Real ideals of compact operators of complex factors

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Abstract Real ideals of compact operators for (complex) factors are investigated. A description (up to isomorphisms) of real two-sided ideals of relatively compact operators of a complex W^* -factors is given. A relative weak $(RW)_r$ convergence in a real Hilbert space is introduced. The classical Hilbert characterization of compactness of operators is extended to the compact operators in semifinite real W^* -algebras.

Keywords W^* -algebras · Real factors · Ideals · Compact operators

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1 Introduction

In the present paper we investigate the real ideals of compact operators for complex factors and give a description (up to isomorphisms) of real two-sided ideal of relatively compact operators of the complex W^* -factors. A concept of relative weak (RW) $_r$ -convergence in a real Hilbert space is introduced. The classical Hilbert characterization of compactness of operators is extended to the compact operators in semifinite real W^* -algebras.

2 Preliminaries

Let $B(H)$ be the algebra of all bounded linear operators on a complex separable Hilbert space H . A weakly closed $*$ -subalgebra $\mathfrak{A}$ containing the identity operator $\mathbf{1}$ in $B(H)$ is called a W^* -algebra. A real $*$ -subalgebra $R \subset B(H)$ is called a *real W^* -algebra* if it is closed in the weak operator topology, $\mathbf{1} \in R$ and $R \cap iR = \{0\}$. A real W^* -algebra R is called a *real factor* if its center $Z(R)$ consists of the elements $\{\lambda \mathbf{1}, \lambda \in \mathbb{R}\}$, where $\mathbb{R}$ is the field of all real numbers. We say that a real W^* -algebra R is of the type $I_{fin}, I_\infty, II_1, II_\infty$, or III_λ , ($0 \leq \lambda \leq 1$) if the enveloping W^* -algebra $\mathfrak{A}(R)$ has the corresponding type in the ordinary classification of W^* -algebras. A linear mapping α of an algebra into itself with $\alpha(x^*) = \alpha(x)^*$ is called an *$*$ -automorphism* if $\alpha(xy) = \alpha(x)\alpha(y)$; it is called an *involutive $*$ -antiautomorphism* if $\alpha(xy) = \alpha(y)\alpha(x)$ and $\alpha^2(x) = x$. If α is an involutive $*$ -antiautomorphism of a W^* -algebra M , we denote by (M, α) the real W^* -algebra generated by α , i.e. $(M, \alpha) = \{x \in M : \alpha(x) = x^*\}$. Conversely, every real W^* -algebra R is of the form (M, α) , where M is the complex envelope of R and α is an involutive $*$ -antiautomorphism of M (see [1, 2, 5, 9]). Therefore we shall identify from now on the real von Neumann algebra R with the pair (M, α) .

A *trace* on a (complex or real) W^* -algebra N is a linear function τ on the set N^+ of positive elements of N with values in $[0, +\infty]$, satisfying $\tau(uxu^*) = \tau(x)$, for an arbitrary unitary u and for any x in N .

The trace τ is said to be *finite*, if $\tau(\mathbf{1}) < +\infty$; *semifinite*, if given any $x \in N^+$ there is a nonzero $y \in N^+$, $y \leq x$ with $\tau(y) < +\infty$.

3 Real and complex ideals of W^* -algebras

Definition 1 Let M be a W^* -algebra. A real subspace I of M is called a *real ideal* of M if $I \cdot M \subset I_c$, where I_c is the smallest complex subspace of M , containing I .

It is easy to see that the subspace I_c is equal to $I + iI$, therefore a real subspace I is a real ideal if and only if $I \cdot M \subset I + iI$.

Since each complex subspace of M is a real subspace, any complex ideal is automatically a real ideal of M . Let I be a real ideal of M . If there exists a real W^* -subalgebra R of M with $R + iR = M$, such that $I \subset R$, then I is called a *pure real ideal* of M . In this case, it is obvious that we have $I \cdot R \subset I$. Note that, the reverse is not true, i.e. from $I \cdot R \subset I$ it does not follow $I \subset R$. But a complex subspace $J = I + iI$ always is a complex ideal of M . On the other hand if $I \subset R$ is a real subspace of M and $I + iI$ is a complex ideal, then I is a pure real ideal, i.e. we obtain $I \cdot R \subset I$.

Let, now I and Q be pure real ideals of M . In general, the set $I + iQ$ is not a (complex) subspace. More precisely the set $I + iQ$ is a complex subspace if and only if $I = Q$. Therefore we consider the smallest complex subspace J of M , containing I and Q . Obviously J is equal to $(I + Q) + i(I + Q)$. Thus, if I and Q are real ideals, then $J = (I + Q) + i(I + Q)$ is a complex ideal.

4 Ideals of compact operators

Let (M, α) be a real factor and let τ be an α -invariant semifinite trace on M . A subspace $K \subset H$ is called τ -finite (or finite relative to τ), if $\tau(P_K) < +\infty$, where P_K is the canonical projection of H on K with $P_K \in M$.

Now, let K be a subset of H . A subset K is called τ -compact (or compact relative to τ), if K is approximated in the norm $\|\cdot\|_H$ by a bounded sequence of τ -finite subspaces.

A real operator A on H (i.e. $A \in (M, \alpha)$) is called *real compact* relative to τ if it is an operator mapping bounded sets into relatively compact sets. We denote by I (respectively, by J) the set of all relatively compact operators of (M, α) (respectively, of M). Let us recall the following result.

Theorem 1 ([3, 10, 11]) *Let M be a semifinite factor and let α be an involutive *-antiautomorphism of M . Then I (respectively, J) is a unique (nonzero) uniformly closed two-sided ideal of (M, α) (respectively, of M), and $I + iI = J$.*

Now, let us recall [4] the notion of the crossed product of a W^* -algebra M by a locally compact topological group G by its *-automorphism. Let $\gamma : G \rightarrow \text{Aut}(M)$ be a group homomorphism such that the map $g \rightarrow \gamma_g$ is strongly continuous. Let $L_2(G, H)$ be the Hilbert space of all H -valued square integrable functions on G . We consider a *-algebra $N \subset B(L_2(G, H))$, generated by operators of the form: $\pi_\gamma(a)$ and $u(g)$, where $a \in M$, $g \in G$, and

$$(\pi_\gamma(a)\xi) = \gamma_h^{-1}(a)\xi(h), \quad (u(g)\xi)(h) = \xi(g^{-1}h),$$

$\xi = \xi(h) \in L_2(G, H)$, $g, h \in G$. The algebra N is called the *crossed product* of M by G , and denoted by $W^*(M, G)$ or $M \times_\gamma G$. Moreover, there exists a canonical embedding $\pi_\gamma : M \rightarrow \pi_\gamma(M) \subset N$. Each element $x \in N$ has the form: $x = \sum_{g \in G} \pi_\gamma(x(g))u(g)$, where $x(\cdot)$ is an M -valued function on G . If θ is a *-automorphism of M , then for the action $\{\theta^n\}$ of the group $\mathbb{Z}$ on M we denote by $W^*(\theta, M)$ or $M \times_\theta \mathbb{Z}$ the crossed product of M by θ . Similarly one can define the notion of crossed product for real W^* -algebras (see [1, 12–14]).

Let M be a factor of type III_λ ($\lambda \neq 1$) and let α be an involutive *-antiautomorphism of M . Then by [12] (see also [1]), either

- there exist a factor N of type II_∞ and an α -invariant automorphism θ of N such that (M, α) is isomorphic to the (real) crossed product $(N, \alpha) \times_\theta \mathbb{Z}$ or
- there exist a factor N of type II_∞ and an antiautomorphism σ of N such that (M, α) is isomorphic to $((N \oplus N^{op}) \times_\sigma \mathbb{Z}, \beta)$, where N^{op} is the opposite W^* -algebra for N and $\beta(x, y) = (y, x)$, for all $x, y \in N$.

Let's consider the first case. Let (M, α) be isomorphic to the (real) crossed product $(N, \alpha) \rtimes_{\theta} \mathbb{Z}$. It is known that $(N, \alpha) \rtimes_{\theta} \mathbb{Z} + i \cdot (N, \alpha) \rtimes_{\theta} \mathbb{Z} = N \rtimes_{\theta} \mathbb{Z}$ (see [12–14]). Let $E : M \rightarrow N$ be the unique α -invariant faithful normal conditional expectation (see [15, 16]). Let us state an auxiliary lemma whose proof immediately follows from the linearity and α -invariance of E .

Lemma 1 *If S is an ideal in (M, α) and $S_c = S + iS$, then*

$$E^{-1}(S) + iE^{-1}(S) = E^{-1}(S_c), \quad E^{-1}(S)^+ \subset E^{-1}(S)^+,$$

where $E^{-1}(A) = \{x : E(x) \in A\}$.

Recall that a cone $K \subset A^+$ is called *hereditary* if $x \in A^+$, $y \in K$ and $x \leq y$ implies $x \in K$; a subalgebra $B \subset A$ is called *hereditary* if the cone B^+ is hereditary. It is easy to see that any two-sided ideal is hereditary.

Lemma 2 *If the cone $E^{-1}(S_c)^+$ is hereditary, then the cone $E^{-1}(S)^+$ is also hereditary.*

Proof If $x \in (M, \alpha)^+ \subset M^+$, $y \in E^{-1}(S)^+ \subset E^{-1}(S_c)^+$ and $x \leq y$, then $x \in E^{-1}(S_c)^+$, since $E^{-1}(S_c)^+$ is hereditary. Therefore $\alpha(x) = x^*$ implies $x \in E^{-1}(S)^+$. $\square$

Proposition 1 *Let M be a semifinite factor. If $S \subset (M, \alpha)$ is a two-sided ideal, then the linear span of $E^{-1}(S)^+$ denoted as $\text{span}(E^{-1}(S)^+)$ is a hereditary *-subalgebra of (M, α) and a two-sided module over (M, α) . Moreover, if S is a norm-closed, then $\text{span}(E^{-1}(S)^+)$ is also norm-closed.*

Proof By Lemma 1 and Proposition 3.3 [6] the cone $E^{-1}(S_c)^+$ is hereditary and $\text{span}(E^{-1}(S_c)^+)$ is a hereditary *-subalgebra of M , where $S_c = S + iS$. By Lemma 2 the cone $E^{-1}(S)^+$ is also hereditary. Using the hereditariness of $\text{span}(E^{-1}(S_c)^+)$ one can easily check the hereditariness of $\text{span}(E^{-1}(S)^+)$.

If S is norm-closed, then S^+ is also norm-closed. The continuity of E implies that $E^{-1}(S)^+$ is closed. Therefore $\text{span}(E^{-1}(S)^+)$ is also closed.

Let $x \in E^{-1}(S)^+$ and $y \in (M, \alpha)$. From $x \in E^{-1}(S_c)^+$ and $y \in M$, by Proposition 3.3 [6] we obtain that $yx \in \text{span}(E^{-1}(S_c)^+)$, since $\text{span}(E^{-1}(S_c)^+)$ is a two-sided module over M . On the other hand $yx \in (M, \alpha)$ and $\{a \in \text{span}(E^{-1}(S_c)^+) : \alpha(a) = a^*\} = \text{span}(E^{-1}(S)^+)$. Hence $yx \in \text{span}(E^{-1}(S)^+)$. Since the element x is arbitrary from $E^{-1}(S)^+$ by linearity we obtain $yx \in \text{span}(E^{-1}(S)^+)$ for any $x \in \text{span}(E^{-1}(S)^+)$. Therefore $\text{span}(E^{-1}(S)^+)$ is a left-sided module over (M, α) . Similarly one can show, that it is a right-sided (M, α) -module. $\square$

Now, we put $\mathcal{I} = \text{span}\{x \in (M, \alpha) : E(x) \in I\}$, where I is the unique (nonzero) uniformly closed two-sided ideal of the semifinite real factor (N, α) (see Theorem 1). By Proposition 1, $\mathcal{I}$ is hereditary.

Lemma 3 *The following is valid*

$$\mathcal{I}^+ = \{x \in (M, \alpha)^+ : x \leq y, \text{ for some } y \in I^+\}.$$

Proof Let $x \in \mathcal{I}^+$ and $J = I + iI$, $\mathcal{J}^+ = \{x \in M^+ : x \leq z, \text{ for some } z \in J^+\}$. Since $x \in \mathcal{J}^+$ by Proposition 3.7 d) [6], $x \leq z$, for some $z \in J^+$. Let $z = y + it$,

$y, t \in (N, \alpha)$. Then $y \geq 0$ (because $z \geq 0$) and $z - x = (y - x) + it \geq 0$, hence $y - x \geq 0$, i.e., $x \leq y$. Since $I^+ \subset J^+$, $y \in I^+$.

Conversely, if $x \in (M, \alpha)^+$ and $x \leq y$, for some $y \in I^+$, then again by Proposition 3.7 d) [6] we have $x \in \mathcal{J}^+$. From $\alpha(x) = x^*$ we have $x \in \mathcal{I}^+$. $\square$

From Lemma 3, in particular, it follows, that a projection from (M, α) is finite if and only if it majorized by some finite projection of (N, α) .

Let I_1 be the norm closure of $\mathcal{I}$. If we apply Proposition 1, Lemma 3 and the scheme of the proofs of Propositions 4.1, 4.5 and Theorem 4.3 [6], then we can prove the following real analogue of Halpern-Kaftal's theorem.

Theorem 2 *Let M be a factor of type III_λ ($\lambda \neq 1$) and let α be an involutive *-antiautomorphism of M . If the real factor (M, α) is isomorphic to the (real) crossed product $(N, \alpha) \times_\theta \mathbb{Z}$, then I_1 is a unique (up to an inner automorphism) smallest hereditary real C*-subalgebra of (M, α) , containing the ideal I , and it is a two-sided module over (N, α) .*

Let us consider the second case. Let (M, α) be isomorphic to $((N \oplus N^{op}) \times_\sigma \mathbb{Z}, \beta)$, where N is a II_∞ -factor, N^{op} is the opposite W*-algebra for N and $\beta(x, y) = (y, x)$, for all $x, y \in N$.

Recall that, a factor N is generated by the fixed point algebra of the one parameter group $\{\sigma_t^\psi : t \in \mathbb{R}\}$ of modular automorphisms, associated with some α -invariant faithful normal semifinite weight ψ . More precisely, the W*-subalgebra $M_\psi = \{x \in M : \sigma_t^\psi(x) = x, t \in \mathbb{R}\}$ contains a central projection p such that N is isomorphic to factor pM_ψ . In this case the real W*-algebra (M_ψ, α) is isomorphic to $(N \oplus N^{op}, \beta)$ (for more details see [1, 12–14]).

Let $E : (M, \alpha) \rightarrow (N \oplus N^{op}, \beta)$ be a faithful normal conditional expectation (see [15, 16]) and let J be the unique (nonzero) uniformly closed two-sided ideal of the semifinite (complex) factor N (see Theorem 1). Similarly to the first case, we denote by I_2 the norm closure of $\text{span}\{x \in (M, \alpha) : E(x) \in (J \oplus J^{op}, \beta)\}$.

Applying the same reasonings, as in the first case, and the scheme of proofs of Propositions 4.1, 4.5 and Theorem 4.3 [6], we obtain one more real analogue of Halpern-Kaftal's theorem.

Theorem 3 *Let M be a factor of type III_λ ($\lambda \neq 1$) and let α be an involutive *-antiautomorphism of M . If the real factor (M, α) is isomorphic to $((N \oplus N^{op}) \times_\sigma \mathbb{Z}, \beta)$, then I_2 is the unique (up to inner automorphism) smallest hereditary real C*-subalgebra of (M, α) , containing the ideal I and is a two-sided module over (N, α) . Here I is the unique (nonzero) uniformly closed two-sided ideal of semifinite real factor (N, α) .*

Thus, summarizing all above, in the injective case, we can describe all (nonzero) uniformly closed two-sided real ideals of semifinite and pure infinite complex factors. Recall that [1, 5], if M is an injective factor of type II, then there exists a unique conjugacy class of involutive *-antiautomorphisms in M ; therefore there exists a unique (up to isomorphisms) real subfactor of M , generating M . If M is an injective factor of type III_λ ($0 < \lambda < 1$), then in M there exist exactly two conjugacy classes of involutive *-antiautomorphism; therefore there exist two (up to isomorphisms) real subfactors of M , generating M . Hence we obtain the following result.

Theorem 4 *Let M be a factor. Then the following assertions are true:*

1. *If M is an injective factor of type II_1 or type II_∞ , then there exist (up to isomorphisms) two (nonzero) uniformly closed two-sided real ideals in M . One of them is the complex ideal J , the other is the pure real ideal I .*
2. *If M is an injective factor of type III_λ ($0 < \lambda < 1$), then there exist (up to isomorphisms) three (nonzero) uniformly closed two-sided real ideals in M . One of them is the complex ideal J , the two others are the pure real ideals I_1 and I_2 .*

5 Relative weak convergence in semifinite real W^* -algebras

In this section, we study the relative weak (RW) convergence in a real Hilbert space. We first recall that the elements of the two-sided closed ideal I generated by the projections which are finite relative to a real W^* -algebra (M, α) are called *compact operators* of (M, α) .

Let H_r be a real Hilbert space with $H_r + iH_r = H$. A sequence $\{\xi_n\} \subset H_r$ (or $\subset H$) is called *weakly converging* to ξ , if for every projection P which is finite relative to $B(H_r)$ (respectively, $B(H)$), $P\xi_n$ converges strongly to $P\xi$ ($P\xi_n \xrightarrow{S} P\xi$). This suggests the following generalization:

Definition 2 Let (M, α) be a real W^* -algebra in $B(H_r) \subset B(H) = B(H_r) + iB(H_r)$. We say that a sequence $\{\xi_n\} \in H_r$ *converges to ξ weakly relative to (M, α)* and briefly say $(RW)_r$ converges or $\xi_n \xrightarrow{(RW)_r} \xi$, if

1. $\|\xi_n\|$ is bounded;
2. for every projection $P \in (M, \alpha)$ which is finite relative to (M, α) , the sequence $\{P\xi_n\}$ converges strongly to the element $P\xi$, i.e. $P\xi_n \xrightarrow{S} P\xi$.

Note that a weakly convergent sequence is necessarily bounded, but following the Example 2 of [7], it is easy to construct an example, in which a unbounded sequence satisfies the second condition of Definition 1.

Example Let H and K be infinite-dimensional separable real Hilbert spaces with orthonormal bases $\{\eta_n\}$, $\{\gamma_n\}$ respectively. We put $R = B(H) \otimes \mathbb{R}\mathbf{1}_K$ and $\xi_n = \sum_{i=1}^{2n} \eta_i \otimes \gamma_i$. Since $\|\xi_n\|^2 = \sum_{i=1}^{2n} \|\eta_i\|^2 \|\gamma_i\|^2 = n$, it is an unbounded sequence. Let P be a finite projection of R . Then there is a finite projection P_0 in $B(H)$ such that $P = P_0 \otimes \mathbf{1}_K$. Without loss of generality we may assume that it is one-dimensional, i.e., that $P_0 = \langle \cdot, \xi \rangle \xi$, for an element $\xi \in H$ with $\|\xi\| = 1$. Then

$$\begin{aligned}
 \|P\xi_n\|^2 &= \left(\sum_{i=1}^{2n} P_0 \eta_i \otimes \gamma_i, \sum_{j=1}^{2n} P_0 \eta_j \otimes \gamma_j \right) \\
 &= \left(\sum_{i=1}^{2n} \langle \eta_i, \xi \rangle \xi \otimes \gamma_i, \sum_{j=1}^{2n} \langle \eta_j, \xi \rangle \xi \otimes \gamma_j \right) \\
 &= \sum_{i,j=1}^{2n} (\langle \eta_i, \xi \rangle \xi, \langle \eta_j, \xi \rangle \xi)_H \cdot (\gamma_i, \gamma_j)_K = \sum_{k=1}^{2n} |\langle \eta_k, \xi \rangle|^2.
 \end{aligned}$$

Since the series $\sum_{k=1}^{\infty} |\langle \eta_k, \xi \rangle|^2$ converges, the sequence $\sum_{k=n}^{2n} |\langle \eta_k, \xi \rangle|^2$ converges to 0, when $n \rightarrow \infty$. Hence $\|P\xi_n\| \rightarrow 0$. Therefore $P\xi_n \xrightarrow{S} P\xi = 0$, but the sequence $\{\xi_n\}$ is unbounded. Moreover it is easy to show that the sequence $\{\eta_n \otimes \gamma\}$ $(RW)_r$ converges to 0, but it does not converge strongly to 0, and the sequence $\{\xi \otimes \gamma_n\}$ converges weakly to 0, but does not $(RW)_r$ converge to 0.

It is easy to see that the following assertions are true.

Lemma 4

$$\xi_n \xrightarrow{RW} \xi \iff \xi_n \xrightarrow{(RW)_r} \xi \quad (1)$$

Here $\xi_n \xrightarrow{RW} \xi$ means that $\|\xi_n\|$ is bounded and $P\xi_n \xrightarrow{S} P\xi$, for every projection $P \in M$ (see [7, Definition 1]).

Proof The proof of implication “ $\Rightarrow$ ” is obvious. Let’s prove the implication “ $\Leftarrow$ ”. Assume, that there is some projection $p \in M$ with $\|p\xi_n\| \not\rightarrow 0$. Since the projections p and $\alpha(p)$ are equivalent we have $\|\alpha(p)\xi_n\| \not\rightarrow 0$ (see [1]). It is easy to see that $a = p + \alpha(p) \in (M, \alpha)$ and $a \geq p$ (because $\alpha(p) \geq 0$), therefore $\|a\xi_n\| \not\rightarrow 0$. Then exists a spectral projection $e \in (M, \alpha)$ of an element a with $\|e\xi_n\| \not\rightarrow 0$. It is a contradiction with the assumption. $\square$

The following theorem is a generalization of Hibert’s characterization of the compact operators.

Theorem 5 (Theorem 7, [7]) *An element A is compact in M iff it maps (RW) converging sequences into strongly converging ones.*

We have the following real analogue of the above characterization.

Theorem 6 *An element A is compact in (M, α) iff it maps $(RW)_r$ converging sequences into strongly converging ones.*

Proof Firstly, let us prove the sufficiency. Let I be the two-sided closed (pure real) ideal generated by the projections which are finite relative to a real W^* -algebra (M, α) and put $J = I + iI$. As it was noticed above J is the two-sided closed (complex) ideal, generated by the projections which are finite relative to a W^* -algebra M . If we put

$$I_1 = \{A \in (M, \alpha) : A\xi_n \xrightarrow{S} 0, \text{ for any sequence } \{\xi_n\} \subset H_r \text{ with } \xi_n \xrightarrow{(RW)_r} 0\},$$

then by (1) for

$$J_1 = \{A \in M : A\xi_n \xrightarrow{S} 0, \text{ for any sequence } \{\xi_n\} \subset H \text{ with } \xi_n \xrightarrow{(RW)} 0\},$$

we obtain $I_1 \subset J_1 = J = I + iI$. Here the equality $J_1 = J$ is valid by Theorem 5. Since $I_1 \subset (M, \alpha)$, we obtain $I_1 \subset I$. Therefore $A \in I$, i.e. A is compact relatively to (M, α)

Now we shall prove the necessity. It suffices to show that $I \subset I_1$. Suppose that $K \in I$ and $\xi_n \xrightarrow{(RW)_r} \xi$. Without loss of generality we can assume that $\xi = \theta$. We repeat step by step the scheme of proof of Theorem 1.3 [8], to obtain the real analogue of the generalized Rellich condition for K , i.e., for every $\lambda > 0$ there is a projection P of (M, α) such that $\|K P\| \leq \lambda$ and $\mathbf{1} - P$ is finite. From $\xi_n \xrightarrow{(RW)_r} 0$, by definition, we obtain $(\mathbf{1} - P)\xi_n \xrightarrow{S} 0$, and hence $K(\mathbf{1} - P)\xi_n \xrightarrow{S} 0$. Since $\|K - K(\mathbf{1} - P)\| = \|K P\| \leq \lambda$ and $\|\xi_n\|$ is bounded by definition, we have $K\xi_n \xrightarrow{S} 0$. Thus, we have shown that for any $K \in I$ the condition $\xi_n \xrightarrow{(RW)_r} \xi$ implies $K\xi_n \xrightarrow{S} K\xi$. Hence $K \in I_1$, and therefore $I_1 \subset I$. $\square$

Theorem 6 shows that Hibert's characterization of the compact operators remains valid in semifinite real W^* -algebras.

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