Erratum to: Strong extensions for q -summing operators acting in p -convex Banach function spaces for

$$1 \leq p \leq q$$

Weak* compactness of the closed unit ball of a Köthe dual space

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Abstract Let X be a saturated Banach function space and denote by X' its Köthe dual. In the paper (Delgado and Sánchez Pérez in Positivity 20:999–1014, 2016) referenced in the title it is implicitly used that the closed unit ball $B_{X'}$ of X' is compact for the topology $\sigma(X', X)$ on X' defined by the elements of X . This fact could be not true in general if X is not σ -order continuous.

Keywords Banach function space · Köthe dual · Weak* compactness · σ -Order continuity

Mathematics Subject Classification 46B50 · 46E30 · 46B42

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Let (Ω, Σ, μ) be a σ -finite measure space and denote by $L^0(\mu)$ the space of real measurable functions defined on Ω . Consider a *saturated Banach function space* X , that is, a Banach lattice that is an ideal in $L^0(\mu)$ such that there is no $A \in \Sigma$ with $\mu(A) > 0$ satisfying that $f\chi_A = 0$ μ -a.e. for all $f \in X$.

The space X is said to be σ -order continuous if for every $(f_n) \subset X$ with $f_n \downarrow 0$ μ -a.e. it follows that $\|f_n\|_X \downarrow 0$. The *Köthe dual* X' of X is the space of functions $g \in L^0(\mu)$ such that $\int |fg| d\mu < \infty$ for all $f \in X$. It is a saturated Banach function space with norm $\|g\|_{X'} = \sup_{f \in B_X} \left| \int fg d\mu \right|$, where B_X denotes the closed unit ball of X . The space X' can be identified with a closed subspace of the topological dual X^* of X via the linear isometry $\eta: X' \rightarrow X^*$ given by $\langle \eta(g), f \rangle = \int fg d\mu$ for all $g \in X'$ and $f \in X$. The map η is surjective if and only if X is σ -order continuous. The Köthe bidual X'' of X is defined as $X'' := (X')'$.

Consider now the weak topology $\sigma(X', X)$ defined by X on its Köthe dual X' . Through all the recently published paper [1] referenced in the title, the fact that the closed unit ball $B_{X'}$ of X' is $\sigma(X', X)$ -compact is used, but *this is not in general true*. For getting a counterexample we just have to take $X = L^\infty(\mu)$ for which $X' = L^1(\mu)$ and note that, by the Banach–Bourbaki theorem, $B_{L^1(\mu)}$ is not compact for the topology $\sigma(L^1(\mu), L^\infty(\mu))$.

However $B_{X'}$ is $\sigma(X', X)$ -compact whenever X is σ -order continuous, since in this case B_{X^*} is identified with $B_{X'}$ via the map $\eta^{-1}: X^* \rightarrow X'$ and we can apply the Banach–Alaoglu theorem. Therefore, requiring X to have the natural condition of σ -order continuity all the results in [1] hold.

A characterization of the weak* compactness of $B_{X'}$ follows as a particular case of a known result for Riesz spaces, see [2, Theorem 82G and Proposition 82B]. Recall that the σ -order continuous part X_a of X is defined as the largest σ -order continuous closed solid subspace of X and can be described as

$$X_a = \{f \in X : |f| \geq f_n \downarrow 0 \text{ implies } \|f_n\|_X \downarrow 0\}.$$

Then we have that $B_{X'}$ is $\sigma(X', X)$ -compact if and only if

$$\|f_n\|_{X''} \downarrow 0 \text{ whenever } (f_n) \subset X \text{ with } f_n \downarrow 0 \mu\text{-a.e.,}$$

that is, if and only if $X \subset (X'')_a$.

Finally, note that we can obtain compactness for $B_{X'}$ if we consider the weak topology on X' defined by the elements of the σ -order continuous part X_a of X . Indeed, in the case when X_a has a weak unit we have that X_a is super order dense in X and so it can be proved that $(X_a)' = X'$ with equal norms. Then, since X_a is σ -order continuous, it follows that $B_{X'}$ is $\sigma(X', X_a)$ -compact.

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