Erratum to: Single projection Kaczmarz extended algorithms

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Published online: 14 August 2017
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Erratum to: Numerical Algorithms, 73(3)(2016), 791–806
<https://doi.org/10.1007/s11075-016-0118-7>

Comments and notations

We made corrections only on section “Convergence analysis” of the original paper. We will refer to the equations, results, and references from the original paper by adding the sign (*). Else, these are related to the Erratum version of section “Convergence analysis”.

1 Erratum - convergence analysis

In order to prove the convergence of the two algorithms MREK and ACEK, we next examine how the distance to any fixed least-squares solution changes. We followed the general ideas of considering projection based algorithms together with relaxation

The online version of the original article can be found at doi:[10.1007/s11075-016-0118-7](https://doi.org/10.1007/s11075-016-0118-7).

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parameters, as in the original paper by Kaczmarz [*3] and further developments, see e.g. the monographs [*1], [*4].

To this end, we denote by $x_*^k = P_{H_{i_k}}^\omega(x^{k-1})$, where H_{i_k} is the unperturbed hyperplane from (*18), given by

$$x_*^k = x^{k-1} - \omega \frac{\langle A_{i_k}, x^{k-1} \rangle - b_{i_k}}{\|A_{i_k}\|^2} A_{i_k}. \quad (1)$$

Proposition 1 For any $x \in LSS(A; \hat{b})$ and for all $k \in \mathbb{N}$, we have for every iterate x^k generated by the algorithm MREK or ACEK, respectively and for any $i_k \in [m]$

(i)

$$\langle x_*^k - x, \omega \gamma_{i_k} \rangle = (1 - \omega) \left\langle \frac{\langle A_{i_k}, x^{k-1} - x \rangle}{\|A_{i_k}\|^2} A_{i_k}, \omega \gamma_{i_k} \right\rangle, \quad (2)$$

(ii)

$$\begin{aligned} \|x^k - x\|^2 &= \|x^{k-1} - x\|^2 + \frac{\omega^3(2 - \omega) + (1 - \omega)^2\omega^2}{\omega(2 - \omega)} \|\gamma_{i_k}\|^2 \\ &\quad - \|\sqrt{\omega(2 - \omega)} \frac{\langle A_{i_k}, x^{k-1} - x \rangle}{\|A_{i_k}\|^2} A_{i_k} - \frac{1 - \omega}{\sqrt{\omega(2 - \omega)}} \omega \gamma_{i_k}\|^2, \end{aligned} \quad (3)$$

(iii)

$$\|x^k - x\|^2 \leq \|x^{k-1} - x\|^2 + \frac{\omega^3(2 - \omega) + (1 - \omega)^2\omega^2}{\omega(2 - \omega)} \|\gamma_{i_k}\|^2, \quad (4)$$

with γ_{i_k} from (*19).

Proof (i) Choose $x \in LSS(A; \hat{b})$ arbitrarily. Then $Ax = b$ and, in particular, $x \in H_{i_k}$. If $S_{i_k} = \{x : \langle A_{i_k}, x \rangle = 0\}$ is the vector subspace parallel with H_{i_k} and

$$P_{S_{i_k}}^\omega(z) = z - \omega \frac{\langle A_{i_k}, z \rangle}{\|A_{i_k}\|^2} A_{i_k}, \quad P_{S_{i_k}}(z) = z - \frac{\langle A_{i_k}, z \rangle}{\|A_{i_k}\|^2} A_{i_k}$$

are the corresponding projections operators, from (1) and (*19) it results

$$x_*^k - x = P_{S_{i_k}}^\omega(x^{k-1} - x) = P_{S_{i_k}}(x^{k-1} - x) + (1 - \omega) \frac{\langle A_{i_k}, x^{k-1} - x \rangle}{\|A_{i_k}\|^2} A_{i_k} \quad (5)$$

and that the vectors $P_{S_{i_k}}(x^{k-1} - x)$ and γ_{i_k} are orthogonal. These give us the equality (2).

- (ii) First of all, because the application $P_{S_{i_k}}^\omega$ has similar properties with φ_j^α (see (*38)), from Lemma *3 (see (*40)) applied to $P_{S_{i_k}}^\omega$ and $P_{S_{i_k}}$ we get (by also using the fact that the projection $P_{S_{i_k}}$ is an idempotent operator)

$$\begin{aligned} \|x_*^k - x\|^2 &= \|P_{S_{i_k}}^\omega(x^{k-1} - x)\|^2 \\ &= \omega(2 - \omega) \left(\|P_{S_{i_k}}(x^{k-1} - x)\|^2 - \|x^{k-1} - x\|^2 \right) \\ &\quad + \|x^{k-1} - x\|^2 \\ &= \omega(2 - \omega) \langle P_{S_{i_k}}(x^{k-1} - x), x^{k-1} - x \rangle \\ &\quad + (1 - \omega(2 - \omega)) \|x^{k-1} - x\|^2 \\ &= \|x^{k-1} - x\|^2 - \omega(2 - \omega) \frac{\langle A_{i_k}, x^{k-1} - x \rangle^2}{\|A_{i_k}\|^2}. \end{aligned} \quad (6)$$

Then, from (*16), (1), (6) and (2) we successively have

$$\begin{aligned} \|x^k - x\|^2 &= \|(x_*^k - x) + \omega\gamma_{i_k}\|^2 = \|x_*^k - x\|^2 + \omega^2 \|\gamma_{i_k}\|^2 + 2\langle x_*^k - x, \omega\gamma_{i_k} \rangle \\ &= \|x^{k-1} - x\|^2 - \omega(2 - \omega) \frac{\langle A_{i_k}, x^{k-1} - x \rangle^2}{\|A_{i_k}\|^2} + \omega^2 \|\gamma_{i_k}\|^2 \\ &\quad + 2(1 - \omega) \left\langle \frac{\langle A_{i_k}, x^{k-1} - x \rangle}{\|A_{i_k}\|^2} A_{i_k}, \omega\gamma_{i_k} \right\rangle. \end{aligned} \quad (7)$$

Now, for simplifying the presentation, we will introduce the notations

$$D = \frac{\langle A_{i_k}, x^{k-1} - x \rangle}{\|A_{i_k}\|^2} A_{i_k}, \quad B = \omega\gamma_{i_k}, \quad (8)$$

which gives us

$$\|D\|^2 = \frac{\langle A_{i_k}, x^{k-1} - x \rangle^2}{\|A_{i_k}\|^2}, \quad \|B\|^2 = \omega^2 \|\gamma_{i_k}\|^2. \quad (9)$$

By using these notations we compute as follows (see the second and fourth terms in the last sum from (7))

$$\begin{aligned} &- \omega(2 - \omega) \|D\|^2 + 2(1 - \omega) \langle D, B \rangle \\ &= - \left[\left\langle \sqrt{\omega(2 - \omega)} D, \sqrt{\omega(2 - \omega)} D \right\rangle - 2 \left\langle \sqrt{\omega(2 - \omega)} D, \frac{1 - \omega}{\sqrt{\omega(2 - \omega)}} B \right\rangle \right] \\ &\quad + \left[\left\langle \frac{1 - \omega}{\sqrt{\omega(2 - \omega)}} B, \frac{1 - \omega}{\sqrt{\omega(2 - \omega)}} B \right\rangle - \left\langle \frac{1 - \omega}{\sqrt{\omega(2 - \omega)}} B, \frac{1 - \omega}{\sqrt{\omega(2 - \omega)}} B \right\rangle \right] \\ &= - \left\| \sqrt{\omega(2 - \omega)} D - \frac{1 - \omega}{\sqrt{\omega(2 - \omega)}} B \right\|^2 + \frac{(1 - \omega)^2}{\omega(2 - \omega)} \|B\|^2. \end{aligned} \quad (10)$$

Then, (3) follows from (7) and (10).

- (iii) It results directly from (3) and the proof is complete. $\square$

The next Lemma is a special case of Lemma 3.1 in [*2]. The corresponding simplified proof is included for completeness.

Lemma 1 Let $(\alpha_k)_{k \in \mathbb{N}} \in \ell_+$ and $(\beta_k)_{k \in \mathbb{N}} \in \ell_+$ be two nonnegative sequences, and $(\varepsilon_k)_{k \in \mathbb{N}} \in \ell_+ \cap \ell^1$ satisfying

$$\alpha_{k+1} = \alpha_k - \beta_k + \varepsilon_k. \quad (11)$$

Then the following statements hold true.

- (i) $(\beta_k)_{k \in \mathbb{N}} \in \ell^1$. In particular $(\beta_k)_{k \in \mathbb{N}} \in \ell_{c_0}$,
- (ii) $(\alpha_k)_{k \in \mathbb{N}}$ converges.

Proof (i) From (11), we have $\beta_k = \alpha_k - \alpha_{k+1} + \varepsilon_k$. Furthermore,

$$\sum_{k=0}^n \beta_k = \sum_{k=0}^n (\alpha_k - \alpha_{k+1}) + \sum_{k=0}^n \varepsilon_k = \alpha_0 - \alpha_{n+1} + \sum_{k=0}^n \varepsilon_k < \alpha_0 + \sum_{k=0}^n \varepsilon_k,$$

which yields $\sum_{k \in \mathbb{N}} \beta_k < \alpha_0 + \sum_{k \in \mathbb{N}} \varepsilon_k < +\infty$. Hence $(\beta_k)_{k \in \mathbb{N}} \in \ell^1$. Now $\ell^1 \subset \ell_{c_0}$, shows (i).

- (ii) Now, both $(\varepsilon_k)_{k \in \mathbb{N}} \in \ell_{c_0}$ and $(\beta_k)_{k \in \mathbb{N}} \in \ell_{c_0}$. By (11),

$$|\alpha_{k+1} - \alpha_k| = |\varepsilon_k - \beta_k| \leq |\varepsilon_k| + |\beta_k| = \varepsilon_k + \beta_k,$$

with $(\varepsilon_k + \beta_k)_{k \in \mathbb{N}} \in \ell^1$. This shows that $(\alpha_k)_{k \in \mathbb{N}}$ is a Cauchy sequence.¹ Since $(\alpha_k)_{k \in \mathbb{N}} \in \ell_+ \subset \mathbb{R}$ it also converges. $\square$

We are now ready to prove convergence of MREK.

Theorem 1 Let $\alpha, \omega \in (0, 2)$. The sequence $(x^k)_{k \in \mathbb{N}}$ generated by the MREK, algorithm converges to a least-squares solution in $LSS(A; \hat{b})$, for any starting vector $x^0 \in \mathbb{R}^n$.

Proof We split the proof into two parts, showing convergence of $(x^k)_{k \in \mathbb{N}}$, and convergence to a point in $LSS(A; \hat{b})$, respectively.

- (i) Choose any $x \in LSS(A; \hat{b})$ and set

$$\alpha_{k+1} = \|x^k - x\|^2, \quad \varepsilon_k = \frac{\omega^3(2-\omega) + (1-\omega)^2\omega^2}{\omega(2-\omega)} \|\gamma_{i_k}\|^2,$$

$$\beta_k = \left\| \sqrt{\omega(2-\omega)} \frac{\langle A_{i_k}, x^{k-1} \rangle - x}{\|A_{i_k}\|^2} A_{i_k} - \frac{1-\omega}{\sqrt{\omega(2-\omega)}} \omega \gamma_{i_k} \right\|^2. \quad (12)$$

The above Lemma 1 asserts convergence of $(\alpha_k)_{k \in \mathbb{N}}$ and $(\beta_k)_{k \in \mathbb{N}} \in \ell^1$, in view of $\varepsilon_k \in \ell^1$, due to Lemma *2 (ii) and Proposition 1 (ii) respectively. From (*16)

¹An arbitrary sequence $(y^k)_{k \in \mathbb{N}}$ is Cauchy, if $\|y^{k+1} - y^k\| \leq a_k$ holds for all $k \in \mathbb{N}$ and $(a_k)_{k \in \mathbb{N}} \in \ell^1 \cap \ell_+$ arbitrary. Indeed, $\|y^{m+k} - y^m\| = \|\sum_{j=m}^{m+k-1} (y^{j+1} - y^j)\| \leq \sum_{j=m}^{m+k-1} \|y^{j+1} - y^j\| \leq \sum_{j=m}^{m+k-1} a_j = s_{m+k-1} - s_{m-1}$, with $s_n := \sum_{j=1}^n a_j$. Now $(s_n)_{n \in \mathbb{N}}$ is Cauchy since it converges due to $(a_k)_{k \in \mathbb{N}} \in \ell^1$.

and by using again the notations (8)-(9) we get (by also using the elementary inequality $(a + b)^2 \leq 2(a^2 + b^2)$)

$$\begin{aligned} \|x^k - x^{k-1}\|^2 &= \omega^2 \left\| -\frac{\langle A_{i_k}, x^{k-1} \rangle - b_{i_k}}{\|A_{i_k}\|^2} A_{i_k} + \gamma_{i_k} \right\|^2 = \omega^2 \left\| D - \frac{1}{\omega} B \right\|^2 \\ &= \frac{\omega^2}{\omega(2-\omega)} \left\| \sqrt{\omega(2-\omega)} D - \frac{\sqrt{\omega(2-\omega)}}{\omega} B \right\|^2 \\ &= \frac{\omega}{2-\omega} \left\| \left(\sqrt{\omega(2-\omega)} D - \frac{1-\omega}{\sqrt{\omega(2-\omega)}} B \right) + \left(\frac{1-\omega}{\sqrt{\omega(2-\omega)}} - \frac{\sqrt{\omega(2-\omega)}}{\omega} \right) B \right\|^2 \\ &\leq 2 \frac{\omega}{2-\omega} \left(\left\| \sqrt{\omega(2-\omega)} D - \frac{1-\omega}{\sqrt{\omega(2-\omega)}} B \right\|^2 + \frac{1}{\omega(2-\omega)} \|B\|^2 \right) \\ &\leq \frac{2\omega}{2-\omega} \beta_k + \frac{2\omega}{(2-\omega)(\omega + (1-\omega)^2)} \varepsilon_k. \end{aligned} \quad (13)$$

Now $\left(\frac{2\omega}{2-\omega} \beta_k + \frac{2\omega}{(2-\omega)(\omega + (1-\omega)^2)} \varepsilon_k \right)_{k \in \mathbb{N}} \in \ell^1$ implies that $(x^k)_{k \in \mathbb{N}}$ is a Cauchy sequence² and converges as well: assume that $x^k \rightarrow \bar{x}$. In particular, using again (*16), $\|x^k - x^{k-1}\| \rightarrow 0$ and $\gamma_{i_k} \rightarrow 0$ (see e.g. (*33)) we obtain

$$\frac{(\langle A_{i_k}, x^{k-1} \rangle - \hat{b}_{i_k}^k)^2}{\|A_{i_k}\|^2} \rightarrow 0. \quad (14)$$

- (ii) We show that $\bar{x} \in LSS(A, \hat{b})$. Fix any $i \in [m]$. Due to the particular choice of i_k in (*21), we have

$$\begin{aligned} |\langle A_i, x^{k-1} \rangle - b_i| - |r_i - y_i^k| &\leq |\langle A_i, x^{k-1} \rangle - b_i - (r_i - y_i^k)| \\ &= |\langle A_i, x^{k-1} \rangle - b_{i_k}^k| \\ &\stackrel{(*24)}{\leq} |\langle A_{i_k}, x^{k-1} \rangle - b_{i_k}^k|. \end{aligned} \quad (15)$$

Thus $|\langle A_i, x^{k-1} \rangle - b_i| \rightarrow 0$, due to $|r_i - y_i^k| \rightarrow 0$ by Lemma *2 (iii) and (14), respectively. Summarizing, we get $\lim_{k \rightarrow \infty} \|Ax^{k-1} - b\| = 0 = \|\bar{x} - b\|$. Thus, $\bar{x} \in S(A; b) = LSS(A, \hat{b})$. $\square$

The main result concerning convergence of ACEK is stated next.

Theorem 2 *The sequence $(x^k)_{k \in \mathbb{N}}$ generated by ACEK, converges to a least-squares solution in $LSS(A; \hat{b})$, for any starting vector $x^0 \in \mathbb{R}^n$.*

Proof Choose any $x \in LSS(A; \hat{b})$ and set α_{k+1} , β_k , ε_k as in (12). The proof of convergence $x^k \rightarrow \bar{x}$ is identically to the first part of the proof of Theorem 1, with the only difference that we have $(\varepsilon_k)_{k \in \mathbb{N}} \in \ell^1$ due to Lemma *4 (ii). Therefore, it

²Argument as above.

suffices to prove that $\bar{x} \in S(A; b) = LSS(A; \hat{b})$. Let the index i be arbitrary fixed in $[m]$. From (*21) it results that

$$\forall k \geq 0, \exists s(k) \in \{k+1, \dots, k+m_0\} \text{ s.t. } i_{s(k)} = i,$$

hence

$$\langle A_{i_{s(k)}}, x^{s(k)-1} \rangle - b_{i_{s(k)}}^{s(k)} = \langle A_i, x^{s(k)-1} \rangle - b_i^{s(k)} \rightarrow 0. \quad (16)$$

From the inequalities in (15) it results that

$$\langle A_i, x^{s(k)-1} \rangle - b_i \leq \langle A_i, x^{s(k)-1} \rangle - b_i^{s(k)} + |r_i - y_i^{s(k)}|,$$

which together with (16) give us that $\langle A_i, \bar{x} \rangle = b_i$ and completes the proof. $\square$