Lieb–Thirring inequalities for complex finite gap Jacobi matrices

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Abstract We establish Lieb–Thirring power bounds on discrete eigenvalues of Jacobi operators for Schatten class complex perturbations of periodic and more generally finite gap almost periodic Jacobi matrices.

Keywords Finite gap Jacobi matrices · Complex perturbations · Eigenvalues estimates

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1 Introduction

In this paper, we consider bounded non-selfadjoint Jacobi operators on $\ell^2(\mathbb{Z})$ represented by tridiagonal matrices

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$$J = \begin{pmatrix} \ddots & \ddots & \ddots & & & \\ & a_0 & b_1 & c_1 & & \\ & & a_1 & b_2 & c_2 & \\ & & & a_2 & b_3 & c_3 \\ & & & & \ddots & \ddots & \ddots \end{pmatrix} \quad (1.1)$$

with bounded complex parameters $\{a_n, b_n, c_n\}_{n \in \mathbb{Z}}$. Our goal is to obtain Lieb–Thirring inequalities for complex perturbations of periodic and, more generally, almost periodic Jacobi operators with absolutely continuous finite gap spectrum.

Lieb–Thirring inequalities for selfadjoint and complex perturbations of the discrete Laplacian have been studied extensively in the last decade [1, 7, 15, 17, 19, 21]. The original work of Lieb and Thirring [25, 26] was carried out in the context of continuous Schrödinger operators, motivated by their study of the stability of matter. We refer to [6, 9, 10, 12, 13, 23, 31] for more recent developments on Lieb–Thirring inequalities for Schrödinger operators and to [20] for a review and history of the subject.

Much less is known for perturbations (especially complex ones) of operators with gapped spectrum. Lieb–Thirring inequalities for selfadjoint perturbations of periodic and almost periodic Jacobi operators with absolutely continuous finite gap spectrum have been established only recently [4, 5, 11, 22]. Analogs of these finite gap Lieb–Thirring inequalities for complex perturbations are not known. The aim of the present work is to fill this gap. What is currently known in the case of complex perturbations is the closely related class of Kato inequalities [16, 18]. Such inequalities have larger exponents on the eigenvalue side when compared to Lieb–Thirring inequalities [cf. (1.3) vs. (1.2) and (1.6) vs. (1.7)] and hence are not optimal for small perturbations of Jacobi operators.

To put our new results in perspective, we first discuss the best currently known results on eigenvalue power bounds for Jacobi operators in more detail. The spectral theory for perturbations of the *free* Jacobi matrix, J_0 , (i.e., the case of $a_n = c_n \equiv 1$ and $b_n \equiv 0$) is well understood, see [29]. Let $\mathbf{E} = \sigma(J_0) = [-2, 2]$ and suppose J is a selfadjoint Jacobi matrix (i.e., $a_n = c_n > 0$) such that $\delta J = J - J_0$ is a compact operator, that is, J is a compact selfadjoint perturbation of J_0 . Hundertmark and Simon [21] proved the following Lieb–Thirring inequalities,

$$\sum_{\lambda \in \sigma_d(J)} \text{dist}(\lambda, \mathbf{E})^{p-\frac{1}{2}} \leq L_{p, \mathbf{E}} \sum_{n=-\infty}^{\infty} |\delta a_n|^p + |\delta b_n|^p, \quad p \geq 1, \quad (1.2)$$

with some explicit constants $L_{p, \mathbf{E}}$ independent of J . Here, $\sigma_d(J)$ is the discrete spectrum of J . It was also shown in [21] that the inequality is false for $p < 1$.

More recently, (1.2) was extended to selfadjoint perturbations of periodic and almost periodic Jacobi matrices with absolutely continuous finite gap spectrum [4, 5, 11, 22]. When $\mathbf{E}$ is a finite gap set (i.e., a finite union of disjoint, compact intervals), the role of J_0 as a natural background operator is taken over by the so-called isospectral torus, denoted $\mathcal{T}_{\mathbf{E}}$. See, e.g., [2, 3, 27, 29, 30] for a deeper discussion of this object. For $J' \in \mathcal{T}_{\mathbf{E}}$ and a compact selfadjoint perturbation $J = J' + \delta J$, Frank and Simon [11]

proved (1.2) for $p = 1$ while the case of $p > 1$ is established in [4]. The constant $L_{p,E}$ is now independent of J and J' and only depends on p and the underlying set E .

As alluded to above, there is a general result of Kato [24] which applies to compact selfadjoint perturbations of arbitrary bounded selfadjoint operators. Specialized to the case of perturbations of Jacobi matrices from $\mathcal{T}_E$, it states that

$$\sum_{\lambda \in \sigma_d(J)} \text{dist}(\lambda, E)^p \leq \|\delta J\|_p^p \leq \sum_{n=-\infty}^{\infty} (4|\delta a_n| + |\delta b_n|)^p, \quad p \geq 1, \quad (1.3)$$

where $\|\cdot\|_p$ denotes the Schatten norm. In contrast to the Lieb–Thirring bounds, the power on the eigenvalues in (1.3) is the same as on the perturbation and so is larger than the power on the eigenvalues in (1.2) by $1/2$. Kato's inequality is optimal for perturbations with large sup norm. On the other hand, the Lieb–Thirring bound with $p = 1$ is optimal for perturbations with small sup norm (cf. [21]). A fact that seemingly went unnoticed is that one can combine (1.2) and (1.3) into one ultimate inequality which is optimal for both large and small perturbations (at least when $p = 1$). This inequality takes the form

$$\sum_{\lambda \in \sigma_d(J)} \text{dist}(\lambda, E)^{p-\frac{1}{2}} (1 + |\lambda|)^{\frac{1}{2}} \leq C_{p,E} \sum_{n=-\infty}^{\infty} |\delta a_n|^p + |\delta b_n|^p, \quad p \geq 1, \quad (1.4)$$

where the constant $C_{p,E}$ is independent of J and J' , $J = J' + \delta J$, δJ is compact, $J' \in \mathcal{T}_E$, and E is a finite gap set.

In recent years, several results have also been established for non-selfadjoint perturbations of selfadjoint Jacobi matrices [1, 16–19]. For compact non-selfadjoint perturbations $J = J_0 + \delta J$ of the free Jacobi matrix J_0 , a near generalization (with an extra ε) of the Lieb–Thirring bound (1.2) was obtained by Hansmann and Katriel [19] using the complex analytic approach developed in [1]. Their non-selfadjoint version of the Lieb–Thirring inequalities takes the following form: For every $0 < \varepsilon < 1$,

$$\sum_{z \in \sigma_d(J)} \frac{\text{dist}(z, [-2, 2])^{p+\varepsilon}}{|z^2 - 4|^{\frac{1}{2}}} \leq L_{p,\varepsilon} \sum_{n=-\infty}^{\infty} |\delta a_n|^p + |\delta b_n|^p + |\delta c_n|^p, \quad p \geq 1, \quad (1.5)$$

where the eigenvalues are repeated according to their algebraic multiplicity and the constant $L_{p,\varepsilon}$ is independent of J . Whether or not this inequality continues to hold for $\varepsilon = 0$ is an open problem.

For non-selfadjoint perturbations of Jacobi matrices from finite gap isospectral tori $\mathcal{T}_E$, an eigenvalue power bound was first obtained by Golinskii and Kupin in [16]. Shortly thereafter, this bound was superseded by a generalization of Kato's inequality to non-selfadjoint perturbations of arbitrary bounded selfadjoint operators (see Hansmann [18]). In the special case of a compact non-selfadjoint perturbation $J = J' + \delta J$ of $J' \in \mathcal{T}_E$, Hansmann's result reads

$$\sum_{z \in \sigma_d(J)} \text{dist}(z, \mathbf{E})^p \leq K_p \sum_{n=-\infty}^{\infty} |\delta a_n|^p + |\delta b_n|^p + |\delta c_n|^p, \quad p > 1, \quad (1.6)$$

where the eigenvalues are repeated according to their algebraic multiplicity and K_p is a universal constant that depends only on p .

The purpose of the present article is to generalize the Lieb–Thirring bound (1.4) to the case of compact non-selfadjoint perturbations $J = J' + \delta J$ of Jacobi matrices J' from finite gap isospectral tori $\mathcal{T}_{\mathbf{E}}$. Let $\partial \mathbf{E}$ denote the set of endpoints of the intervals that form $\mathbf{E}$. Then, our main result can be formulated as follows: *For every $p \geq 1$ and any $\varepsilon > 0$,*

$$\sum_{z \in \sigma_d(J)} \frac{\text{dist}(z, \mathbf{E})^{p+\varepsilon} (1 + |z|)^{\frac{1-3\varepsilon}{2}}}{\text{dist}(z, \partial \mathbf{E})^{\frac{1}{2}}} \leq L_{\varepsilon, p, \mathbf{E}} \sum_{n=-\infty}^{\infty} |\delta a_n|^p + |\delta b_n|^p + |\delta c_n|^p, \quad (1.7)$$

where the eigenvalues are repeated according to their algebraic multiplicity and the constant $L_{\varepsilon, p, \mathbf{E}}$ is independent of J' and J . We note that for the eigenvalues that accumulate to $\partial \mathbf{E}$, the inequality (1.7) gives a qualitatively better estimate than (1.6). We also point out that (1.7) is new even for perturbations of the free Jacobi matrix J_0 since, unlike (1.5), it is nearly optimal not only for small but also for large perturbations. As with (1.5), it is an open problem whether or not (1.7) remains true for $\varepsilon = 0$.

2 Schatten norm estimates

In this section, we establish the fundamental estimates that are needed to prove our main result, Theorem 3.3. Throughout, $\mathcal{S}_p$ will denote the Schatten class and $\|\cdot\|_p$ the corresponding Schatten norm for $p \geq 1$. To clarify our application of complex interpolation, we occasionally use $\|\cdot\|_{\infty}$ to denote the operator norm.

Theorem 2.1 *Suppose J' is a selfadjoint Jacobi matrix and $D \geq 0$ is a diagonal matrix of Schatten class $\mathcal{S}_p$ for some $p \geq 1$. Denote by $d\rho_n$ the spectral measure of (J', δ_n) , that is, the measure in the Herglotz representation of the n th diagonal entry of $(J' - z)^{-1}$,*

$$\langle \delta_n, (J' - z)^{-1} \delta_n \rangle = \int_{\sigma(J')} \frac{d\rho_n(t)}{t - z}, \quad z \in \mathbb{C} \setminus \sigma(J'). \quad (2.1)$$

Then,

$$\|D^{1/2} (J' - z)^{-1} D^{1/2}\|_p^p \leq \frac{\sqrt{2} \|D\|_p^p}{\text{dist}(z, \sigma(J'))^{p-1}} \sup_{n \in \mathbb{Z}} \int_{\sigma(J')} \frac{d\rho_n(t)}{|t - z|} \quad (2.2)$$

for $z \in \mathbb{C} \setminus \sigma(J')$.

Proof We consider first the case $p = 1$. Let $\{P(t)\}_{t \in \mathbb{R}}$ denote the projection-valued spectral family of the selfadjoint operator J' . Recall that for any measurable and

bounded function f on $\sigma(J')$, the bounded operator $f(J')$ is given by the functional calculus,

$$f(J') = \int_{\sigma(J')} f(t) dP(t). \quad (2.3)$$

Taking $f(t) = 1/(t - z)$ in (2.3), substituting into (2.1), and recalling that the measure in the Herglotz representation is unique yield

$$\langle \delta_n, dP(t) \delta_n \rangle = d\rho_n(t). \quad (2.4)$$

Applying (2.3) to $f(t) = 1/|t - z|$ and using (2.4) then imply

$$\langle \delta_n, |J' - z|^{-1} \delta_n \rangle = \int_{\sigma(J')} \frac{d\rho_n(t)}{|t - z|}, \quad z \in \mathbb{C} \setminus \sigma(J'). \quad (2.5)$$

We also note that if, in addition, the function $f(t)$ in (2.3) is nonnegative, then $f(J')$ is a bounded, selfadjoint, and nonnegative operator.

Fix $z \in \mathbb{C} \setminus \sigma(J')$. In the following, we assume without loss of generality that $\operatorname{Im}(z) \geq 0$. Define the nonnegative functions

$$\begin{aligned} f(t) &= \operatorname{Im}\left(\frac{1}{t - z}\right), \quad f_+(t) = \operatorname{Re}\left(\frac{1}{t - z}\right) \chi_{(\operatorname{Re}(z), \infty)}(t), \\ f_-(t) &= -\operatorname{Re}\left(\frac{1}{t - z}\right) \chi_{(-\infty, \operatorname{Re}(z)]}(t), \end{aligned} \quad (2.6)$$

and note that

$$f_+(t) - f_-(t) + if(t) = \frac{1}{t - z}, \quad (2.7)$$

$$f_+(t) + f_-(t) + f(t) = \left| \operatorname{Re}\left(\frac{1}{t - z}\right) \right| + \left| \operatorname{Im}\left(\frac{1}{t - z}\right) \right| \leq \frac{\sqrt{2}}{|t - z|}, \quad t \in \mathbb{R}. \quad (2.8)$$

Then, we have $f(J') \geq 0$, $f_{\pm}(J') \geq 0$, and

$$(J' - z)^{-1} = f_+(J') - f_-(J') + if(J'), \quad (2.9)$$

$$f_+(J') + f_-(J') + f(J') \leq \sqrt{2} |J' - z|^{-1}. \quad (2.10)$$

Using (2.9), the triangle inequality, and the fact that for nonnegative operators the trace norm coincides with the trace, we obtain the estimate

$$\begin{aligned} &\|D^{1/2} (J' - z)^{-1} D^{1/2}\|_1 \\ &\leq \|D^{1/2} f_+(J') D^{1/2}\|_1 + \|D^{1/2} f_-(J') D^{1/2}\|_1 + \|D^{1/2} f(J') D^{1/2}\|_1 \\ &= \operatorname{tr}(D^{1/2} f_+(J') D^{1/2}) + \operatorname{tr}(D^{1/2} f_-(J') D^{1/2}) + \operatorname{tr}(D^{1/2} f(J') D^{1/2}). \end{aligned} \quad (2.11)$$

Let $D_{n,n}$ denote the diagonal entries of D . Since D is nonnegative and diagonal, we have

$$\sum_{n \in \mathbb{Z}} D_{n,n} = \operatorname{tr}(D) = \|D\|_1. \quad (2.12)$$

Hence, by linearity of the trace it follows from (2.11), (2.10), and (2.5) that

$$\begin{aligned} \|D^{1/2} (J' - z)^{-1} D^{1/2}\|_1 &\leq \operatorname{tr}(D^{1/2} [f_+(J') + f_-(J') + f(J')] D^{1/2}) \\ &\leq \sqrt{2} \operatorname{tr}(D^{1/2} |J' - z|^{-1} D^{1/2}) \\ &= \sqrt{2} \sum_{n \in \mathbb{Z}} D_{n,n} \int_{\sigma(J')} \frac{d\rho_n(t)}{|t - z|} \\ &\leq \sqrt{2} \|D\|_1 \sup_{n \in \mathbb{Z}} \int_{\sigma(J')} \frac{d\rho_n(t)}{|t - z|}. \end{aligned} \quad (2.13)$$

This is exactly the case $p = 1$ in (2.2).

To obtain (2.2) for $p > 1$, we use complex interpolation. Define the map

$$\zeta \mapsto T(\zeta) = D^{\zeta p/2} |J' - z|^{-1} D^{\zeta p/2} \quad (2.14)$$

from the strip $0 \leq \operatorname{Re}(\zeta) \leq 1$ into the space of bounded operators. Then, for any $u, v \in \ell^2(\mathbb{Z})$, the scalar function

$$\zeta \mapsto \langle u, T(\zeta)v \rangle \quad (2.15)$$

is continuous on the strip $0 \leq \operatorname{Re}(\zeta) \leq 1$, analytic in its interior, and bounded. In addition, since $\|D^{iy}\| \leq 1$ and

$$D^{x+iy} = D^{iy} D^x = D^x D^{iy} \quad \text{for all } x \geq 0, \quad y \in \mathbb{R}, \quad (2.16)$$

it follows that

$$\|T(iy)\|_\infty \leq \| |J' - z|^{-1} \| \leq \frac{1}{\operatorname{dist}(z, \sigma(J'))}, \quad y \in \mathbb{R}, \quad (2.17)$$

and by [28, Theorem 2.7] and (2.13),

$$\begin{aligned} \|T(1 + iy)\|_1 &\leq \|D^{p/2} |J' - z|^{-1} D^{p/2}\|_1 \\ &\leq \sqrt{2} \|D^p\|_1 \sup_{n \in \mathbb{Z}} \int_{\sigma(J')} \frac{d\rho_n(t)}{|t - z|}, \quad y \in \mathbb{R}. \end{aligned} \quad (2.18)$$

Thus, by the complex interpolation theorem (see [28, Theorem 2.9], [14, Theorem III.13.1]), we have

$$\|T(x)\|_{1/x} \leq \sup_{y \in \mathbb{R}} \|T(iy)\|_\infty^{1-x} \sup_{y \in \mathbb{R}} \|T(1 + iy)\|_1^x, \quad 0 < x < 1. \quad (2.19)$$

Taking $x = 1/p$, raising both sides to the power p , and noting that $T(1/p) = D^{1/2} (J' - z)^{-1} D^{1/2}$ and $\|D^p\|_1 = \|D\|_p^p$ finally yield (2.2). $\square$

In what follows, $E \subset \mathbb{R}$ will denote a finite gap set, that is,

$$E = \bigcup_{n=1}^N [\alpha_n, \beta_n], \quad \alpha_1 < \beta_1 < \alpha_2 < \cdots < \alpha_N < \beta_N, \quad N \geq 1, \quad (2.20)$$

and ∂E will be the set of endpoints of E , that is,

$$\partial E = \{\alpha_n, \beta_n\}_{n=1}^N. \quad (2.21)$$

For a probability measure $d\rho$ supported on E , we define the associated m -function by

$$m(z) = \int_E \frac{d\rho(t)}{t - z}, \quad z \in \mathbb{C} \setminus E. \quad (2.22)$$

The measure $d\rho$ is called *reflectionless* (on E) if

$$\operatorname{Re}[m(x + i0)] = 0 \quad \text{for a.e. } x \in E. \quad (2.23)$$

Reflectionless measures appear prominently in spectral theory of finite and infinite gap Jacobi matrices (see, e.g., [2, 27, 29, 30]). In particular, the isospectral torus associated with E is the set of all Jacobi matrices J' that are reflectionless on E (i.e., the spectral measure of (J', δ_n) is reflectionless for every $n \in \mathbb{Z}$) and for which $\sigma(J') = E$. It is well known (see for example [30]) that $d\rho$ is a reflectionless probability measure on E if and only if $m(z)$ is of the form

$$m(z) = \frac{-1}{\sqrt{(z - \beta_N)(z - \alpha_1)}} \prod_{j=1}^{N-1} \frac{z - \gamma_j}{\sqrt{(z - \beta_j)(z - \alpha_{j+1})}}, \quad (2.24)$$

for some $\gamma_j \in [\beta_j, \alpha_{j+1}]$, $j = 1, \dots, N - 1$.

We now provide an estimate for the variant of the m -function for $d\rho_n$ that appear in Theorem 2.1.

Theorem 2.2 *Let $E \subset \mathbb{R}$ be a finite gap set and suppose $d\rho$ is a reflectionless probability measure on E . Then, for every $p > 1$,*

$$\int_E \frac{d\rho(t)}{|t - z|^p} \leq \frac{K_{p,E}}{\operatorname{dist}(z, E)^{p-1} \operatorname{dist}(z, \partial E)^{\frac{1}{2}} (1 + |z|)^{\frac{1}{2}}}, \quad z \in \mathbb{C} \setminus E, \quad (2.25)$$

where the constant $K_{p,E}$ is independent of $d\rho$. In addition, for every $\varepsilon > 0$,

$$\int_E \frac{d\rho(t)}{|t - z|} \leq \frac{K_{\varepsilon,E}}{\operatorname{dist}(z, E)^{\varepsilon} \operatorname{dist}(z, \partial E)^{\frac{1}{2}} (1 + |z|)^{\frac{1}{2} - \varepsilon}}, \quad z \in \mathbb{C} \setminus E, \quad (2.26)$$

where the constant $K_{\varepsilon, \mathbf{E}}$ is independent of $d\rho$.

Proof Denote the bands of $\mathbf{E}$ as in (2.20). Since $d\rho$ is reflectionless on a finite gap set, it follows from the Stieltjes inversion formula and (2.24) that $d\rho$ is absolutely continuous with density given by

$$w(t) = \frac{1}{\pi} \operatorname{Im}[m(t + i0)] = \frac{1/\pi}{\sqrt{|t - \beta_N| |t - \alpha_1|}} \prod_{j=1}^{N-1} \frac{|t - \gamma_j|}{\sqrt{|t - \beta_j| |t - \alpha_{j+1}|}} \quad (2.27)$$

for some $\gamma_j \in [\beta_j, \alpha_{j+1}]$, $j = 1, \dots, N-1$. Fix $1 \leq k \leq N$ and rearrange the terms in (2.27) as follows

$$w(t) = \frac{1/\pi}{\sqrt{|t - \alpha_k| |t - \beta_k|}} \prod_{j=1}^{k-1} \frac{|t - \gamma_j|}{\sqrt{|t - \alpha_j| |t - \beta_j|}} \prod_{j=k+1}^N \frac{|t - \gamma_{j-1}|}{\sqrt{|t - \alpha_j| |t - \beta_j|}}. \quad (2.28)$$

Since $\alpha_j < \beta_j \leq \gamma_j \leq \alpha_{j+1} < \beta_{j+1}$ for $j = 1, \dots, N-1$, the two products in (2.28) are at most 1 for every $t \in [\alpha_k, \beta_k]$, and thus,

$$w(t) \leq \frac{1/\pi}{\sqrt{|t - \alpha_k| |t - \beta_k|}}, \quad t \in [\alpha_k, \beta_k]. \quad (2.29)$$

Applying this estimate for the individual bands of $\mathbf{E}$ implies that

$$\int_{\mathbf{E}} \frac{d\rho(t)}{|t - z|^p} \leq \frac{1}{\pi} \sum_{k=1}^N \int_{\alpha_k}^{\beta_k} \frac{1}{|t - z|^p} \frac{dt}{\sqrt{|t - \alpha_k| |t - \beta_k|}}, \quad z \in \mathbb{C} \setminus \mathbf{E}. \quad (2.30)$$

By [19, Lemma 11], for $p > 1$, each integral in the sum can be estimated by

$$\int_{\alpha_k}^{\beta_k} \frac{1}{|t - z|^p} \frac{dt}{\sqrt{|t - \alpha_k| |t - \beta_k|}} \leq \frac{K_p}{\operatorname{dist}(z, [\alpha_k, \beta_k])^{p-1} \sqrt{|z - \alpha_k| |z - \beta_k|}}. \quad (2.31)$$

Since the function $z \mapsto \operatorname{dist}(z, \partial \mathbf{E})(1 + |z|)/|z - \alpha_k| |z - \beta_k|$ is continuous on $\mathbb{C} \setminus \{\alpha_k, \beta_k\}$ and bounded near α_k, β_k , and ∞ , it is bounded on $\mathbb{C} \setminus \mathbf{E}$, and therefore,

$$\int_{\alpha_k}^{\beta_k} \frac{1}{|t - z|^p} \frac{dt}{\sqrt{|t - \alpha_k| |t - \beta_k|}} \leq \frac{K_{p, \mathbf{E}}}{\operatorname{dist}(z, \mathbf{E})^{p-1} \operatorname{dist}(z, \partial \mathbf{E})^{\frac{1}{2}} (1 + |z|)^{\frac{1}{2}}}, \quad p > 1. \quad (2.32)$$

Combining (2.32) with (2.30) yields (2.25).

In order to obtain (2.26), note that since E is a bounded set we have the trivial bound

$$\frac{|t - z|}{1 + |z|} \leq \frac{|t| + |z|}{1 + |z|} \leq K_E, \quad t \in E, \quad z \in \mathbb{C} \setminus E. \quad (2.33)$$

This inequality yields the estimate

$$\begin{aligned} \int_E \frac{d\rho(t)}{|t - z|} &= \int_E \frac{|t - z|^\varepsilon d\rho(t)}{|t - z|^{1+\varepsilon}} \\ &\leq K_E^\varepsilon (1 + |z|)^\varepsilon \int_E \frac{d\rho(t)}{|t - z|^{1+\varepsilon}}, \quad z \in \mathbb{C} \setminus E \end{aligned} \quad (2.34)$$

and hence, (2.26) follows from (2.25). $\square$

3 Lieb–Thirring bounds

We start this section by recalling some results on the distribution of zeros of analytic functions with restricted growth toward the boundary of the domain of analyticity. Let a_+ denote the maximum of a and 0. The following theorem for analytic functions on the unit disk is an alternative form of the extension [19, Theorem 4] of the earlier result [1, Theorem 0.2].

Theorem 3.1 *Let $S \subset \partial\mathbb{D}$ be a finite collection of points and suppose $h(z)$ is an analytic function on $\mathbb{D}$ such that $|h(0)| = 1$ and for some $K, \alpha, \beta, \gamma \geq 0$,*

$$\log |h(z)| \leq \frac{K |z|^\gamma}{(1 - |z|)^\alpha \operatorname{dist}(z, S)^\beta}, \quad z \in \mathbb{D}. \quad (3.1)$$

Then, for every $\varepsilon > 0$, there exists a constant $C_{\alpha, \beta, \gamma, \varepsilon}$ independent of $h(z)$ such that the zeros of $h(z)$ satisfy

$$\sum_{z \in \mathbb{D}, h(z)=0} \frac{(1 - |z|)^{\alpha+1+\varepsilon} \operatorname{dist}(z, S)^{(\beta-1+\varepsilon)_+}}{|z|^{(\gamma-\varepsilon)_+}} \leq C_{\alpha, \beta, \gamma, \varepsilon} K, \quad (3.2)$$

where each zero is repeated according to its multiplicity.

In [16], an analogous result on the distribution of zeros of analytic functions on $\Omega = \mathbb{C} \setminus E$ was obtained via a reduction to the unit disk case. For our purposes, we will need the following extension of [16, Theorem 0.1] where an additional decay assumption at infinity is imposed in exchange for a stronger conclusion. The extension follows from the reduction to the unit disk case developed in [16] combined with the above version (Theorem 3.1) of the unit disk result. We omit the proof as it is a straightforward modification of the one presented in [16].

Theorem 3.2 Let $E \subset \mathbb{R}$ be a finite gap set and suppose $f(z)$ is an analytic function on $\Omega = \overline{\mathbb{C}} \setminus E$ such that $|f(\infty)| = 1$ and for some $K, p, q, r \geq 0$,

$$\log |f(z)| \leq \frac{K}{\text{dist}(z, E)^p \text{dist}(z, \partial E)^q (1 + |z|)^r}, \quad z \in \Omega. \quad (3.3)$$

Then, for every $\varepsilon > 0$, there exists a constant $C_{p,q,r,\varepsilon}$ independent of $f(z)$ such that the zeros of $f(z)$ satisfy

$$\sum_{z \in \Omega, f(z)=0} \text{dist}(z, E)^{p'} \text{dist}(z, \partial E)^{q'} (1 + |z|)^{r'} \leq C_{p,q,r,\varepsilon} K, \quad (3.4)$$

where $p' = p + 1 + \varepsilon$, $q' = \frac{1}{2}[(p + 2q - 1 + \varepsilon)_+ - p']$, $r' = (p + q + r - \varepsilon)_+ - p' - q'$, and each zero is repeated according to its multiplicity.

We are now ready to present our finite gap version of the Lieb–Thirring inequalities for non-selfadjoint perturbations of Jacobi matrices from the isospectral torus $\mathcal{T}_E$.

Theorem 3.3 Let $E \subset \mathbb{R}$ be a finite gap set and suppose J, J' are two-sided Jacobi matrices such that $J' \in \mathcal{T}_E$ and $J = J' + \delta J$ is a compact perturbation of J' . Then, for every $p \geq 1$ and any $\varepsilon > 0$,

$$\sum_{z \in \sigma_d(J)} \frac{\text{dist}(z, E)^{p+\varepsilon} (1 + |z|)^{\frac{1-3\varepsilon}{2}}}{\text{dist}(z, \partial E)^{\frac{1}{2}}} \leq L_{\varepsilon, p, E} \sum_{n=-\infty}^{\infty} |\delta a_n|^p + |\delta b_n|^p + |\delta c_n|^p, \quad (3.5)$$

where the eigenvalues are repeated according to their algebraic multiplicity and the constant $L_{\varepsilon, p, E}$ is independent of J and J' .

Proof Suppose that $\delta J \in \mathcal{S}_p$ for some $p \geq 1$ and define $D \geq 0$ to be the diagonal matrix with the entries

$$D_{n,n} = \max\{|\delta a_{n-1}|, |\delta a_n|, |\delta b_n|, |\delta c_{n-1}|, |\delta c_n|\}, \quad n \in \mathbb{Z}. \quad (3.6)$$

A straightforward verification shows that $\delta J = D^{1/2} B D^{1/2}$, where B is a bounded tridiagonal matrix whose entries lie in the unit disk. This in particular means that $\|B\| \leq 3$. Define

$$f(z) = \det_{[p]}(I + D^{1/2} (J' - z)^{-1} D^{1/2} B), \quad (3.7)$$

where $[p]$ is the smallest integer $\geq p$. This regularized perturbation determinant is analytic on $\Omega = \overline{\mathbb{C}} \setminus E$ (see, e.g., [14, Chapter IV. Sect. 3]). More importantly, the zeros of f coincide with the discrete eigenvalues of J and the multiplicity of the zeros matches the algebraic multiplicity of the corresponding eigenvalues (see [19] and [13, Appendix C] for a proof). By [8, Lemma XI.9.22(d)], we have

$$\log |f(z)| \leq K_p \|D^{1/2} (J' - z)^{-1} D^{1/2} B\|_p^p \leq 3K_p \|D^{1/2} (J' - z)^{-1} D^{1/2}\|_p^p \quad (3.8)$$

for some constant K_p . It thus follows from Theorems 2.1 and 2.2 (with $\varepsilon/2$ instead of ε) that

$$\log |f(z)| \leq \frac{K_{\varepsilon,p,E} \|D\|_p^p}{\text{dist}(z, E)^{p+\frac{\varepsilon}{2}-1} \text{dist}(z, \partial E)^{\frac{1}{2}} (1+|z|)^{\frac{1-\varepsilon}{2}}}, \quad z \in \Omega. \quad (3.9)$$

Applying Theorem 3.2 (with $\varepsilon/2$ instead of ε) and noting that $(1-3\varepsilon)/2 \leq r'$ and $D_{n,n}^p \leq |\delta a_{n-1}|^p + |\delta a_n|^p + |\delta b_n|^p + |\delta c_{n-1}|^p + |\delta c_n|^p$ then yield (3.5). $\square$

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