(0,+\infty)}(x).$$

Corollary 4.3 *The random variable $I_{2,\mu}$ has the inverse gamma density $h_{\mu/2,1/4}$.*

Let $\ell \in (\mathbb{R}^k)^*$ be a linear form on $\mathbb{R}^k$, and let $\sigma_u = w_u - 2\alpha u$ be the k -dimensional Brownian motion with a drift, starting from $a \in \mathbb{R}^k$. Then $\ell \cdot \sigma_u = \ell \cdot w_u - 2\ell \cdot \alpha u$ is the 1-dimensional Brownian motion with a drift $-2\ell \cdot \alpha$ and starting from $\ell \cdot a$. Moreover, $\mathbb{E} \ell \cdot \sigma_u = -2\ell \cdot \alpha u$ and $\text{Var} \ell \cdot \sigma_u = 2\ell^2 u$. Thus, the following theorem is an immediate corollary from Theorem 4.2.

Theorem 4.4 *Let $\sigma_u = w_u - 2\alpha u$ be the k -dimensional Brownian motion with drift, and let $\ell \in (\mathbb{R}^k)^*$ be such that $\ell \cdot \alpha > 0$. Then*

$$\mathbf{E}_a f \left(\int_0^\infty e^{d\ell \cdot \sigma_u} du \right) = c_{d,\ell,\alpha} e^{\gamma \ell \cdot a} \int_0^\infty f(u) u^{-\gamma/d} \exp \left(-\frac{e^{d\ell \cdot a}}{2d^2 \ell^2 u} \right) \frac{du}{u},$$

where $\gamma = 2\ell \cdot \alpha / \ell^2$. In particular, the functional $\int_0^\infty e^{d\ell \cdot \sigma_u} du$ has the inverse gamma density $h_{2\ell \cdot \alpha / (d\ell^2), 1/(d^2 \ell^2)}$.

5 Estimates for v^a

Recall that, for $\rho \in A^+$, $|\cdot|_\rho$ denotes a subadditive, δ_t^ρ -homogeneous norm on N , and

$$v^a(x) = v(a^{-1}xa) \chi(a)^{-1}, \quad (5.1)$$

where v is the Poisson kernel for $\mathcal{L}_\alpha$ and $\chi(a) = e^{\rho_0 \cdot a}$, $\rho_0 = \sum_{j=1}^m \xi_j$.

We use the following notation, for any $a \in A = \mathbb{R}^k$, we write

$$\tilde{a}_j = \xi_j \cdot a. \quad (5.2)$$

The main result of this section is the following estimate for v^a .

Theorem 5.1 *For all $q > 1$ there exists C_q such that for all $s < 0$ and all x with $|x|_\alpha = 1$,*

$$v^{s\alpha}(x) \leq C_q \chi(-s\alpha) e^{\frac{2s}{q} \min_{1 \leq j \leq r} \tilde{\alpha}_j^2 / \xi_j^2}.$$

We will prove Theorem 1.5 in Section 5.2. First we need some notation and lemmas.

5.1 Lemmas

Recall that, in Section 1, we defined a positive “Weyl chamber” A^+ ,

$$A^+ = \{a \in A : \xi_j \cdot a > 0 \text{ for } 1 \leq j \leq r\},$$

and that σ_u denotes the k -dimensional Brownian motion with drift -2α with $\alpha \in A^+$, i.e., $\sigma_u = w_u - 2\alpha u$, where w_u is the standard Brownian motion with normalization given in Eq. 4.1.

For $r+1 \leq j \leq 2r$ we set

$$\xi_j = 2\xi_{j-r}.$$

For $1 \leq j \leq 2r$, let

$$\tilde{\sigma}_j(u) = (\widetilde{\sigma_u})_j = \xi_j \cdot \sigma_u$$

and

$$A_j(s, t) = \int_s^t e^{\tilde{\sigma}_j(u)} du,$$

$$A_j = A_j(0, \infty) = \int_0^\infty e^{\tilde{\sigma}_j(u)} du.$$

For $k \in \mathbb{R}$, let (remember about the notation introduced in Eq. 5.2)

$$\mathcal{X}_k = \{\sigma : e^{\tilde{\alpha}_i k} \geq A_i \text{ for all } i \leq 2r\},$$

$$\mathcal{X}_k^j = \{\sigma : A_j > e^{\tilde{\alpha}_j(k-1)}\} \cap \mathcal{X}_k.$$

Note that

$$\mathcal{X}_k \setminus \mathcal{X}_{k-1} = \bigcup_{1 \leq j \leq 2r} \mathcal{X}_k^j.$$

Hence, the set of all trajectories of σ_u decomposes as the sum

$$\bigcup_{k \in \mathbb{Z}} \bigcup_{1 \leq j \leq 2r} \mathcal{X}_k^j.$$

We start with some properties of $\mathcal{X}_k^j$. In particular, we estimate $\mathbf{P}_a(\mathcal{X}_k^j)$. The following simple lemma allows us to reduce to the case of $\mathbf{P}_0$.

Lemma 5.2 *Let $k_1, k_2 \in \mathbb{R}$. The following equality holds*

$$\mathbf{P}_a(e^{\tilde{\alpha}_j k_1} \leq A_j(s, t) \leq e^{\tilde{\alpha}_j k_2}) = \mathbf{P}_0(e^{\tilde{\alpha}_j k_1 - \tilde{a}_j} \leq A_j(s, t) \leq e^{\tilde{\alpha}_j k_2 - \tilde{a}_j}).$$

Hence, for $t > 0$,

$$\mathbf{P}_{-ta}(\mathcal{X}_k^j) = \mathbf{P}_0(\mathcal{X}_{k+t}^j). \quad (5.3)$$

Proof If σ_u begins at a then $\eta_u = \sigma_u - a$ begins at 0. Also

$$A_j(s, t)(\sigma) = e^{\xi_j a} A_j(s, t)(\eta).$$

The lemma follows. □

The following lemma gives an estimate for $\mathbf{P}_0(\mathcal{X}_k^j)$.

Lemma 5.3 *There exists a constant $C = C_{\xi_j, \alpha} > 0$ such that for all $k \in \mathbb{R}$,*

$$\mathbf{P}_0(\mathcal{X}_k^j) \leq C e^{-2\tilde{\alpha}_j^2 k / \xi_j^2}.$$

Proof It is enough to consider only positive k since the estimate is trivial for $k \leq 0$. By Theorem 4.4 we have

$$\begin{aligned} \mathbf{P}_0(\mathcal{X}_k^j) &\leq \mathbf{P}_0(e^{\tilde{\alpha}_j(k-1)} < A_j \leq e^{\tilde{\alpha}_j k}) \\ &= c_{\xi_j, \alpha} \int_{e^{\tilde{\alpha}_j(k-1)}}^{e^{\tilde{\alpha}_j k}} u^{-2\xi_j \alpha / \xi_j^2} \exp\left(-1/(2\xi_j^2 u)\right) \frac{du}{u} \\ &\leq c_{\xi_j, \alpha} \int_{e^{\tilde{\alpha}_j(k-1)}}^{e^{\tilde{\alpha}_j k}} u^{-1-2\xi_j \alpha / \xi_j^2} du \\ &\leq c_{\xi_j, \alpha} (e^{-\tilde{\alpha}_j 2(k-1)\xi_j \alpha / \xi_j^2} - e^{-\tilde{\alpha}_j 2k\xi_j \alpha / \xi_j^2}) \\ &\leq c_{\xi_j, \alpha} e^{-\tilde{\alpha}_j 2k\xi_j \alpha / \xi_j^2}. \end{aligned}$$

The lemma follows. $\square$

The following lemma is an immediate corollary from Theorem 2.3.

Lemma 5.4 *Let ϑ be as in Theorem 2.3. Then for every $r > 0$ and every $\rho \in A^+$ there exist positive constants C, D and c such that for all $\sigma \in \bigcup_{1 \leq j \leq 2r} \mathcal{X}_k^j$, and all x from the set $K_{\rho, r} = \{|x|_\rho = r\}$ we have the following estimate for the evolution kernel p^σ generated by L_{σ_i} ,*

$$p^\sigma(t, 0)(x) \leq C \left(\int_0^t \chi(\sigma_u)^{2/\vartheta} du \right)^{-\vartheta/2} \exp(-De^{-ck}).$$

The constants C, D and c do not depend on k .

Proof For $\sigma \in \mathcal{X}_k^j$ we have $e^{\tilde{\alpha}_i k} \geq A_i$ for all $1 \leq i \leq 2r$. Then the functional $A^\sigma(s, t)$ defined in Eq. 2.6 satisfies:

$$A^\sigma(0, t) \leq \sum_{1 \leq i \leq 2r} \int_0^t e^{\xi_i \cdot \sigma_u} du \leq \sum_{1 \leq i \leq 2r} A_i \leq \sum_{1 \leq i \leq 2r} e^{\tilde{\alpha}_i k}.$$

Since $\tilde{\alpha}_i > 0$, then if $k > 0$, the above sum is bounded by $2re^{\max_i \tilde{\alpha}_i k}$, whereas if $k \leq 0$ by $2re^{\min_i \tilde{\alpha}_i k}$. Now, we apply Theorem 2.3 and the lemma follows. $\square$

5.2 Proof of Theorem 5.1

Now we are ready to give

Proof of Theorem 5.1 Let, for $0 < t \leq \infty$,

$$B(\sigma, t) = \left(\int_0^t \chi(\sigma_u)^{2/\vartheta} du \right)^{-\vartheta/2},$$

where $\chi(a) = e^{\rho_0 \cdot a}$ and $\rho_0 = \sum_{j=1}^m \xi_j$. This is the term appearing in the estimate given in Theorem 2.3. We split the set of all trajectories of σ_u into

$$\bigcup_{k \in \mathbb{Z}} \bigcup_{1 \leq j \leq 2r} \mathcal{X}_k^j.$$

Let the norm $|x|_\alpha = 1$. By Lemma 5.4 and Lemma 5.2 we have

$$\begin{aligned} \mathbf{E}_{s\alpha} p^\sigma(t, 0)(x^{-1}) &= \sum_{k \in \mathbb{Z}} \sum_{j=1}^{2r} \mathbf{E}_{s\alpha} p^\sigma(t, 0)(x^{-1}) \mathbf{1}_{\mathcal{X}_k^j}(\sigma) \\ &\leq C \sum_{k \in \mathbb{Z}} \exp(-De^{-ck}) \sum_{j=1}^{2r} \mathbf{E}_{s\alpha} B(\sigma, t) \mathbf{1}_{\mathcal{X}_k^j}(\sigma) \\ &= C \sum_{k \in \mathbb{Z}} \exp(-De^{-ck}) \sum_{j=1}^{2r} \chi(-s\alpha) \mathbf{E}_0 B(\sigma, t) \mathbf{1}_{\mathcal{X}_{k-s}^j}(\sigma). \end{aligned}$$

Applying the Hölder inequality we get

$$\begin{aligned} \mathbf{E}_{s\alpha} p^\sigma(t, 0)(x^{-1}) &\leq C \sum_{k \in \mathbb{Z}} \exp(-De^{-ck}) \sum_{j=1}^{2r} \chi(-s\alpha) (\mathbf{E}_0 B^p(\sigma, t))^{1/p} (\mathbf{P}_0 \mathbf{1}_{\mathcal{X}_{k-s}^j}(\sigma))^{1/q}. \end{aligned}$$

Hence, by Eq. 3.2, we get

$$\begin{aligned} v^{s\alpha}(x) &\leq C \sum_{k \in \mathbb{Z}} \exp(-De^{-ck}) \\ &\quad \times \sum_{j=1}^{2r} \chi(-s\alpha) (\mathbf{E}_0 B^p(\sigma, \infty))^{1/p} (\mathbf{P}_0 \mathbf{1}_{\mathcal{X}_{k-s}^j}(\sigma))^{1/q}. \end{aligned}$$

By Theorem 4.4, for every $p > 0$,

$$\mathbf{E}_0 B^p(\sigma, \infty) \leq C_p.$$

Hence, it follows from Lemma 5.3, and the fact that for all $j \leq 2r$, $\tilde{\alpha}_j > 0$, that for all $q > 1$,

$$\begin{aligned} v^{s\alpha}(x) &\leq C_q \sum_{k \in \mathbb{Z}} \exp(-De^{-ck}) \sum_{j=1}^{2r} \chi(-s\alpha) e^{-2\tilde{\alpha}_j^2 k / (q\xi_j^2)} e^{2\tilde{\alpha}_j^2 s / (q\xi_j^2)} \\ &\leq C_q \chi(-s\alpha) e^{\frac{2s}{q} \min_{1 \leq j \leq r} \tilde{\alpha}_j^2 / \xi_j^2}. \end{aligned}$$

□

6 Proof of Theorem 1.1

Using homogeneity this is an easy corollary from Theorem 5.1.

Proof of Theorem 1.1 By definition (Eq. 5.1) of $v^{s\alpha}$ and Theorem 5.1 we have, for $x \in N$ with $|x|_\alpha = 1$, and $s < 0$,

$$v((s\alpha)^{-1}x(s\alpha)) = e^{\rho_0(s\alpha)} v^{s\alpha}(x) \leq C e^{\frac{2s}{q} \min_{1 \leq j \leq r} \tilde{\alpha}_j^2 / \xi_j^2}.$$

Let $\delta_t^\alpha = \text{Ad}((\log t)\alpha)$. Then $|\delta_t^\alpha x|_\alpha = t|x|_\alpha$. Let $y = \delta_{\exp(-s)}^\alpha x$. Then $|y|_\alpha = e^{-s} > 1$, and using the above inequality we get

$$\nu(y) = \nu(\delta_{\exp(-s)}^\rho x) \leq C_q e^{\frac{2s}{q} \min_{1 \leq j \leq r} \tilde{\alpha}_j^2 / \xi_j^2} = C_q |y|_\alpha^{-\frac{2}{q} \min_{1 \leq j \leq r} \tilde{\alpha}_j^2 / \xi_j^2}.$$

Clearly, for y with $|y|_\alpha \leq 1$ we have $\nu(y) \leq C_\alpha$. $\square$

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