Logarithmic Sobolev Inequalities of Diffusions for the L^2 Metric

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We would like to point out that there was a problem in the published version of the proof for Theorem 4.4. However this result remains true, but the following change in its proof is needed. In fact, we keep the step 1 until the following formula (4.7) at page 96:

$$F(X^n) - \mathbb{E}F(X^n) = \int_0^T \langle \mathbb{E} \left(\int_t^T (A_t^{(n)})^* \left[(A_s^{(n)})^* \right]^{-1} \nabla_s^{(n)} F(X^n) | \mathcal{B}_t \right), dW_t \rangle \quad (4.7)$$

where $\nabla_s^{(n)} F$ is the gradient w.r.t. L^2 -norm on $W_{x_0}(M_n)$, and

$$M_t^n = \frac{1}{2} Ric_{U_t^n} - \overline{\nabla V_n}(U_t^n)$$
$$A_0^{(n)} = I, \quad \frac{dA_t^{(n)}}{dt} = A_t^{(n)} M_t^n.$$

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Step 2: convergence in (4.7) At first, for the left-hand side in (4.7), let us make a last localization on the functional F :

$$F_n(X) := \Phi \left(\int_0^{T \wedge \tau_n} f_1(s, X_s) ds, \dots, \int_0^{T \wedge \tau_n} f_m(s, X_s) ds \right).$$

Remarking that $\mathbb{P}(\tau_n \uparrow +\infty) = 1$ by the non explosion of our diffusion under condition (4.1), we have $F_n(X) \rightarrow F(X)$ a.s., because

$$|F_n(X) - F(X)| \leq \sum_{i=1}^m \|\partial_i \Phi\|_\infty \|f_i\|_\infty (T - T \wedge \tau_n) \rightarrow 0.$$

Since we have $F_n(X) = F_n(X^n)$, we obtain that $F(X^n) \rightarrow F(X)$ a.s. and consequently in $L^p(\mathbb{P})$ for all $p \in [1, +\infty)$. Thus the left-hand side of (4.7) converges to $F(X) - \mathbb{E}F(X)$ as desired.

For the right-hand side of (4.7), denoting $\mathbb{E}(\cdot | \mathcal{B}_t)$ by $\mathbb{E}_{\mathcal{B}_t}$, we have :

$$\begin{aligned} & \int_0^T \langle \mathbb{E}_{\mathcal{B}_t} \left(\int_t^T (A_t^{(n)})^* [(A_s^{(n)})^*]^{-1} \nabla_s^{(n)} F(X^n) ds \right), dW_t \rangle \\ &= \int_0^T \langle \mathbb{E}_{\mathcal{B}_t} \left(\int_t^T (A_t)^* [(A_s)^*]^{-1} \nabla_s F(X) 1_{T < \tau_n} \right), dW_t \rangle \\ & \quad + \int_0^T \langle \mathbb{E}_{\mathcal{B}_t} \left(\int_t^T (A_t^{(n)})^* [(A_s^{(n)})^*]^{-1} \nabla_s^{(n)} F(X^n) ds 1_{T \geq \tau_n} \right), dW_t \rangle \\ &:= (I)_n + (II)_n. \end{aligned}$$

Letting $n \rightarrow +\infty$, we have in $L^2(\mathbb{P})$,

$$(I)_n \rightarrow (I) := \int_0^T \langle \mathbb{E}_{\mathcal{B}_t} \left(\int_t^T (A_t)^* [(A_s)^*]^{-1} \nabla_s F(X) ds \right), dW_t \rangle$$

the desired r.h.s. Indeed, we have by the same proof as for Lemma 3.2,

$$\|A_t^* [(A_s)^*]^{-1}\| \leq e^{-K(s-t)}, \quad \forall s \geq t$$

and for the functionals in $\mathcal{D}$

$$|\nabla_s F(\gamma)| \leq \sum_{i=1}^m \|\partial_i \Phi\|_\infty \|\nabla f_i\|_\infty =: C,$$

so that

$$\begin{aligned} \mathbb{E}|(I)_n - (I)|^2 &= \mathbb{E} \int_0^T \left| \int_t^T (A_t)^* [(A_s)^*]^{-1} \nabla_s F(X) ds 1_{T \geq \tau_n} \right|^2 dt \\ &\leq \int_0^T \left| \int_t^T C e^{-(K)(s-t)} ds \right|^2 dt \mathbb{P}(\tau_n \leq T) \\ &\leq C^2 T^3 e^{2|K|T} \mathbb{P}(\tau_n \leq T) \rightarrow 0 \end{aligned}$$

We proceed in the same way for $(II)_n$, since by our assumption (3) in Step 1 and the proof of Lemma 3.2,

$$\|A_t^{(n)*} [(A_s^{(n)})^*]^{-1}\| \leq e^{-(K-1)(s-t)}, \quad \forall s \geq t$$

and also

$$|\nabla_s^{(n)} F(\gamma)| \leq \sum_{i=1}^m \|\partial_i \Phi\|_\infty \|\nabla f_i\|_\infty =: C$$

we have

$$\begin{aligned} \mathbb{E}(II_n)^2 &\leq \mathbb{E} \int_0^T \left| \int_t^T \left(A_t^{(n)} \right)^* \left[\left(A_s^{(n)} \right)^* \right]^{-1} \nabla_s^{(n)} F(X^n) ds \mathbf{1}_{T \geq \tau_n} \right|^2 dt \\ &\leq C^2 T^3 e^{2(|K|+1)T} \mathbb{P}(\tau_n \leq T) \rightarrow 0. \end{aligned}$$

Consequently letting n go to infinity in (4.7) for $F(X^n)$, we get the desired martingale representation for $F(X)$. $\square$

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