Isomorphic copies of l^∞ in Cesàro–Orlicz function spaces

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Abstract We characterize Cesàro–Orlicz function spaces Ces_φ containing isomorphic copy of l^∞ . We also describe the subspaces $(Ces_\varphi)_a$ of all order continuous elements of Ces_φ . Finally, we study the monotonicity structure of the spaces Ces_φ and $(Ces_\varphi)_a$.

Keywords Cesàro–Orlicz function space · Isomorphic copy of l^∞ · Order continuous norm · Monotonicity properties in Banach lattices

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1 Introduction

The structure of different types of Cesàro spaces has been widely investigated during the last decades from the isomorphic as well as isometric point of view. Firstly, the Cesàro sequence spaces ces_p has been studied by many authors starting from 1970. Then the attention has been moved to the Cesàro function spaces Ces_p . It was interesting among others that some properties are fulfilled in the sequence case and do not in the function case. Moreover, it happens that the cases $Ces_p[0, 1]$ and $Ces_p[0, \infty)$ are

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essentially different (see the description of the Köthe dual of Cesàro spaces in [1] and [19]. Note also that isometric description of the dual space to Ces_p when $1 < p < \infty$ can be found in [15]). Probably the most important papers concerning the structure of the spaces Ces_p are two papers by Astashkin and Maligranda (see [1, 2]). Recall that the space $Ces_p[0, 1]$, for $1 \leq p \leq \infty$, consists of those Lebesgue measurable real functions f on $[0, 1]$ for which the Cesàro-means $C|f|(x) = \frac{1}{x} \int_0^x |f(t)|dt$ belong to $L_p[0, 1]$ (similarly for the spaces ces_p and $Ces_p[0, \infty)$). It is natural to investigate the space

$$CX = CX(I) = \{f \in L^0(I) : C|f| \in X\}$$

for any Banach ideal space X on I , where $I = [0, 1]$ or $I = [0, \infty)$. These spaces have been defined in [27] for X being a Banach ideal space on $[0, \infty)$ and, for example, in [12, 19, 20] for X being a symmetric space on I . If we take instead of X the Orlicz sequence space l^φ (the generalization of l^p) then the space CX becomes the Cesàro–Orlicz sequence space denoted by ces_φ . Clearly, $ces_\varphi = ces_p$ if $\varphi(u) = u^p$. The spaces ces_φ has been studied by many authors (e.g. [9–11, 14, 16]). In particular, the criteria for existence of an isomorphic as well as isometric copies of l^∞ in the spaces ces_φ have been established in [14, 16]. If we put $X = L^\varphi(I)$ the Orlicz function space then CX becomes the Cesàro–Orlicz function space $Ces_\varphi(I)$. As far as we know the structure of these spaces has not been investigated until now. We want to characterize these spaces Ces_φ which contain an order isomorphic copy of l^∞ (equivalently, are not order continuous). Clearly, such results are very useful in further studying isomorphic structure of these spaces. We characterize the subspaces $(Ces_\varphi)_a$ of all order continuous elements in Ces_φ . Finally, we prove criteria for strict monotonicity of the spaces $(Ces_\varphi)_a$. We also give a characterization of uniform monotonicity in Cesàro–Orlicz function spaces. We consider the largest possible class of Orlicz functions giving the maximal generality of spaces under consideration.

2 Preliminaries

The symbols $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$ denote the sets of reals, nonnegative reals and natural numbers, respectively. Let $L^0 = L^0(I)$ be the space of all classes of real-valued Lebesgue measurable functions defined on I , where $I = [0, 1]$ or $I = [0, \infty)$. A Banach space $E = (E, \|\cdot\|)$ is said to be a Banach ideal space on I if E is a linear subspace of $L^0(I)$ and satisfies two conditions:

- (i) if $g \in E$, $f \in L^0$ and $|f| \leq |g|$ a.e. on I then $f \in E$ and $\|f\| \leq \|g\|$,
- (ii) there is an element $f \in E$ that is positive on whole I .

Sometimes we write $\|\cdot\|_E$ to be sure in which space the norm has been taken.

For two Banach ideal spaces E and F on I the symbol $E \hookrightarrow F$ means that the embedding $E \subset F$ is continuous, i.e., there exists constant a $C > 0$ such that $\|x\|_F \leq C \|x\|_E$ for all $x \in E$. Moreover, $E = F$ means that the spaces are the same as the sets and the norms are equivalent.

Recall that for $f \in E$ the distribution function d_f is defined by

$$d_f(\lambda) = m(\{t \in I : |f(t)| > \lambda\})$$

for all $\lambda > 0$, where m is the Lebesgue measure. The non-increasing rearrangement of f is denoted by f^* and is defined as

$$f^*(t) = \inf\{\lambda > 0 : d_f(\lambda) < t\}$$

for $t \geq 0$. We say that two functions $f, g \in L^0(I)$ are equimeasurable if they have the same distribution functions $d_f \equiv d_g$. Then we write $f \sim g$. By a symmetric function space (symmetric Banach function space or rearrangement invariant Banach function space) on I we mean a Banach ideal space $(E, \|\cdot\|_E)$ with an additional property that for any two functions $f \in E, g \in L^0(I)$ with $f \sim g$ we have $g \in E$ and $\|f\|_E = \|g\|_E$. In particular, $\|f\|_E = \|f^*\|_E$.

A point $f \in E$ is said to have an order continuous norm (or to be an order continuous element) if for each sequence $(f_n) \subset E$ satisfying $0 \leq f_n \leq |f|$ and $f_n \rightarrow 0$ a.e. on I , one has $\|f_n\| \rightarrow 0$. By E_a we denote the subspace of all order continuous elements of E . It is worth to notice that in case of Banach ideal spaces on I , $x \in E_a$ if and only if $\|x\chi_{A_n}\| \rightarrow 0$ for any decreasing sequence of Lebesgue measurable sets $A_n \subset I$ with empty intersection. A Banach ideal space is called order continuous ($E \in (OC)$ shortly) if every element of E is order continuous, i.e., $E = E_a$.

We say a Banach ideal space X is strictly monotone ($X \in (SM)$), if $\|x\| < \|y\|$ for all $x, y \in X$ such that $0 \leq x \leq y$ and $x \neq y$. A Banach ideal space X is uniformly monotone ($X \in (UM)$) if for each $\epsilon \in (0, 1)$ there exists $\delta(\epsilon) \in (0, 1)$ such that for all $0 \leq y \leq x$, $\|x\| = 1$ and $\|y\| \geq \epsilon$, we have $\|x - y\| \leq 1 - \delta(\epsilon)$. X is called lower locally uniformly monotone ($X \in (LLUM)$) if for any $x \in X$, $\|x\| = 1$ and $\epsilon \in (0, 1)$ there exists $\delta(x, \epsilon) \in (0, 1)$ such that $\|x - y\| \leq 1 - \delta(x, \epsilon)$ whenever $0 \leq y \leq x$ and $\|y\| \geq \epsilon$. Note that monotonicity properties of Banach ideal spaces are useful in dominated best approximation problems, see [8] for further references.

The next well known theorem shows the relation between the order continuity of E and the existing of an isomorphic copy of l^∞ .

Theorem A (G. Ya. Lozanovskii, see [23]) *A Banach ideal space E is order continuous if and only if E contains no isomorphic copy of l^∞ .*

A function $\varphi : [0, \infty) \rightarrow [0, \infty]$ is called an Orlicz function if:

- (i) φ is convex,
- (ii) $\varphi(0) = 0$,
- (iii) φ is neither identically equal to zero nor infinity on $(0, \infty)$,
- (iv) φ is left continuous on $(0, \infty)$, i.e., $\lim_{u \rightarrow b_\varphi^-} \varphi(u) = \varphi(b_\varphi)$ if $b_\varphi < \infty$, where

$$b_\varphi = \sup\{u > 0 : \varphi(u) < \infty\}.$$

For more information about Orlicz functions see [7, 18].

If we denote

$$a_\varphi = \sup\{u \geq 0 : \varphi(u) = 0\},$$

then $0 \leq a_\varphi \leq b_\varphi \leq \infty$, $a_\varphi < \infty$, $b_\varphi > 0$, since an Orlicz function is neither identically equal to zero nor infinity on $(0, \infty)$. The function φ is continuous and non-

decreasing on $[0, b_\varphi)$ and is strictly increasing on $[a_\varphi, b_\varphi)$. We use notations $\varphi > 0$, $\varphi < \infty$ when $a_\varphi = 0$, $b_\varphi < \infty$, respectively.

We say an Orlicz function φ satisfies the condition Δ_2 for large arguments ($\varphi \in \Delta_2(\infty)$ for short) if there exists $K > 0$ and $u_0 > 0$ such that $\varphi(u_0) < \infty$ and

$$\varphi(2u) \leq K\varphi(u)$$

for all $u \in [u_0, \infty)$. Similarly, we can define the condition Δ_2 for small, with $\varphi(u_0) > 0$ ($\varphi \in \Delta_2(0)$) or for all arguments ($\varphi \in \Delta_2(\mathbb{R}_+)$). These conditions play a crucial role in the theory of Orlicz spaces, see [7, 18, 25, 26]. We will write $\varphi \in \Delta_2$ in two cases: $\varphi \in \Delta_2(\infty)$ if $I = [0, 1]$ and $\varphi \in \Delta_2(\mathbb{R}_+)$ if $I = [0, \infty)$.

The Orlicz function space $L^\varphi = L^\varphi(I)$ generated by an Orlicz function φ is defined by

$$L^\varphi = \{f \in L^0(I) : I_\varphi(f/\lambda) < \infty \text{ for some } \lambda = \lambda(f) > 0\},$$

where $I_\varphi(f) = \int_I \varphi(|f(t)|)dt$ is a convex modular (for the theory of Orlicz spaces and modular spaces see [25, 26]). The space L^φ is a Banach ideal space with the Luxemburg–Nakano norm

$$\|f\|_\varphi = \inf\{\lambda > 0 : I_\varphi(f/\lambda) \leq 1\}.$$

It is well known that $\|f\|_\varphi \leq 1$ if and only if $I_\varphi(f) \leq 1$.

The continuous Cesàro operator is defined for $0 < x \in I$ as

$$Cf(x) = \frac{1}{x} \int_0^x f(t)dt.$$

For a Banach ideal space X on I we define an abstract Cesàro space $CX = CX(I)$ as

$$CX = \{f \in L^0(I) : C|f| \in X\}$$

with a norm $\|f\|_{CX} = \|C|f|\|_X$.

The Cesàro–Orlicz function space $Ces_\varphi = Ces_\varphi(I)$ is defined as $Ces_\varphi(I) = CL^\varphi(I)$. Consequently, the norm in the space Ces_φ is given by the formula

$$\|f\|_{Ces(\varphi)} = \inf\{\lambda > 0 : \rho_\varphi(f/\lambda) \leq 1\}$$

where $\rho_\varphi(f) = I_\varphi(C|f|)$ is a convex modular.

The dilation operator D_s , $s > 0$ defined on $L^0(I)$ by

$$D_s x(t) = x(t/s) \chi_I(t/s) = x(t/s) \chi_{[0, \min\{1, s\})}(t),$$

for $t \in I$, is bounded in any symmetric space E on I and $\|D_s\|_{E \rightarrow E} \leq \max\{1, s\}$ (see [4, p. 148]). Moreover, the lower and upper Boyd indices of E are defined by

$$p(E) = \lim_{s \rightarrow 0^+} \frac{\ln \|D_s\|_{E \rightarrow E}}{\ln s},$$

$$q(E) = \lim_{s \rightarrow \infty} \frac{\ln \|D_s\|_{E \rightarrow E}}{\ln s}.$$

In particular, they satisfy the inequalities

$$1 \leq p(E) \leq q(E) \leq \infty.$$

In the case when E is the Orlicz space L^φ , these indices correspond to the so-called Orlicz–Matuszewska indices of Orlicz functions generating the Orlicz spaces, i.e., $\alpha_\varphi = p(L^\varphi)$ and $\beta_\varphi = q(L^\varphi)$, where α_φ and β_φ are the lower and upper Orlicz–Matuszewska indices of the Orlicz space L^φ (see [22, Proposition 2.b.5 and Remark 2 on page 140]). For more details see [5, 6, 24–26].

In this paper we accept the convention that $\sum_{n=m}^k x_n = 0$ if $k < m$. Let us mention the important result about boundedness of the operator C .

Theorem B [17, p. 127] *For any symmetric space E on I the operator $C : E \rightarrow E$ is bounded if and only if the lower Boyd index satisfies $p(E) > 1$.*

The immediate consequence of Theorem B and the above discussion is a next corollary.

Corollary 1 *The embedding $L^\varphi \hookrightarrow Ces_\varphi$ holds if and only if $\alpha_\varphi > 1$.*

Remark Let X, Y be Banach ideal spaces on I . If $X \hookrightarrow Y$ then $CX \hookrightarrow CY$. Indeed, suppose $X \hookrightarrow Y$. Then for all $x \in CX$

$$\|x\|_{CY} = \|C|x|\|_Y \leq A \|C|x|\|_X = A \|x\|_{CX},$$

for some constant $A > 0$. This means that $CX \hookrightarrow CY$.

Let L^φ and L^ψ be the Orlicz spaces with $\varphi, \psi < \infty$. The criteria for the embeddings $L^\varphi \hookrightarrow L^\psi$ can be found in [25, Theorem 3.4 p. 18]. Consequently, we conclude that:

- (i) if there exists $k > 0$ with $\psi(u) \leq \varphi(ku)$ for all $u \in [0, \infty)$ then $Ces_\varphi[0, \infty) \hookrightarrow Ces_\psi[0, \infty)$,
- (ii) if there exists $k, u_0 > 0$ such that $\psi(u) \leq \varphi(ku)$ for all $u \in [u_0, \infty)$ then $Ces_\varphi[0, 1] \hookrightarrow Ces_\psi[0, 1]$,
- (iii) if there exists $k, u_0 > 0$ such that $\psi(u) \leq \varphi(ku)$ for all $u \in [0, u_0]$ then $ces_\varphi \hookrightarrow ces_\psi$.

Note that case (iii) is exactly Proposition 1 in [16] but now it is an immediate consequence of our above remark.

3 Results

Fact Let X be a Banach ideal space on I . By the definition, the order continuity of X gives the same for CX (for proof see [21, Lemma 1]). The case of strict monotonicity and uniform monotonicity is similar. The converse is not true in general (see [12, Proposition 2.1] and [21, Example 1]).

Proof We proof only the implication for uniform monotonicity. We apply Theorem 6 (ii) in [13]. Suppose that $X \in (\text{UM})$. Take $\epsilon > 0$, $x, y \in CX$, $x, y \geq 0$, $\|x\|_{CX} = 1$ and $\|y\|_{CX} \geq \epsilon$. We have

$$\|x + y\|_{CX} = \|C|x + y|\|_X = \|C|x| + C|y|\|_X.$$

Since X is uniformly monotone we have that there is a $\sigma(\epsilon) > 0$ such that

$$\|C|x| + C|y|\|_X \geq 1 + \sigma(\epsilon).$$

But this means that $CX \in (\text{UM})$. □

Proposition 2 Suppose X is a symmetric space on $I = [0, 1]$ and $C : X \rightarrow X$. Then $X \in (\text{OC})$ if and only if $CX \in (\text{OC})$.

Proof Necessity It follows from Fact above.

Sufficiency We thank Professor Karol Leśnik for giving the proof. Suppose that $X \notin (\text{OC})$, i.e. there exists $0 < f \in X \setminus X_a$. Because X is symmetric we can assume that $f = f^*$ (see [8, Lemma 2.6]). Then $Cf \in X$ and $Cf \geq f$. Applying the symmetry of X and passing to subsequence, if necessary, we can assume that there is a $\delta > 0$ such that

$$\|f \chi_{[0, 1/n)}\|_X \geq \delta,$$

for all $n \in \mathbb{N}$. Since

$$C(f \chi_{[0, 1/n)})(t) \geq f \chi_{[0, 1/n)}(t),$$

for all $t \in I$ and $n \in \mathbb{N}$, we have

$$\|f \chi_{[0, 1/n)}\|_{CX} = \|C(f \chi_{[0, 1/n)})\|_X \geq \|f \chi_{[0, 1/n)}\|_X \geq \delta > 0,$$

which means that $X \notin (\text{OC})$. □

Proposition 3 Let φ be an Orlicz function. The following conditions are equivalent:

- (i) the space $Ces_\varphi[0, \infty) \neq \{0\}$,
- (ii) there exists $\lambda_0 > 0$ and $x_0 \in [0, \infty)$ such that $\int_{x_0}^{\infty} \varphi(\lambda_0/t) dt < \infty$,
- (iii) for each $\lambda > 0$ there exist $y_0 \in [0, \infty)$ with $\int_{y_0}^{\infty} \varphi(\lambda/t) dt < \infty$.

Proof The equivalence of conditions (i) and (ii) follows from [19, Theorem 1 (a)]. Clearly, (iii) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii). Let $\lambda > 0$. We have to consider two cases. If $\lambda \leq \lambda_0$, there is nothing to proof. Now suppose $\lambda > \lambda_0$. Set $y_0 = \lambda x_0 / \lambda_0$. Then

$$\begin{aligned} \int_{y_0}^{\infty} \varphi(\lambda/t) dt &= \int_{y_0}^{\infty} \varphi\left(\frac{\lambda_0}{\lambda_0 t / \lambda}\right) dt = \frac{\lambda}{\lambda_0} \int_{\lambda_0 y_0 / \lambda}^{\infty} \varphi\left(\frac{\lambda_0}{u}\right) du \\ &= \frac{\lambda}{\lambda_0} \int_{x_0}^{\infty} \varphi\left(\frac{\lambda_0}{u}\right) du < \infty. \end{aligned}$$

□

Remark 3.1 Let φ be an Orlicz function satisfying condition (S):

$$\liminf_{t \rightarrow 0} t\varphi'(t)/\varphi(t) > 1,$$

where the symbol φ' means the right derivative of the function φ . Then $Ces_\varphi[0, \infty) \neq \{0\}$.

Proof Firstly, condition (S) implies (ii) in Proposition 3. Indeed, we can use the same argument as in the proof of implication (a) $\Rightarrow$ (c) in [9, Theorem 2.2]. But condition (ii) is equivalent to $Ces_\varphi[0, \infty) \neq \{0\}$. $\square$

Remark 4 Let φ be an Orlicz function. Then $Ces_\varphi[0, 1] \neq \{0\}$.

Proof It follows from [19, Theorem 1 (b)] that $Ces_\varphi[0, 1] \neq 0$ if and only if there exists $0 < a < 1$ such that $\chi_{[a, 1]} \in L^\varphi[0, 1]$. But $L^\varphi[0, 1]$ is a symmetric space and

$$L^\infty[0, 1] = L^1[0, 1] \cap L^\infty[0, 1] \hookrightarrow L^\varphi[0, 1] \hookrightarrow L^1[0, 1] + L^\infty[0, 1] = L^1[0, 1]$$

(see [4]). Therefore $\chi_{[a, 1]} \in L^\varphi[0, 1]$ for all $a \in (0, 1)$. $\square$

The following theorem gives a similar information as in the sequence space ces_φ one can get from [9, Theorem 2.3]. Before we formulate the theorem define a set

$$C_\varphi = C_\varphi(I) = \{x \in Ces_\varphi(I) : \rho_\varphi(kx) < \infty \text{ for all } k > 0\}.$$

Note that $C_\varphi = \{0\}$ whenever $b_\varphi < \infty$.

Theorem 5 Suppose $\varphi < \infty$. Then the following conditions are true:

- (i) C_φ is the subspace of all order continuous elements of Ces_φ ,
- (ii) C_φ is a closed separable subspace of Ces_φ .

Proof (i). We will show that $C_\varphi \hookrightarrow (Ces_\varphi)_a$. Take any $x \in C_\varphi$ and a sequence (A_n) with $A_n \searrow \emptyset$. It is enough to show, that $\|x\chi_{A_n}\|_{Ces(\varphi)} \rightarrow 0$ or equivalently $\rho_\varphi(kx\chi_{A_n}) \rightarrow 0$ for any $k > 0$. Set $k > 0$. Clearly,

$$\lim_{n \rightarrow \infty} m((0, t] \cap A_n) = 0,$$

for each $t > 0$. Then for almost each $t > 0$

$$0 \leq C(|x\chi_{A_n}|)(t) = \frac{1}{t} \int_0^t |x(s)\chi_{A_n}(s)| ds \rightarrow 0.$$

Moreover,

$$\rho_\varphi(kx\chi_{A_n}) \leq \rho_\varphi(kx) < \infty,$$

for each k . Thus, by the Lebesgue dominated convergence theorem, we have $\rho_\varphi(kx\chi_{A_n}) \rightarrow 0$, so $x \in (Ces_\varphi(I))_a$.

Next we prove the reverse inclusion. Define a set

$$B_\varphi = B_\varphi(I) = \text{cl}\{f \in Ces_\varphi(I) : f \text{ is simple function and } m(\text{supp}(f)) < \infty\}.$$

We claim that:

$$C_\varphi = B_\varphi. \quad (A)$$

Consider the inclusion $B_\varphi \subset C_\varphi$. Of course, if $B_\varphi = \emptyset$ then $B_\varphi \subset C_\varphi$ so we can assume that $B_\varphi \neq \emptyset$. We divide the proof in three cases.

- (1) Suppose $x = \chi_{[a,b]}$, where $[a, b] \subset I$, $0 \leq a < b < \infty$. Since $\varphi < \infty$ and $Ces_\varphi \neq \{0\}$, by Proposition 3 we conclude that $\rho_\varphi(kx) < \infty$ for all $k > 0$.
- (2) Let $x = \sum_{k=1}^n c_k \chi_{A_k}$, $A_k = [a_k, b_k] \subset I$ for $n, k \in \mathbb{N}$ and $0 \leq a_k < b_k < \infty$. Using the convexity of modular ρ_φ and argument from case (1), we obtain $\rho_\varphi(\lambda x) < \infty$ for all $\lambda > 0$.
- (3) Assume that there exists a sequence (x_n) such that $x_n \rightarrow x$ in Ces_φ and x_n are of the form (2). Let $k > 0$. For sufficiency large $n \in \mathbb{N}$ we have

$$\begin{aligned} \rho_\varphi(kx) &= \rho_\varphi(k(x - x_n + x_n)) = \rho_\varphi\left(\frac{1}{2}2k(x - x_n) + \frac{1}{2}2kx_n\right) \\ &\leq \frac{1}{2}\rho_\varphi(2k(x - x_n)) + \frac{1}{2}\rho_\varphi(2kx_n) \leq 1 + \rho_\varphi(2kx_n) < \infty. \end{aligned}$$

Therefore $B_\varphi \subset C_\varphi$.

Now, we will prove the reverse inclusion $C_\varphi \subset B_\varphi$. Let $x \in C_\varphi$, $k > 0$ and $\epsilon > 0$. Since $\int_I \varphi(C(k|x|))dt < \infty$ there exists $\beta = \beta(k) \in \mathbb{R}_+$ with

$$\int_\beta^\infty \varphi(C(2k|x(t)|))dt < \epsilon/2.$$

Put $z_n = \varphi(C(2k|x|\chi_{[0,1/n]}))$ and $z = \varphi(C(2k|x|))$. Then $z \in L^1$, $0 \leq z_n \leq z$ and $z_n \rightarrow 0$ a.e. on I . Note that $L^1 \in (OC)$ whence $\|z_n\|_{L^1} \rightarrow 0$. Consequently, there exists $\alpha = \alpha(k) \in \mathbb{R}_+$ such that

$$\int_0^\infty \varphi(C(2k|x|\chi_{[0,\alpha]}))dt < \epsilon/2.$$

Define an element $x_n = x\chi_{[1/n,n]}$ for $n \in \mathbb{N}$. For sufficiently large $n \in \mathbb{N}$ satisfying $1/n < \alpha$ and $n > \beta$ we have

$$\begin{aligned} \rho_\varphi(k(x_n - x)) &= \rho_\varphi(kx\chi_{[0,1/n)\cup(n,\infty)}) = \int_0^\infty \varphi(C(k|x(t)|\chi_{[0,1/n)\cup(n,\infty)}(t)))dt \\ &= \int_0^\infty \varphi(C(k|x(t)|\chi_{[0,1/n)}(t)) + C(k|x(t)|\chi_{(n,\infty)}(t)))dt \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{2} \int_0^\infty \varphi(C(2k|x(t)|\chi_{[0,1/n)}(t)))dt + \frac{1}{2} \int_0^\infty \varphi(C(2k|x(t)|\chi_{(n,\infty)}(t)))dt \\ &\leq \frac{1}{2} \int_0^\infty \varphi(C(2k|x(t)|\chi_{[0,\alpha)}(t)))dt + \frac{1}{2} \int_0^\infty \varphi(C(2k|x(t)|\chi_{(\beta,\infty)}(t)))dt < \epsilon \end{aligned}$$

i.e. $\|x_n - x\|_{Ces(\varphi)} \rightarrow 0$. Now, it is enough to prove that for each interval $[a, b]$, $0 \leq a < b < \infty$, there is a sequence of functions y_n of the form (2) with

$$\|x\chi_{[a,b]} - y_n\|_{Ces(\varphi)} \rightarrow 0,$$

as $n \rightarrow 0$. Without loss of generality we may assume that $x\chi_{[a,b]} \geq 0$. Since $x \in L^0$ we can find a sequence (y_n) of simple functions such that $y_n \nearrow x\chi_{[a,b]}$. Thus

$$0 \leq x\chi_{[a,b]} - y_n \leq x\chi_{[a,b]},$$

and $x\chi_{[a,b]} - y_n \rightarrow 0$ a.e. on I . But $x \in C_\varphi \subset (Ces_\varphi(I))_a$, whence

$$\|x\chi_{[a,b]} - y_n\|_{Ces(\varphi)} \rightarrow 0.$$

This proves the claim (A).

Take $x \in (Ces_\varphi(I))_a$. Define an element $x_n = x\chi_{[1/n,n]}$. Then $\|x_n - x\|_{Ces(\varphi)} \rightarrow 0$ by the definition. Moreover, $x_n \in B_\varphi$ for all $n \in \mathbb{N}$ by the above proof. Since B_φ is closed, so $x \in B_\varphi = C_\varphi$.

(ii) Clearly, since ρ_φ is a convex modular, C_φ is a linear subspace of Ces_φ . C_φ is closed, since $B_\varphi = C_\varphi$ and B_φ is closed by the definition (note that X_a is closed for each Banach ideal space X , see [4, proof of Theorem 3.8, p. 16]). Separability of the space C_φ follows from (i) and the fact, that Lebesgue measure is separable, see [4, Theorem 5.5, p. 27]. $\square$

Remark 6 Theorem 5 (i) can be shortened. First note that $(Ces_\varphi)_a \subset B_\varphi$ (see [4, Theorem 3.11, p. 18], the set B_φ is denoted by $(Ces_\varphi)_b$ in [4], cf. [4, Definition 3.9, p. 17]). Moreover, $B_\varphi \subset C_\varphi \subset (Ces_\varphi)_a$ as we showed above. Thus $(Ces_\varphi)_a = C_\varphi$.

Theorem 7 Let φ be an Orlicz function and suppose $C : L^\varphi \rightarrow L^\varphi$. Then the following conditions are equivalent:

- (i) $\varphi \in \Delta_2$,
- (ii) $Ces_\varphi \in (OC)$,
- (iii) Ces_φ contains no isomorphic copy of l^∞ .

Proof Equivalence (ii) $\Leftrightarrow$ (iii) follows from Theorem A. Now we will proof the equivalence of conditions (i) and (ii).

(i) $\Rightarrow$ (ii). If $\varphi \in \Delta_2$, then L^φ is order continuous, see e.g. [25, pp. 21–22]. Therefore also Ces_φ is order continuous by the Fact.

(ii) $\Rightarrow$ (i). We have to consider two cases.

I. Suppose $I = [0, \infty)$.

- (1) Let $b_\varphi < \infty$ and $\varphi(b_\varphi) = \infty$. Let $(u_n) \subset \mathbb{R}_+$ be the sequence with $u_n \nearrow b_\varphi^-$. Of course, $\varphi(u_n) \rightarrow \infty$. Passing to a subsequence if necessary, we can assume that $\varphi(u_n) \geq 1$ for all $n \in \mathbb{N}$. Let $a_n = 1/2^n \varphi(u_n)$ for $n \in \mathbb{N}$ and denote $a = \sum_{n=1}^{\infty} a_n$. Define a sequence of pairwise disjoint open intervals $(A_n)_{n \in \mathbb{N}} \subset [0, 1]$ as follows

$$A_n = \left(a - \sum_{k=1}^n \frac{1}{2^k \varphi(u_k)}, a - \sum_{k=1}^{n-1} \frac{1}{2^k \varphi(u_k)} \right),$$

for all $n \in \mathbb{N}$. Then $m(A_n) = a_n$ for all $n \in \mathbb{N}$. Let

$$x = \sum_{n=1}^{\infty} u_n \chi_{A_n}.$$

We claim that $x \in Ces_\varphi$. In fact,

$$\begin{aligned} I_\varphi(x) &= \int_0^\infty \varphi(x) dm = \int_0^\infty \sum_{n=1}^{\infty} \varphi(u_n) \chi_{A_n} dm \\ &= \sum_{n=1}^{\infty} \int_{A_n} \varphi(u_n) dm = \sum_{n=1}^{\infty} \varphi(u_n) m(A_n) = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1. \end{aligned}$$

The assumption $C : L^\varphi \rightarrow L^\varphi$ implies that $L^\varphi \hookrightarrow Ces_\varphi$, so $x \in Ces_\varphi$. Note that there exists $N \in \mathbb{N}$ such that for all $k \geq N$, $2u_k > b_\varphi$. Consequently, since x is non-increasing

$$\rho_\varphi(2x) = I_\varphi(C(2x)) = I_\varphi(2Cx) \geq I_\varphi(2x) = \sum_{n=1}^{\infty} \varphi(2u_n) m(A_n) = \infty,$$

and, by Theorem 5 (i), we conclude that $Ces_\varphi \notin (OC)$.

- (2) Assume that $b_\varphi < \infty$ and $\varphi(b_\varphi) < \infty$. Define an element $x = b_\varphi \chi_{[0,1]}$. Since $I_\varphi(x) < \infty$ so $x \in L^\varphi$ and $x \in Ces_\varphi$ because $C : L^\varphi \rightarrow L^\varphi$. Moreover,

$$\rho_\varphi(2x) = I_\varphi(C(2x)) \geq I_\varphi(2x) = \varphi(2b_\varphi) = \infty,$$

thus, $Ces_\varphi[0, \infty) \notin (OC)$, by Theorem 5 (i).

- (3) Let $a_\varphi > 0$. Put $x = a_\varphi \chi_{[0,\infty)}$. Then $x \in Ces_\varphi[0, \infty)$ and

$$\rho_\varphi(2x) = I_\varphi(C(2x)) = I_\varphi(2x) = \varphi(2a_\varphi) m([0, \infty)) = \infty.$$

Therefore, $Ces_\varphi[0, \infty) \notin (OC)$ by Theorem 5 again.

- (4) Now we assume that $\varphi > 0$, $\varphi < \infty$ and $\varphi \notin \Delta_2(\mathbb{R}_+)$. This means, that $\varphi \notin \Delta_2(0)$ or $\varphi \notin \Delta_2(\infty)$. If $\varphi \notin \Delta_2(\infty)$ then

$$\varphi(2u_n) \geq 2^n \varphi(u_n),$$

for some sequence $(u_n) \nearrow \infty$. Therefore in this case we can use the arguments from I (1) above. Now assume that $\varphi \notin \Delta_2(0)$. That means, that there exist a decreasing sequence $(u_n)_{n \in \mathbb{N}} \subset \mathbb{R}_+$, $u_n \searrow 0$ and $\varphi(2u_n) \geq 2^n \varphi(u_n)$ for all $n \in \mathbb{N}$. Define an element

$$x = \sum_{n=1}^{\infty} u_n \chi_{B_n},$$

where $B_n = (\sum_{k=1}^{n-1} \frac{1}{2^k \varphi(u_k)}, \sum_{k=1}^n \frac{1}{2^k \varphi(u_k)})$. Then

$$\begin{aligned} I_\varphi(x) &= \int_0^\infty \varphi(x) dm = \int_0^\infty \sum_{n=1}^\infty \varphi(u_n) \chi_{B_n} dm \\ &= \sum_{n=1}^\infty \int_{B_n} \varphi(u_n) dm = \sum_{n=1}^\infty \varphi(u_n) m(B_n) = \sum_{n=1}^\infty \frac{1}{2^n} = 1, \end{aligned}$$

so $x \in \text{Ces}_\varphi$ by the assumption $C : L^\varphi \rightarrow L^\varphi$. Furthermore, since x is non-increasing,

$$\begin{aligned} \rho_\varphi(2x) &= I_\varphi(C(2x)) \geq I_\varphi(2x) \\ &= \sum_{n=1}^\infty \varphi(2u_n) m(B_n) \geq \sum_{n=1}^\infty 2^n \varphi(u_n) m(B_n) = \sum_{n=1}^\infty 1 = \infty, \end{aligned}$$

which means that the space $\text{Ces}_\varphi[0, \infty)$ is not order continuous by Theorem 5 above.

II. Suppose $I = [0, 1]$.

- (1) Let $b_\varphi < \infty$ and $\varphi(b_\varphi) = \infty$. In this case we can use construction from case I (1) to show that $\text{Ces}_\varphi \notin (\text{OC})$.
- (2) Assume that $b_\varphi < \infty$ and $\varphi(b_\varphi) < \infty$. Now we follow as in the proof of case I (2) but in this case we don't need the assumption that $C : L^\varphi \rightarrow L^\varphi$. In fact, put $x = b_\varphi \chi_{[0,1]}$. Then

$$\rho_\varphi(x) = I_\varphi(Cx) = I_\varphi(x) = \int_0^1 \varphi(b_\varphi) dm = \varphi(b_\varphi) < \infty,$$

so $x \in \text{Ces}_\varphi$. Furthermore,

$$\rho_\varphi(2x) = I_\varphi(C(2x)) = I_\varphi(2C(x)) = I_\varphi(2x) = \varphi(2b_\varphi) = \infty,$$

which means that $\text{Ces}_\varphi \notin (\text{OC})$.

- (3) Now suppose $\varphi < \infty$ and $\varphi \notin \Delta_2(\infty)$. It is well known that $L^\varphi[0, 1] \in (\text{OC})$ if and only if $\varphi \in \Delta_2(\infty)$ (see, e.g. [25, pp. 21–22]). Therefore, from Proposition 2,

$Ces_\varphi[0, 1] \notin (OC)$. Additionally, the same direct proof of case I (4) above works also for $I = [0, 1]$. $\square$

An immediate consequence of Fact, Theorems 5 and 7 is the next corollary.

Corollary 8 *Let φ be an Orlicz function.*

- (i) *If $\varphi \in \Delta_2$, then $Ces_\varphi = (Ces_\varphi)_a = C_\varphi$.*
- (ii) *If $C : L^\varphi \rightarrow L^\varphi$ then $C_\varphi = Ces_\varphi$ if and only if $\varphi \in \Delta_2$.*

Lemma 9 *If $x \in C_\varphi$, then $\|x\|_{Ces(\varphi)} = 1$ if and only if $\rho_\varphi(x) = 1$.*

Proof See the proof of Lemma 2.1 in [9]. Note that the implication: $\rho(x) = 1$, then $\|x\|_\rho = 1$ for $x \in X_\rho$, holds in any modular space X_ρ generated by a convex modular ρ , see [26]. $\square$

Theorem 10 *Let φ be an Orlicz function satisfying $\varphi < \infty$.*

- (i) *If $I = [0, \infty)$, then the space C_φ is strictly monotone if and only if $\varphi > 0$.*
- (ii) *If $I = [0, 1]$ and $\lim_{u \rightarrow \infty} \varphi(u)/u = \infty$, then the space C_φ is strictly monotone if and only if $\varphi > 0$.*

Proof Necessity Assume that $a_\varphi > 0$. We have to consider two cases:

- (i) If $I = [0, \infty)$ then take $y_t = t\chi_{[0,1)}$ for $t \in \mathbb{R}_+$. We have

$$(Cy_t)(u) = t\chi_{[0,1)}(u) + \frac{t}{u}\chi_{[1,\infty)}(u).$$

We define the function

$$f(t) = \int_0^\infty \varphi \left(t\chi_{[0,1)}(u) + \frac{t}{u}\chi_{[1,\infty)}(u) \right) du = \rho_\varphi(y_t),$$

for $t > 0$. Since $t/x \rightarrow 0$ as $x \rightarrow \infty$ and $a_\varphi > 0$, so there exists $x_0 = x_0(t) \in \mathbb{R}_+$ such that $t/x_0 \leq a_\varphi$. Consequently, $f(t) \leq \varphi(t)x_0$, which means that $f(t)$ takes finite values for $t > 0$. Moreover, f is convex function, so f is continuous function on $\mathbb{R}_+$. This means that $f[\mathbb{R}_+] = \mathbb{R}_+$. In that case, from the Darboux property we find $\lambda > 0$ satisfying $f(\lambda) = 1$. Take $x_1 > 1$ with $\lambda/x_1 \leq a_\varphi/2$. Let $z = a_\varphi\chi_{[x_1, x_1+1)}/2$. Then for any $x \geq x_1$ we have

$$C(y_\lambda + z)(x) = Cy_\lambda(x) + Cz(x) \leq a_\varphi,$$

whence $\rho_\varphi(y_\lambda + z) = 1$ and, from Lemma 9, we get $\|y_\lambda + z\|_{Ces(\varphi)} = 1$. Summing up, we built elements $u = y_\lambda$ i $v = y_\lambda + z$ such that $u \neq v$, $u \leq v$ i $\|u\|_{Ces(\varphi)} = \|v\|_{Ces(\varphi)} = 1$. Moreover, $u, v \in C_\varphi$. This follows easily from the equality $C_\varphi = B_\varphi$ (see Proof of Theorem 5). This means, that the space C_φ isn't strictly monotone.

(ii) Suppose $I = [0, 1]$. For $a, b \in \mathbb{R}$ set $x = b\chi_{[0,a]}$. Then

$$(Cx)(t) = b\chi_{[0,a]}(t) + \frac{ab}{t}\chi_{[a,1]}(t).$$

Since $\lim_{u \rightarrow \infty} \varphi(u)/u = \infty$, there is a number $b_0 \in \mathbb{R}$ with $b_0 > a_\varphi$ and $a_\varphi \varphi(b_0)/b_0 > 1$. We define the function

$$f(a) = a\varphi(b_0) + \int_a^{\frac{ab_0}{a_\varphi}} \varphi\left(\frac{ab_0}{t}\right) dt,$$

for $a \in [0, a_\varphi/b_0]$. Clearly, $f(0) = 0$ and $f(a_\varphi/b_0) > 1$. Moreover, we claim that f is continuous on $[0, a_\varphi/b_0]$. Take $a_n \rightarrow a$ with $a \in (0, a_\varphi/b_0]$ and put

$$A = \left[a, \frac{ab_0}{a_\varphi} \right],$$

$$A_n = \left[a_n, \frac{a_nb_0}{a_\varphi} \right],$$

for $n \in \mathbb{N}$. Then

$$\begin{aligned} 0 &\leq |f(a_n) - f(a)| \leq |a_n - a|\varphi(b_0) + \left| \int_a^{\frac{ab_0}{a_\varphi}} \varphi\left(\frac{ab_0}{t}\right) dt - \int_{a_n}^{\frac{a_nb_0}{a_\varphi}} \varphi\left(\frac{a_nb_0}{t}\right) dt \right| \\ &= |a_n - a|\varphi(b_0) + \left| \int_{A \setminus A_n} \varphi\left(\frac{ab_0}{t}\right) dt \right. \\ &\quad \left. + \int_{A \cap A_n} \varphi\left(\frac{ab_0}{t}\right) - \varphi\left(\frac{a_nb_0}{t}\right) dt - \int_{A_n \setminus A} \varphi\left(\frac{a_nb_0}{t}\right) dt \right| \\ &\leq |a_n - a|\varphi(b_0) + \left| \int_{A \setminus A_n} \varphi\left(\frac{ab_0}{t}\right) dt \right| + \left| \int_{A \cap A_n} \varphi\left(\frac{ab_0}{t}\right) - \varphi\left(\frac{a_nb_0}{t}\right) dt \right| \\ &\quad + \left| \int_{A_n \setminus A} \varphi\left(\frac{a_nb_0}{t}\right) dt \right|. \end{aligned}$$

Now we have

$$0 \leq \left| \int_{A \setminus A_n} \varphi\left(\frac{ab_0}{t}\right) dt \right| \leq m(A \setminus A_n) \sup_{t \in A} \varphi\left(\frac{ab_0}{t}\right) = m(A \setminus A_n) \varphi(b_0) \rightarrow 0,$$

as $n \rightarrow \infty$. Similarly, we can show that $\left| \int_{A_n \setminus A} \varphi\left(\frac{a_nb_0}{t}\right) dt \right| \rightarrow 0$ as $n \rightarrow \infty$. Moreover, for n large enough $a/2 < a_n < 2a$. Therefore, for $t \in A$

$$\left| \varphi\left(\frac{ab_0}{t}\right) - \varphi\left(\frac{a_nb_0}{t}\right) \right| \leq \max_{t_1, t_2 \in [a_\varphi, 2b_0]} |\varphi(t_1) - \varphi(t_2)| = \eta(a, b_0) = \eta < \infty,$$

because φ is continuous on the compact set $[a_\varphi, 2b_0]$. Since $|\varphi(\frac{ab_0}{t}) - \varphi(\frac{a_nb_0}{t})| \rightarrow 0$ pointwise on A , $\eta\chi_A$ is integrable majorant and $L^1[0, 1]$ is order continuous, so from Dominated Convergence Theorem we get

$$0 \leq \left| \int_{A \cap A_n} \varphi\left(\frac{ab_0}{t}\right) - \varphi\left(\frac{a_nb_0}{t}\right) dt \right| \leq \left| \int_A \varphi\left(\frac{ab_0}{t}\right) - \varphi\left(\frac{a_nb_0}{t}\right) dt \right| \rightarrow 0,$$

as $n \rightarrow \infty$. This means that $|f(a_n) - f(a)| \rightarrow 0$, so f is continuous. This proves the claim.

By the Darboux property of f , there exists $a_1 \in (0, a_\varphi/b_0)$ such that $f(a_1) = 1$. Consider two elements

$$x_1 = b_0\chi_{[0, a_1]},$$

and

$$x_2 = b_0\chi_{[0, a_1]} + \left(a_\varphi - \frac{a_1b_0}{\delta}\right)\chi_{[\frac{\delta+1}{2}, 1]},$$

where $\delta > 0$ is such that $a_1b_0/a_\varphi < \delta < 1$. Obviously, $x_1 \leq x_2$ and $x_1 \neq x_2$ and $x_1, x_2 \in C_\varphi$. Furthermore, $\rho_\varphi(x_1) = \rho_\varphi(x_2) = 1$. Indeed, we have the following inequalities

$$a_1 < \frac{b_0}{a_\varphi}a_1 < \delta < \frac{\delta+1}{2} < 1,$$

so for $t \in [(\delta+1)/2, 1)$

$$(Cx_2)(t) = \frac{a_1b_0}{t} + \left(a_\varphi - \frac{a_1b_0}{\delta}\right)\frac{1}{t}\int_{\frac{\delta+1}{2}}^t ds < \frac{a_1b_0}{t} + a_\varphi - \frac{a_1b_0}{\delta} \leq a_\varphi.$$

Thus, by the Lemma 9, $\|x_1\|_{Ces(\varphi)} = \|x_2\|_{Ces(\varphi)} = 1$, whence the space C_φ is not strictly monotone.

Sufficiency Assume, that $a_\varphi = 0$, $0 \leq x \leq y$, $x \neq y$ and $x, y \in C_\varphi$. Without loss of generality, we can assume, that $\|x\|_{Ces(\varphi)} = 1$. From Lemma 9, $\rho_\varphi(x) = 1$. We need only to show that $\rho_\varphi(y) > 1$. Since φ is superadditive, so for any $u, v \in (C_\varphi)_+$ we have

$$\rho_\varphi(u+v) \geq \rho_\varphi(u) + \rho_\varphi(v).$$

Since $y - x \geq 0$, $y - x \neq 0$ and $\varphi > 0$, we have $\rho_\varphi(y - x) > 0$, and consequently

$$\rho_\varphi(y) = \rho_\varphi(x + (y - x)) \geq \rho_\varphi(x) + \rho_\varphi(y - x) = 1 + \rho_\varphi(y - x) > 1.$$

□

Theorem 11 Let φ be an Orlicz function with $\varphi < \infty$ and $C : L^\varphi \rightarrow L^\varphi$. Suppose additionally $\lim_{u \rightarrow \infty} \varphi(u)/u = \infty$ if $I = [0, 1]$. The following conditions are equivalent:

- (i) $Ces_\varphi \in (UM)$,
- (ii) $Ces_\varphi \in (LLUM)$,
- (iii) $\varphi > 0$ and $\varphi \in \Delta_2$.

Proof (i) $\Rightarrow$ (ii). This implication follows by the definition.

(ii) $\Rightarrow$ (iii). If $a_\varphi > 0$ then $C_\varphi \notin (SM)$ by Theorem 10. Moreover, see [3, Proposition 2.1], the following implication is true: if $X \in (LLUM)$ then $X \in (OC)$. Therefore, if $\varphi \notin \Delta_2$ then $Ces_\varphi \notin (OC)$ by Theorem 7, whence $Ces_\varphi \notin (LLUM)$.

(iii) $\Rightarrow$ (i). If $\varphi > 0$ and $\varphi \in \Delta_2$ then $L_\varphi \in (UM)$, see e.g. [13, Theorem 7]. Consequently $Ces_\varphi \in (UM)$ by Fact. $\square$

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