Deformation Quantization on the Closure of Minimal Coadjoint Orbits

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Abstract. We consider a complex simple Lie algebra $\mathfrak{g}$, with the action of its adjoint group. Among the three canonical nilpotent orbits under this action, the minimal orbit is the non zero orbit of smallest dimension. We are interested in equivariant deformation quantization: we construct $\mathfrak{g}$-invariant star-products on the minimal orbit and on its closure, a singular algebraic variety. We shall make use of Hochschild homology and cohomology, of some results about the invariants of the classical groups, and of some interesting representations of simple Lie algebras. To the minimal orbit is associated a unique, completely prime two-sided ideal of the universal enveloping algebra $U(\mathfrak{g})$. This ideal is primitive and is called the Joseph ideal. We give explicit expressions for the generators of the Joseph ideal and compute the infinitesimal characters.

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Contents

1. Introduction

1.1. GENERAL FRAMEWORK

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$, $\mathfrak{g}'$ its $\mathbb{C}$-vector space dual and G its adjoint Lie group. The group acts smoothly on $\mathfrak{g}'$ and each orbit comes fitted with a natural symplectic structure (see, e.g. [64]). Moreover, the closure of each coadjoint orbit is an algebraic variety of the type

$$\{(x_1, \ldots, x_n) \in \mathbb{C}^n; \ \forall r \in R, \ r(x_1, \ldots, x_n) = 0\},$$

where R is a set of polynomials. We are concerned with the graded coordinate algebra $\mathbb{C}[x_1, \ldots, x_N]/\langle R \rangle$.

As shown by Kirillov, there are interesting relationships between these coadjoint orbits and representation theory; Kostant [59] and Souriau [66] brought in ideas from classical and quantum mechanics and especially from quantization. Meanwhile Gerstenhaber had developed his deformation theory [38], and eventually it was understood [5] that quantization on a symplectic manifold is a deformation of an algebra of functions. One can go further, to regard every such deformation as quantization.

It is curious, we think, that the greater part of recent work on this subject seemed to downplay the special features that coadjoint orbits inherit from the Lie algebra. This paper is a contribution to the study of algebraic invariant star-products, a type of deformation quantization that makes more intimate contact with Lie structure. Invariant star-products were defined in [5], Definition 4.2.1. We shall borrow from the analogy with quantization of mechanical and field theoretic systems. It turns out that the view of quantization introduced by Weyl, intimately tied to Bohr's correspondence principle, is a most efficient one; it supplements cohomological methods precisely where those become inefficient.

The goal of this paper is *not* to prove the existence of star-products, invariant or not. In the context of this paper a star-product on a space S is a map from $S \times S$ to $S[[\hbar]]$, or at least an algorithm for evaluating such a map.

The present paper is a new and mathematically more precise version of the original posting by Christian Frønsdal arXiv-math/0510580v1 [math.RT]. It was edited (following Daniel Sternheimer's request to Claude Roger) with Claude Roger and Frédéric Butin.

1.2. HISTORY OF STAR-PRODUCTS

A first correspondence between quantum mechanical and classical observables was established by Weyl [70] (classical-to-quantum) and Wigner [71] (the other way). The formula for pulling back the non commutative commutator of operators to classical observables is due to Moyal [62]. The resulting deformation of the Poisson bracket, and of the product of functions on phase space, was subjected to mathematical analysis by Vey [68], following the study of one-differentiable deformations in [26]. The papers [5] generalized this notion and proposed a new axiomatic approach to quantization, interpreted as a formal deformation and formulated in terms of a general type of associative star-product.

One considers the space of formal series, in a parameter $\hbar$, of $\mathcal{C}^\infty$ functions on a symplectic manifold W, with an associative product of the form

$$f, \; g \in \mathcal{C}^\infty(W), \;\; f * g = fg + \frac{i\hbar}{2}\{f, \, g\} + \sum_{n=2}^{\infty} \hbar^n C_n(f, \, g),$$

where fg is the ordinary product of functions, $\{\cdot, \cdot\}$ is the Poisson bracket, and the cochains C_n are often taken to be bi-differential operators. There were at least four different developments.

1. The problem of classification of star-products on symplectic spaces, up to a natural but very weak form of equivalence, was investigated by Gutt and others [43]. The existence of star-products on an arbitrary symplectic manifold was established by De Wilde and Lecomte [20] and by Fedosov [21], culminating with the results of Kontsevich who demonstrated the existence of differentiable star-products on an arbitrary Poisson manifold [55,57,67]. The importance of these results is that they are global statements about (smooth) manifolds. Another interesting theorem is due to Cahen et al. [15]: they showed that if $\mathfrak{g}$ is a semi-simple Lie algebra, there is no differential star-product (the coefficients of a differential star product are bidifferential operators) on any neighborhood of the origin in $\mathfrak{g}'$ that is tangential to the coadjoint orbits.

2. A generalization of Weyl's original correspondence is required in field theories, "the ordering problem". There have been applications to mechanical problems [2]. Attention is called to the power of this method in the local algebraic context. From this point of view *the existence* of associative star-products is not surprising, although a concise expression for $f * g$ may be difficult to obtain.

3. The concept of invariant star-products on coadjoint orbits of Lie algebras was proposed in [5]. The existence of invariant star-products on any "regular" coadjoint orbit of a semi-simple Lie algebra was demonstrated in the same paper. This result was obtained by setting up an explicit, equivariant type of Weyl correspondence. The original Weyl correspondence yields, in particular, an invariant star-product for the linear symplectic algebra of the manifold. Invariant star-products are implicit in recent studies of nilpotent orbits, especially those that deal with the Joseph ideal, e.g. [13]. There was an important parallel development in the work of Berezin [8].

4. Invariant star quantization was used as a tool in representation theory, to generalize the method of geometric quantization of Kostant [59] and Souriau [66]. See for example [29,43,69] (and also the book [41] about the Virasoro group for an introduction to geometric quantization). This idea has not yet realized its full potential.

Recently differential and algebraic star-products on regular orbits [24] and on singular orbits [22,25,60] of compact semi-simple Lie groups were studied. These authors study an injective map [23] $(\mathbb{C}[X], *_{\mathrm{alg}}) \hookrightarrow (\mathcal{C}^\infty(X), *_{\mathrm{diff}})$, where X is a coadjoint orbit. In the non compact case, they extend their construction of star-products to singular semi-simple orbits. But the nilpotent singular orbit, that is our main concern here, is not dealt with.

1.3. TERMINOLOGY

We recall the definitions of orbits and make precise the terminology that we use throughout the paper. The base field is $\mathbb{C}$ (unless otherwise stated).

If $\mathfrak{g}$ is a complex semi-simple Lie algebra and G a connected semi-simple Lie group with Lie algebra $\mathfrak{g}$, the *coadjoint orbits* are the orbits in the dual $\mathfrak{g}'$ under the adjoint action of G. A coadjoint orbit is said to be *semi-simple* (resp. *nilpotent*) if it is the orbit of a semi-simple (resp. nilpotent) element of $\mathfrak{g}'$. We will see in Section 5.1 that, unlike the set of semi-simple orbits, the set of nilpotent orbits is finite and contains three canonical orbits, the *principal* (or *regular*) orbit, the *subregular* orbit and the *minimal* orbit. In this article, we focus our attention on the minimal orbit, denoted $\mathcal{O}_{\min}$. Coadjoint orbits are (smooth) differentiable manifolds. The Zariski closure of the minimal nilpotent orbit is the disjoint union of this orbit and the unique zero dimensional orbit (the origin); it is a singular algebraic variety. In this paper we construct $\mathfrak{g}$-invariant star-products on the Zariski closure $\overline{\mathcal{O}_{\min}}$ of $\mathcal{O}_{\min}$. Since the minimal orbit is determined by its closure, we shall for simplicity often call $\overline{\mathcal{O}_{\min}}$ "the minimal orbit" instead of "the Zariski closure of the minimal orbit". More generally, the closure $\overline{\mathcal{O}}$ of a *non-generic* orbit $\mathcal{O}$ (a generic orbit is an orbit of maximal dimension) is a singular algebraic variety that is stratified in several orbits of different dimensions.

1.4. DEFORMATIONS AND HOCHSCHILD COHOMOLOGY

Since Gerstenhaber [38], it has been clear that the deformations of an algebra of functions on a smooth manifold are classified by its Hochschild cohomology, which is known (for Lie algebras at least since the 70s [26,27,68] and in fact since [47]) to be isomorphic to the space of multivector fields on the manifold. Less well known is the role played by the BGS (Barr–Gerstenhaber–Schack) decomposition of the Hochschild complex. In particular the non existence of (non trivial) abelian deformations in the case of smooth manifolds follows from the vanishing of the Harrison component of Hochschild cohomology. Algebraic varieties offer more room for deformations. The lifting of a Poisson structure to a star-product is governed by components of cohomology that are purely local, associated with the singularities. This strongly suggests that global Poisson structures on varieties more general than smooth manifolds lift to global star-products [33,56]. But the results reported here are local.

1.5. RESULTS OF THE ARTICLE

This paper is intended as a preliminary study of the algebraic deformations of the coordinate algebras of some algebraic varieties with singularities, a context in which the BGS decomposition [3,4,28,36,40,46] can be expected to have some interesting applications. The coadjoint orbits of simple Lie groups are related by closure to a family of algebraic varieties, and the coordinate algebras of these varieties do in fact have interesting deformations. In that setting our work focuses on invariant star-products to the exclusion of all others.

A familiar reduction paradigm was used to reduce the cohomology to a complex of closed, linear chains. It had been expected that this would lead to an easy

classification of essential deformations. It turned out, however, to be difficult to obtain enough information about the homology. Before us, experts [13], applying much heavier machinery [7,9] to the same problem, had the same experience and were forced to fall back on heuristic arguments. See below, and in Section 6.1.

Returning to the point of view that sees a star-product as a correspondence between ordinary polynomials and star polynomials, we were able to complete the calculations.

The main results of the paper are the following:

(a) The Hochschild (co)homology of the graded coordinate algebra of an algebraic variety defined by quadratic relations is isomorphic to that of the restriction to linear, closed chains. Obstructions to extending a first order star-product (an infinitesimal deformation) to a formal, associative star-product, to all orders in the deformation parameter, can therefore be reduced to a study of the star-products $x * x$ and $x * x * x$ for x of degree 1.

(b) The minimal, nilpotent, coadjoint orbit of a complex, simple Lie algebra $\mathfrak{g}$ different from $\mathfrak{sl}_n$ admits a one-parameter family of invariant star-products/ deformations. In the case of $\mathfrak{sl}_n$ we determine a two-parameter family of deformations, including an interesting one-parameter subfamily of abelian deformations. In all the other cases there is a unique, invariant star-product such that

$$\forall u, v \in \mathfrak{g}', \ u * v - v * u = \hbar\{u, v\}, \quad \text{and} \quad u * u = uu + k,$$

for some unique $k \in \mathbb{C}[[\hbar]]$; $\{u, v\}$ is the Poisson bracket, see Section 6. The value of k is determined by an examination of the next case,

$$u * u * u = uuu + \phi(u),$$

where ϕ is a polynomial of degree one. Both k and ϕ are uniquely determined by the relations that define the orbit. A uniform calculation covers the five exceptional algebras.

For $\mathfrak{sl}_n$ and the other classical simple Lie algebras we calculate the generators of the Joseph ideals and determine the associated highest weight modules. The uniqueness of the Joseph ideal is a corollary; see [35].

1.6. SOME RECENT RELATED WORKS

Recently, after arXiv-math/0510580v1 [math.RT] was posted, some reprints appeared that have some relations with it.

In [42], Gukov and Witten study the quantization of a symplectic manifold in the framework of brane theory. They use two-dimensional sigma-models and are mainly interested in the case where the symplectic manifold M admits a complexification Y that is an affine variety, i. e. the set of zeros of a finite subset of

$\mathbb{C}[x_1, \ldots, x_n]$. An important point is the existence of a flat Spin_c structure on M. As a consequence of that, they notice in particular that neither their method, nor the one of Brylinski [14], is applicable in the case of the minimal nilpotent orbit of $\mathfrak{so}(p, q)$ with $p + q$ odd and $p, q \geq 4$. But the results obtained in the present paper for $sl_n(\mathbb{C})$ can be applied to that case.

Another interesting study is that of Halbout, Oudom and Tang [44,45]. Their point of view in [45] is different of ours, because they consider a symplectic manifold M and compute the Gerstenhaber bracket of the semi-direct product $\mathcal{C}^\infty(M) \rtimes \Gamma$, where Γ is a finite group acting on M. This allows them to classify the noncommutative Poisson structures on $\mathcal{C}^\infty(M) \rtimes \Gamma$. But there is a link with our study. In fact, in the case of minimal singular nilpotent orbits (which we study in this article), if a manifold M is a finite dimensional covering of a singular orbit, the orbit can be represented by a quotient of this covering by a discrete group. As for [44], it is the continuation of [45] in the case of generalized quadratic relations in $T(V) \rtimes \Gamma$, where $T(V)$ is the tensor algebra of a vector space V.

1.7. OUTLINE OF THE PAPER

Section 2 is an introduction to formal star-products and the BGS decomposition of the Hochschild complex; see [3,4,28,36,40] and [46].

Section 3 is a study of the Hochschild cohomology of algebraic varieties defined by sets of quadratic relations. A main tool used here is a reduction of the Hochschild complex to a subcomplex of closed, linear chains; see Theorem 3.4. The method is effective when the underlying algebra is finitely generated and graded, as it is when the relations are quadratic (homogeneous), with only positive degrees. The BGS decomposition is used throughout.

Section 4 is a brief introduction to invariant star-products on coadjoint orbits of a semi-simple Lie algebra. Section 4.3 gives an example of the appearance of finite representations within the program of star quantization. Included here is the first example (known to us) of harmonic polynomials in the enveloping algebra of a simple Lie algebra. We show an "instance of a deformation" (not a formal deformation) in which the deformed variety (the spectrum of the deformation of the ring of coordinate functions) is a finite union of disconnected varieties.

In Section 5 we study the minimal nilpotent orbits of a semi-simple Lie algebra. We recall some important results about the structure of these orbits. In Section 5.2 we also give the definition of the Joseph ideals.[1] Then we describe some very special, "unitarizable" representations that are associated with these orbits, and with the Joseph ideals. These representations play a conspicuous role in physics.

Section 6 examines invariant star-products on the most interesting nilpotent coadjoint orbits, those of minimal dimension, with their Joseph ideals. Attempting to calculate the cohomology, we encounter a difficulty that had already been met

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[1]We thank Birne Binegar for information about Joseph ideals.

by Braverman and Joseph [13], and fail to obtain a sufficiently detailed description of the third homology space HH_3 of the coordinate algebra. For the solution of this problem we offer only Conjecture 6.1, but we circumvent the difficulty by an independent, direct calculation. It is done by regarding an invariant star-product as a correspondence principle, in the spirit of Weyl's symmetric ordering. Detailed knowledge of the cohomology of the restricted complex is not needed. Generators of the Joseph ideals are determined.

The Lie algebras $\mathfrak{sl}_n$ and $\mathfrak{so}_n$ are treated separately and all the calculations are included, with some proofs relegated to an Appendix. The case of $\mathfrak{sp}_{2n}$ is too well known to warrant much attention. The five exceptional simple Lie algebras are handled uniformly together, all the calculations are in the main text, they are both short and easy, because of the small number of invariants of these Lie algebras.

Within a family of generally non commutative, invariant star-products there may be a subfamily of non trivial abelian ones. According to Braverman and Joseph, this would contradict the fact that the minimal orbits—excepting the case of $\mathfrak{sl}_n$—are rigid. Granted that a deformation of the ring of coordinate functions implies a deformation of its spectrum. But is it known that the deformed spectrum of an abelian deformation is always an algebraic variety?[2] Is every equivariant deformation of a coadjoint orbit a coadjoint orbit? In the event, we confirm that abelian deformations of the coordinate algebra exist only in the case of $\mathfrak{sl}_n$, and that in that case the spectrum can be identified with a neighboring orbit of the same dimension.

2. Star-products and Hochschild (Co)Homology

2.1. HOCHSCHILD (CO)HOMOLOGY AND DEFORMATIONS

• Given an associative and commutative $\mathbb{C}$-algebra, denoted by A, the homological *reduced* Hochschild chain complex of A is

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[2]We are grateful to Dmitry Kaledin for the following instructive remarks on that question. Given an algebraic variety X and a one-parameter commutative deformation $\mathcal{O}_\hbar$ of the structure sheaf $\mathcal{O}_X$, one can ask whether it is true that there exists an algebraic variety Y containing X such that sections of $\mathcal{O}_\hbar$ are formal germs of functions on Y near X. The answer to this depends on what one understands by an "algebraic variety". If one just means "scheme", and X is affine, then the answer is *yes* for trivial reasons (take $Y = \operatorname{Spec} H_0(X, \mathcal{O}_\hbar)$). However, usually one wants varieties to be "nice"—for example, schemes of finite type (algebraically, finitely generated algebras). If this is what one wants from Y, then the problem is non-trivial; in some cases the answer is still *yes*, but this requires heavy machinery such as Grothendieck's algebraization theorem for formal schemes. Furthermore, in reference to results of Braverman–Joseph on rigidity of nilpotent orbits, it is reasonable to understand by "variety" something to which rigidity applies. And then, there is another context where there are no problems: one can replace a (closed) orbit X by its completion at 0 and treat it as a Noetherian local formal scheme (algebraically, this corresponds to working with m-adically complete local algebras (A, m) such that the associated graded quotient $\operatorname{gr} A = \bigoplus_i (m/m^2)^i$ is finitely generated). Then Y also trivially exists as a Noetherian formal local scheme, and the rigidity still applies.

$$\cdots \xrightarrow{d_5} C_4(A) \xrightarrow{d_4} C_3(A) \xrightarrow{d_3} C_2(A) \xrightarrow{d_2} C_1(A),$$

where the p-chains are defined by $C_p(A) = A^{\otimes p}$. The differential $d_\bullet = \bigoplus_{p=1}^\infty d_p$ is given by $d_1(a) = 0$ and for $p > 1$

$$d_p(a_1 \otimes a_2 \otimes \cdots \otimes a_p) = a_1 a_2 \otimes a_3 \otimes \cdots \otimes a_p +$$
$$+ \sum_{i=2}^{p-1} (-1)^{i+1} a_1 \otimes a_2 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_p.$$

Then we define the Hochschild homology of A by

$$Z_p(A) = \text{Ker } d_p, \quad B_p(A) = \text{Im } d_{p+1} \quad \text{and} \quad HH_p(A) = Z_p(A)/B_p(A).$$

- Similarly, the cohomological Hochschild complex of A is the following

$$C^0(A) \xrightarrow{d^{(0)}} C^1(A) \xrightarrow{d^{(1)}} C^2(A) \xrightarrow{d^{(2)}} C^3(A) \xrightarrow{d^{(3)}} C^4(A) \xrightarrow{d^{(4)}} \cdots,$$

where the space $C^p(A)$ of p-cochains is, for $p \in N^*$, the space of $\mathbb{C}$-linear maps from $A^{\otimes p}$ to A, and $C^0 = A$. The differential $d^\bullet = \bigoplus_{p=0}^\infty d^{(p)}$ is given by

$$\forall C \in C^p(A), \ d^{(p)} C(a_0, \ldots, a_p) = a_0 C(a_1, \ldots, a_p) - C(d_{p+1}(a_0 \otimes \cdots \otimes a_p)) +$$
$$+ (-1)^{p-1} C(a_0, \ldots, a_{p-1}) a_p.$$

We may write it in terms of Gerstenhaber bracket[3] $[\cdot, \cdot]_G$ and the product μ of A, as follows:

$$d^{(p)} C = (-1)^{p+1} [\mu, C]_G.$$

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[3]Recall that for $F \in C^p(A)$ and $H \in C^q(A)$, the Gerstenhaber product is the element $F \bullet H \in C^{p+q-1}(A)$ defined by

$$F \bullet H(a_1, \ldots, a_{p+q-1}) = \sum_{i=0}^{p-1} (-1)^{i(q+1)} F(a_1, \ldots, a_i, H(a_{i+1}, \ldots, a_{i+q}), a_{i+q+1}, \ldots, a_{p+q-1}).$$

It is not associative, but the associator $\text{Ass}(F, H, K) := (F \bullet H) \bullet K - F \bullet (H \bullet K)$ verifies the relation

$$\text{Ass}(F, H, K) = (-1)^{(q-1)(r-1)} \text{Ass}(F, K, H).$$

Therefore, the Gerstenhaber bracket, defined by $[F, H]_G := F \bullet H - (-1)^{(p-1)(q-1)} H \bullet F$, is antisymmetric,

$$[F, H]_G = -(-1)^{(p-1)(q-1)} [H, F]_G,$$

and satisfies the graded Jacobi identity

$$[F, [H, K]_G]_G + (-1)^{(p-1)(q+r-2)} [H, [K, F]_G]_G + (-1)^{(r-1)(p+q-2)} [K, [F, H]_G]_G = 0.$$

See for example [37], and [16, p. 38].

Then we define the Hochschild cohomology of A by $HH^0(A) = \mathrm{Ker}\; d^{(0)} = A$ and

$$\forall p \in \mathbb{N}^*, \quad Z^p(A) = \mathrm{Ker}\; d^{(p)}, \quad \mathsf{B}^p(A) = \mathrm{Im}\; d^{(p-1)}, \quad \text{and}$$
$$HH^p(A) = Z^p(A)/\mathsf{B}^p(A).$$

This paper makes use of a pairing between homology and cohomology that requires that we deal with the restriction of A to a non-unital subalgebra A_+. (Section 3.1.)

We denote by $\mathbb{C}[[\hbar]]$ (resp. $A[[\hbar]]$) the algebra of formal power series in the parameter $\hbar$, with coefficients in $\mathbb{C}$ (resp. A). A star-product of the algebra A is defined as a map $*$ from $A[[\hbar]] \times A[[\hbar]]$ to $A[[\hbar]]$ that is $\mathbb{C}[[\hbar]]$-bilinear and such that, $\forall k \in \mathbb{C},\, f \in A$,

$$k * f = f * k = f,$$

and $\forall f, g, h \in A$,

$$f * g \equiv fg \mod \hbar A[[\hbar]], \quad f * (g * h) = (f * g) * h.$$

This means that there exists a sequence of bilinear maps[4] C_j from $A \times A$ to A of which the first term C_0 is the product of A and such that

$$\forall a, b \in A, \quad a * b = \sum_{j=0}^{\infty} C_j(a, b)\hbar^j, \tag{1}$$

$$\forall n \in \mathbb{N}, \quad \sum_{i+j=n} C_i(a, C_j(b, c)) =$$
$$= \sum_{i+j=n} C_i(C_j(a, b), c), \quad \text{that is} \quad \sum_{i+j=n} [C_i, C_j]_G = 0. \tag{2}$$

We call this a star-product of order p if the Equation (2) is satisfied (only) for $n \le p$.

A first order star-product is a product

$$f * g = fg + \hbar C_1(f, g) \tag{3}$$

associative to first order in $\hbar$, which makes C_1 a closed Hochschild cochain, namely

$$d^{(2)}C_1(f, g, h) := f C_1(g, h) - C_1(fg, h) + C_1(f, gh) - C_1(f, g)h = 0.$$

If C_1 is exact, that is, if there is a 1-cochain such that

$$C_1(f, g) = d^{(1)}E(f, g),$$

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[4]Often, in the definition of a star-product (see for example [16]), the bilinear maps C_j's are bidifferential operators. Here, we do not make this assumption.

then, to first order in $\hbar$, Equation 3 can be written

$$(f - \hbar E(f)) * (g - \hbar E(g)) = fg - \hbar E(fg).$$

Essential first order deformations are classified by $HH^2(A)$ (see [37]).

If $m = \sum_{j=0}^{p} C_j \hbar^j$ is a star-product of order p, then we may extend m to a star-product of order $p+1$ if and only if there exists C_{p+1} such that $\forall a, b, c \in A$,

$$\sum_{i=1}^{p} \left(C_i(C_{p+1-i}(a, b), c) - C_i(a, C_{p+1-i}(b, c)) \right) = d^{(2)} C_{p+1}(a, b, c),$$

$$\sum_{i=1}^{p} [C_i, C_{p+1-i}]_G = 2 d^{(2)} C_{p+1}. \tag{4}$$

The left hand side is closed; this minor miracle is responsible for the fact that commutative algebras are not isolated in the family of associative algebras. (For this easy but important fact, see for example [5,16,38]). *So $HH^3(A)$ contains the obstructions to extend a deformation of order p to a deformation of order $p+1$.*

2.2. THE BGS DECOMPOSITION OF HOCHSCHILD (CO)HOMOLOGY

In this section, we explain how the Hochschild cohomological complex splits into a finite or infinite sum of direct summands. This decomposition is based on the action of the symmetric group $\mathfrak{S}_n$ on n-cochains, and on the existence of idempotents. In the case where the algebra is generated by N generators, there are only N nonzero summands. The complete decomposition of the Hochschild cohomology of a commutative algebra was found by Gerstenhaber and Schack [40], after the pioneering work of Harrison [46] and Barr [3,4,39] (See also [61]).

Consider the rational group algebra $\mathbb{Q}(\mathfrak{S}_n)$, and let $L(\mathsf{C}^{(n)}(A))$ be the algebra of linear maps from $\mathsf{C}^{(n)}(A)$ to $\mathsf{C}^{(n)}(A)$. There is a natural injective morphism

$$\mathbb{Q}(\mathfrak{S}_n) \hookrightarrow L(\mathsf{C}^{(n)}(A))$$

that allows us to identify the elements of $\mathbb{Q}(\mathfrak{S}_n)$ with linear maps from $\mathsf{C}^{(n)}(A)$ to $\mathsf{C}^{(n)}(A)$. The following result is the key point for the decomposition of the Hochschild cohomology.

THEOREM 2.1. (Barr–Gerstenhaber–Schack) *For each $n \geq 2$, there exist n elements $e_n(1), \ldots, e_n(n)$ of $\mathbb{Q}(\mathfrak{S}_n)$ that verify the following properties*:

(1) $e_n(k)^2 = e_n(k)$,
(2) $e_n(k)e_n(l) = 0$ for $k \neq l$ *(the $e_n(k)$'s are mutually orthogonal)*,
(3) $\sum_{k=1}^{n} e_n(k) = 1$,
(4) $d^{(n)} \circ e_n(k) = e_{n+1}(k) \circ d^{(n)}$.

The elements $e_n(k)$ defined above are called BGS idempotents or Eulerian idempotents. It is interesting to note that these elements are some Lagrange interpolation polynomials. See [4] and [40] for the proof of Theorem 2.1 and for more details about the BGS idempotents.

There are thus decompositions

$$\mathsf{C}^n = \bigoplus_{k=1}^n \mathsf{C}^{n,k}, \quad \mathsf{Z}^n = \bigoplus_{k=1}^n \mathsf{Z}^{n,k}, \quad \mathsf{B}^n = \bigoplus_{k=1}^n \mathsf{B}^{n,k}$$

of the spaces of n-cochains, n-cocycles and n-coboundaries, and we have the BGS decomposition

$$HH^n = \bigoplus_{k=1}^n HH^{n,k}$$

with $Harr^n := HH^{n,1}$.

A generating function for the idempotents was found by Garsia [36]:

$$\sum_{k=1}^n z^k e_n(k) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} (z - \delta_\sigma)(z - \delta_\sigma + 1) \cdots (z - \delta_\sigma + n - 1)(-1)^\sigma \sigma,$$

where $(-1)^\sigma$ is the sign of σ and δ_σ is the number of descents of σ, $\delta_\sigma = |\{(i, i+1); \ \sigma(i) > \sigma(i+1)\}|$. For example, the permutation (1342) has two descents, from 1 to 2 and from 3 to 4.

The simplest idempotents are

$$e_2(1)12 = \frac{1}{2}(12 + 21),$$

$$e_2(2)12 = \frac{1}{2}(12 - 21),$$

$$e_3(1)123 = \frac{1}{6}(2(123 - 321) + 132 - 231 + 213 - 312),$$

$$e_3(2)123 = \frac{1}{2}(123 + 321),$$

$$e_n(n) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} (-1)^\sigma \sigma.$$

The spaces of Hochschild n-chains, n-cycles and n-boundaries decompose in the same way, with

$$d_n \circ e_n(k) = e_{n-1}(k) \circ d_n,$$

and

$$\mathsf{C}_n = \bigoplus_{k=1}^n \mathsf{C}_{n,k}, \quad \mathsf{Z}_n = \bigoplus_{k=1}^n \mathsf{Z}_{n,k}, \quad \mathsf{B}_n = \bigoplus_{k=1}^n \mathsf{B}_{n,k},$$

so that

$$HH_n = \bigoplus_{k=1}^{n} HH_{n,k},$$

with $Harr_n := HH_{n,1}$.

2.3. APPLICATION OF THE BGS DECOMPOSITION TO STAR-PRODUCTS

According to the preceding section, every 2-cochain C has a decomposition

$$C = e_2(1)(C) + e_2(2)(C) = C^+ + C^- \quad \text{with} \quad C^+ \in \mathsf{C}^{2,1} \quad \text{and} \quad C^- \in \mathsf{C}^{2,2};$$

that is, C^+ (resp. C^-) is a symmetric (resp. skew-symmetric) bilinear map.

Associativity of the star-product Equation (1) to first order is the requirement that both 2-cochains be closed, $d^{(2)}C^+ = d^{(2)}C^- = 0$. A first order deformation of the algebra is inessential if both forms are exact. In the case of a smooth manifold, $Harr^2$ is empty and choosing $C_1^+ = 0$ entails no essential loss.

Returning now to Equation (4), we have mappings

$$(5)$$

$$
\begin{array}{l}
\mathsf{Z}^{2,1} \times \mathsf{Z}^{2,1} \\
\qquad\qquad\searrow \\
\qquad\qquad\quad \mathsf{Z}^{3,1} \xleftarrow{\ d^{(2)}\ } \mathsf{Z}^{2,1} \\
\qquad\qquad\nearrow \\
\mathsf{Z}^{2,2} \times \mathsf{Z}^{2,2} \\
\qquad\qquad\searrow \\
\qquad\qquad\quad \mathsf{Z}^{3,3} \\
\\
\mathsf{Z}^{2,1} \times \mathsf{Z}^{2,2} \\
\qquad\qquad\searrow \\
\qquad\qquad\quad \mathsf{Z}^{3,2} \xleftarrow{\ d^{(2)}\ } \mathsf{Z}^{2,2} \\
\qquad\qquad\nearrow \\
\mathsf{Z}^{2,2} \times \mathsf{Z}^{2,1}
\end{array}
$$

where the first column of arrows represents the construction on the left hand side of Equation (4) (with the Gerstenhaber bracket) and the second column of arrows is the mapping by the differential $d^{(2)}$.

Because $\mathsf{B}^{3,3}$ is empty, the obstruction in $\mathsf{Z}^{3,3} = HH^{3,3}$ demands that the skew-symmetric part of the left hand side of Equation (4) vanish. That is the Jacobi identity, satisfied if C_1^- is a Poisson bracket. The first line shows that the obstruction to abelian deformations is the Harrison component $Harr^3 = HH^{3,1} \subset HH^3$.

In the case of smooth manifolds $HH^n = HH^{n,n} = 0$, and abelian deformations are inessential, according to the HKR Theorem. The familiar deformations with $C_1 = C_1^-$ encounter no additional obstructions to second order because, if $C_1^+ = 0$, the brackets of the first column belong to $\mathsf{Z}^{3,1}$ and $\mathsf{Z}^{3,3}$. But $HH^{3,3} = 0$, so the only obstruction is $Harr^3$.

We now turn to a preliminary investigation of algebraic varieties with singularities. We shall find varieties for which $\mathsf{Z}^{3,1}$, $\mathsf{Z}^{3,2}$ and $\mathsf{Z}^{3,1}$ are all non empty.

3. Hochschild (Co)Homology of Some Varieties with Singularities

3.1. CASE OF QUADRATIC RELATIONS: THE CONIC VARIETIES

We call conic variety an algebraic variety of the type $\mathbb{C}^N/R$, where $R = \{g_\alpha\}_{\alpha=1,2,\dots}$ is a set of homogeneous, quadratic polynomials,

$$g_\alpha = \sum_{i,j=1}^{N} g_\alpha^{ij} x_i x_j.$$

Let A be the graded coordinate algebra, $A = \mathbb{C}[x_1, \dots, x_N]/\langle R \rangle$, and A_+ the subalgebra of elements of *positive degree* (so we shall use the reduced Hochschild complex, called here for simplicity the Hochschild complex). The restriction to positive degrees is essential: in fact, cochains on A_+ extend naturally to A, but the homology of A_+ is richer than that of A (the homology of A is zero). In this section, we make use of some results of [30].

DEFINITION 3.1. A Hochschild chain $a = a_1 \otimes a_2 \otimes \cdots \otimes a_p$ will be said to be *linear* if a_k is of degree 1 for every $k \in \{1, \dots, p\}$.

The *restricted homological Hochschild complex* is the restriction of the homological Hochschild complex of A to closed, linear chains. We denote by $\widetilde{C}_p(A)$ the space of p-chains of the restricted complex.

The differential of this complex is zero, so that the homology space of degree p is $\widetilde{C}_p(A)$ itself.

Remark 3.2. In the case of a smooth manifold, the chains of the restricted complex are the skew-symmetric ones; the next theorem reduces in that case to the well-known result of Hochschild, Kostant and Rosenberg [47]. For other generalizations see [32,33], and Section 3.3.

DEFINITION 3.3. Given two complexes, a quasi-isomorphism between them is a morphism of complexes that induces an isomorphism in cohomology.

THEOREM 3.4. *The Hochschild complex of A is quasi-isomorphic to the* (co)*homological restricted complex.*

Proof. (see also [30].)

The restriction of a closed/exact form is closed/exact. Conversely, every closed/exact restricted cochain is the restriction of a closed/exact Hochschild cochain. It is enough to consider homogeneous chains; that is, $a = a_1 \otimes \cdots \otimes a_n$ such that each factor a_k, $k = 1, \dots, n$ is of well defined degree. (The only grading that we use is the total polynomial degree.) To show that every restricted (=closed, linear) n-cochain extends to a closed, Hochschild n-cochain we consider the formula

$$d^{(n)}C(a_1, \ldots, a_{n+1}) = a_1 C(a^1) - C(d_n a) + (-1)^{n+1} C(a^{n+1}) a_{n+1}, \tag{6}$$

where $a^1 = a_2 \otimes \cdots \otimes a_{n+1}$ and $a^{n+1} = a_1 \otimes \cdots \otimes a_n$. Evidently the degree of a is higher than the degrees of a^1 and a^{n+1}. This formula can therefore be used to try to extend closedness, recursively to higher degrees. The obstruction is $d_n a = 0$, but it is easy to verify that the remaining terms in Equation (6) vanish when a is exact and C is closed on lower degrees. The obstruction comes from homology; if the restricted n-cochain C has the property that $d^{(n)} C(h) = 0$ for a representative h of every homology class of $(n+1)$-chains, then it extends to a closed, Hochschild n-cochain. $\qquad\square$

In the absence of singularities the restricted cochains are alternating and the cochains are defined on alternating chains only. These n-cochains are closed and not homologous to zero because the differential does not connect the projections $e_n(n)$ and $e_{(n+1)}(n+1)$. That is, the differential is zero. In the presence of singularities there are additional, homologically non trivial chains, and the closedness of a cochain, even if alternating, has to be tested on them. There is a net loss of alternating cohomology and non zero cohomology in the other BGS sectors.

3.2. CASE OF ONE QUADRATIC RELATION: THE SIMPLE CONE

In this section, we study a particular case of conic variety. We retain all the definitions but we suppose that $R = \{g\}$ is just one quadratic polynomial. In this case the following holds.

PROPOSITION 3.5. *Every closed chain is homologous to a linear chain and no linear chain is exact. The space Z_{2k+l} of closed, linear $(2k+l)$-chains is spanned by the following $(2k+l)$-chains, with $Z_{2k+l,k+l} \in Z_{2k+1,k+l}$:*

$$\left(Z_{2k+l,k+l}\right)_{m_1 \ldots m_l} = g^{i_1 j_1} \ldots g^{i_k j_k} \sum_\sigma (-1)^\sigma \sigma (x_{i_1} \otimes \cdots \otimes x_{j_k} \otimes x_{m_1} \otimes \cdots \otimes x_{m_l}),$$

$$\forall m_1, \ldots, m_l = 1, \ldots, N,$$

where the sum is over all permutations of the set $\{i_1, \ldots, i_k, j_1, \ldots, j_k, m_1, \ldots, m_l\}$ that preserve the internal order of each pair $(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)$.

EXAMPLE 3.6.

$$(Z_{1,1})_i = x_i, \quad i = 1 \ldots N,$$

$$(Z_{2,2})_{ij} = x_i \wedge x_j,$$

$$Z_{2,1} = g^{ij} x_i \otimes x_j,$$

$$(Z_{3,3})_{ijk} = x_i \wedge x_j \wedge x_k,$$

$$(Z_{3,2})_k = g^{ij} (x_i \otimes x_j \otimes x_k - x_i \otimes x_k \otimes x_j + x_k \otimes x_i \otimes x_j),$$

$$(Z_{4,4})_{ijkl} = x_i \wedge x_j \wedge x_k \wedge x_l,$$

$$(Z_{4,3})_{kl} = g^{ij}(x_i \otimes x_j \otimes x_k \otimes x_l + x_k \otimes x_i \otimes x_j \otimes x_l + x_i \otimes x_l \otimes x_j \otimes x_k +$$
$$+ x_k \otimes x_l \otimes x_i \otimes x_j + x_i \otimes x_k \otimes x_l \otimes x_j + x_l \otimes x_i \otimes x_k \otimes x_j - (k \leftrightarrow l)),$$

$$(Z_{4,2}) = g^{ij}g^{kl}(x_i \otimes x_k \otimes x_l \otimes x_j - x_i \otimes x_k \otimes x_j \otimes x_l + x_i \otimes x_j \otimes x_k \otimes x_l).$$

From this point onwards, the notation $\mathsf{Z}_{k,l}$, $\mathsf{B}_{k,l}$, $\mathsf{Z}^{k,l}$, $\mathsf{B}^{k,l}$ stands for spaces defined with reference to the restricted complex. The dimension of $\mathsf{Z}_{2k+l,k+l} \subset HH_{2k+l,k+l}$ is $\binom{l}{N}$ or 0. We note that $\mathsf{Z}_{3,1}$ is empty. To lowest order, a star-product is determined by the 2-cochain C_1, and up to equivalence by the restricted 2-cochains; that is, by their values $C_1^-(x_i \wedge x_j)$ and $g^{ij}C_1^+(x_i, x_j)$ on the homological basis. The differential $d^{(2)}C_1^+$ is in $\mathsf{B}^{3,1}$, and since $\mathsf{Z}^{3,1}$ is empty, every restricted, symmetric 2-cochain is closed. The differential $d^{(2)}C_1^-$ is in $\mathsf{B}^{3,2}$; C_1^- is closed if and only if $d^{(2)}C_1^-$ vanishes on $\mathsf{Z}_{3,2}$,

$$d^{(2)}C_1^-(Z_{3,2}) = 4g^{ij}x_i C_1^-(x_j, x_k) = 0.$$

This will be interpreted as the statement that the "hamiltonian" vector field $C_1^-(x_k, \cdot)$ is tangential to the constraint surface.

The addition of an exact form $d^{(2)}E$ to C_1 (we like to think of it as a 'gauge transformation') does not affect C_1^- but it adds $2g^{ij}x_i E(x_j)$ to $C_1^+(Z_{2,1}) = g^{ij}C_1^+(x_i, x_j)$.

EXAMPLE 3.7. Suppose that the 2-form g is non degenerate; then C_1^+ is fixed up to equivalence by its degree zero term (in the case of regular functions on the closure, by its value $c = g^{ij}C_1^+(x_i, x_j)|_{x=0}$ at $x_1 = x_2 = \cdots = 0$). That is, $HH^{2,2}$ is the space of tangential vector fields on the cone and $HH^{2,1} = \mathbb{C}$. (See the end of Section 3.2 for equivariant cohomology.)

We examine the obstructions to extending the star-product to all orders in $\hbar$, referring to Equations (4) and (5).

The emptiness of $\mathsf{B}^{3,3}$ is an obstruction that must be circumvented by imposing the Jacobi identity on C_1^-. Because C_1^- is closed this entails that it extends to a unique Poisson bracket on A. Recall that, if $(a, b) \mapsto \{a, b\}$ is a Poisson bracket on A then for every $f \in A$, the mapping $f^\sharp : A \to A$ defined by $a \mapsto \{f, a\}$ is a derivation. The projection of Equation (4) on $\mathsf{Z}_{2,3}$ is now solved by taking $C_n^-(x_i, x_j) = 0$, $\forall i, j = 1 \ldots N$ and $\forall n \geq 2$. This choice is implicit in the context of differentiable deformations, and it is one of the axioms of invariant star-products. We do not investigate alternatives.

● **Summary:** *The projection of Equation* (5) *on* $\mathsf{Z}_{3,3}$ *leads to*

(1) $C_1^-(x_i, x_j) = \{x_i, x_j\}$ *extends to a Poisson bracket on A,*
(2) $\{x_i, g(x, x)\} = 0$ *is the condition that C_1 be closed,*

(3) $C_n^-(x_i, x_j) = 0 \ \forall i, \ j = 1 \dots N$ *and* $\forall n \geq 2$.

The projection of Equation (4) on $Z_{3,2}$ takes the form, for $k = 1, \dots, N$,

$$\sum_{m+n=p+1} \left[-4g^{ij}C_m^+(C_n^-(x_i, x_k), x_j) + 2g^{ij}C_m^-(C_n^+(x_i, x_j), x_k) \right] =$$
$$= d^{(2)}C_{p+1}^-(Z_{3,2})_k = 4g^{ij}x_iC_{p+1}^-(x_j, x_k). \tag{7}$$

If g is non degenerate, then we can restrict the value of $C_1^+(x_i, x_j)$ to $\mathbb{C}$; see Example 3.7. The obstruction is then the term of degree zero on the left hand side. In view of Summary (3) above, Equation (7) simplifies to

$$2g^{ij}C_p^+(\{x_i, x_k\}, x_j) - g^{ij}\{C_p^+(x_i, x_j), x_k\} = 0. \tag{8}$$

EXAMPLE 3.8. If $\{x_i, x_k\} = \varepsilon_{ik}^m x_m$, the coefficients ε_{ik}^m being the structure constants of a simple Lie algebra $\mathfrak{g}$, then Equation (8) is satisfied when C_p^+ is the Killing form of $\mathfrak{g}$.

EXAMPLE 3.9. In the case of $\mathfrak{sl}_n$, there is a well-known equivariant 3-tensor $f : A \otimes A \to A$ and $C_p^+ = f$ also solves Equation (8). Explicitly, we have $f(a \otimes b) = ab + ba - \frac{2}{n}tr(ab)I_n$.

EXAMPLE 3.10. Choose coordinates such that $g(x, x) = x_N^2 - \rho(x)$, where $\rho(x)$ is a polynomial in $x_1, \dots, x_N$, at most linear in x_N. A regular function on the cone, being the restriction of a polynomial in $x_1, \dots, x_N$, has a unique decomposition of the form $f = f_1 + x_N f_2$, where f_1, f_2 are polynomials in $x_1, \dots, x_{N-1}$. Define ([31])

$$f * g = fg + \hbar f_2 g_2 = fg|_{x_N^2 = \rho(x) + \hbar}. \tag{9}$$

Then the bilinear map $*$ is a star-product; it can be interpreted as the ordinary product of functions on the hyperboloid $g(x, x) = \hbar$. This deformation, in which C_1 is symmetric, can be followed by another deformation in which C_1 is skew-symmetric, leading to a Poisson bracket such that the vector fields $\{x_i, \cdot\}$; are tangent to the hyperboloid $g(x, x) = \hbar$; no Harrison cohomology intervenes in either stage. When both deformations are combined we note that, with C_1^+ as we have defined it in Equation (9), $C_1^+(f, g) = f_2 g_2$, the only contribution to the left hand side of Equation (8) at $x = 0$ comes from the linear term in $\{x_i, x_k\}$. Closure of C_m^- implies that the vector field $\{\cdot, x_k\}$ is tangent to $g(x, x) = \hbar$.

• **Equivariant cohomology:** in the context of Lie algebras and invariant star-products all maps will be equivariant. This affects the question of exactness, as in Example 3.7, and will be taken into account as the occasion arises. Theorem 3.4 is not affected.

3.3. CASE OF ONE POLYNOMIAL RELATION

Let $A = \mathbb{C}[x_1, \ldots, x_N]/\langle g \rangle$, where g is a polynomial without constant or linear terms.

Grading: we can no longer restrict our attention to homogeneous chains. By an appropriate linear transformation of variables we can bring the polynomial g to the form $x_1^t + h(x_1, \ldots, x_N)$, where h is a polynomial of degree less than t in x_1.

A *normalized* polynomial is one that is of degree less than t in x_1; to each regular function on $\mathbb{C}/g$ there is just one normalized polynomial and the degree of a regular function is defined to be the degree of this normalized polynomial. The normalized polynomial is obtained by a Euclidean division in $(\mathbb{C}[x_2, \ldots, x_N])[x_1]$. Let g be a normalized polynomial in $x_1, \ldots, x_n$, without a constant term and without linear terms. Choose a presentation

$$g = \sum_{a,\,b=1}^{K} g^{ab} y_a y_b, \ \text{ with } g^{ab} = g^{ba} \in \mathbb{C}, \ a, b = 1 \ldots K,$$

where for each $a = 1 \ldots K$, the polynomial factor y_a has no constant term, and let A be the filtered algebra $A = \mathbf{C}[x_1, \ldots, x_N]/\langle g \rangle$ with A_+ the sum of the positive degrees of A. With each element of A_+ is associated a unique, normalized polynomial.

Let Z_{2k+l} denote the space of closed p-chains, spanned by the following p-chains, for $p = 2k + l = 1, 2 \ldots$, and $m_1, \ldots, m_l = 1, \ldots, N$,

$$(Z_{2k+l,k+l})_{m_1 \ldots m_l} = g^{a_1 b_1} \ldots g^{a_k b_k} \sum_\sigma (-)^\sigma \sigma (y_{a_1} \otimes \cdots \otimes y_{b_k} \otimes x_{m_1} \otimes \cdots \otimes x_{m_l}),$$

where the sum is over all permutations that preserve the internal order of each pair $(a_1, b_1), \ldots, (a_k, b_k)$.

THEOREM 3.11. *The restriction of the Hochschild complex to such chains is a quasi-homomorphism; that is, the cohomologies are isomorphic.*

The proof of this theorem is as the proof of Theorem 3.4, but it needs the following lemma. Let A_+ be the filtered algebra without unit as above. The degree of an A_+-chain $a_1 \otimes \cdots \otimes a_p$ is the sum of the degrees of its factors.

LEMMA 3.12. *If an A_+ p-chain a is exact, then there is a $(p+1)$-chain b, with the same degree as a, such that $a = d_{p+1} b$. See* [33].

4. Introduction to g-Invariant Star-Products on Coadjoint Orbits

4.1. BACKGROUND

The origin of the problem is as follows (see [1]). Let W be a symplectic space with Poisson tensor Λ and consider an action by $\mathfrak{so}_3$, generated by Hamiltonian vector fields $\Lambda(dL_i)$, $i=1\ldots3$ where L_1, L_2, $L_3 \in \mathcal{C}^\infty(W)$ satisfy the following Poisson bracket relations,

$$\{L_i, L_j\} = \epsilon_{ijk} L_k.$$

The Casimir element

$$Q := \sum_{i=1}^3 L_i^2 \in \mathcal{C}^\infty(W)$$

is an invariant of the adjoint action,

$$\{L_i, Q\} = 0, \quad i = 1\ldots3.$$

The Hamiltonian vector fields leave invariant each surface $\{Q = k\}$ ($k \in \mathbb{C}$), and each such surface is a symplectic leaf with an induced Poisson structure.

The problem is to invent an equivariant ordering (an invariant star-product $*$) such that the star-polynomial $\mathcal{W}(Q) := \sum_{i=1}^3 L_i * L_i$ is *central*,

$$L_i * \mathcal{W}(Q) = \mathcal{W}(Q) * L_i,$$

and *fixed*,

$$\exists q \in \mathbb{C}; \ \forall f \in \mathcal{C}^\infty(W), \ \mathcal{W}(Q) * f = qf.$$

4.2. DEFINITION OF g-INVARIANT STAR-PRODUCTS

Let G be a Lie group, G_e the connected component of G containing the neutral element, $\mathfrak{g}$ the Lie algebra of G, and $\mathfrak{g}'$ the dual vector space. Given an orbit W of the coadjoint action of G_e in $\mathfrak{g}'$ we have a homomorphism from the symmetric algebra $S(\mathfrak{g})$ into $\mathcal{C}^\infty(W)$. We identify $\mathfrak{g}$ with $\mathfrak{g}'$ as a vector space. Then there is a natural Poisson structure $\{\cdot, \cdot\}$ on W (Berezin–Kirillov–Kostant–Souriau Poisson structure scaled by $\hbar$), such that

$$\forall a, b \in \mathfrak{g}, \ \hbar\{a, b\} = [a, b].$$

DEFINITION 4.1. A star-product on a coadjoint orbit W is g-invariant if, for all $k \in \mathbb{C}$, $a, b \in \mathfrak{g}$, $f, g \in \mathcal{C}^\infty(W)$, we have the three following properties

(1) $k * a = a * k = ka$,
(2) $a * b - b * a = \hbar\{a, b\}$,
(3) $\{a, f * g\} = \{a, f\} * g + f * \{a, g\}$.

Throughout, we are interested in algebraic star-products, star-products that are defined (only) on S($\mathfrak{g}$), the set of polynomial functions on $\mathfrak{g}'$.

Remark 4.2. Coadjoint orbits provide a plethora of symplectic manifolds, but to invoke the assistance of a Lie group *for that purpose alone* is somewhat odd. It seems more natural, in this context, to investigate star-products that incorporate additional elements of group theoretical structure.

Given an associative star-product $*$ on W, a linear map $\mathcal{W}$ from the symmetric algebra into $\mathcal{C}^\infty(W)[[\hbar]]$ is defined as follows,

$$S(\mathfrak{g}) \to \mathcal{C}^\infty(W)[[\hbar]], \quad a^n \mapsto \mathcal{W}(a^n) := (a*)^n, \quad a \in \mathfrak{g}.$$

For any polynomial $P(a)$ we write $P(a, *)$ for $\mathcal{W}(P(a))$. Conversely, any invertible linear map $\mathcal{W}$ that associates an element of $\mathcal{C}^\infty(W)$ to each symmetrized formal star monomial defines an associative star-product on $\mathcal{C}^\infty(W)$.

The original Moyal product is the unique associative, invariant product for $W = \mathbb{R}^{2N}$ with the standard Poisson bracket such that $\mathcal{W}(a^n) = a^n$ for every a that is linear in the natural coordinates. It is invariant under the Lie algebra of affine symplectic transformations. The domain includes the space of regular functions (the space of polynomials in the generators).

A recipe for the construction of all invariant star-products for any compact, semi-simple Lie algebra, on any regular coadjoint orbit, was formulated 30 years ago (see [5,29]). For non regular orbits of a compact group see [60].

DEFINITION 4.3. A star-product on W is non degenerate if the space of star-polynomials (actually, the image by $\mathcal{W}$) is dense in the space $\mathcal{C}^\infty(W)$.

DEFINITION 4.4. ([53, p. 185], [54, p. 368]) Let π be an irreducible representation of the universal enveloping algebra $U(\mathfrak{g})$, and let $Z(U(\mathfrak{g}))$ be the center of $U(\mathfrak{g})$. According to Schur's lemma, for every $X \in \mathfrak{g}$, we have $\pi(X) = \lambda_\pi(X)\mathbf{1}$, and the map

$$Z(U(\mathfrak{g})) \to \mathbb{C}, \quad X \mapsto \lambda_\pi(X)$$

is a morphism of algebras that is called the infinitesimal character of π.

DEFINITION 4.5. [54, p. 155] A regular (or generic) coadjoint orbit is a coadjoint orbit of maximal dimension (we will see in Section 5.1 that there exists a unique regular nilpotent orbit).

4.3. EXAMPLES RELATED TO $\mathfrak{so}_n$

The following theorem describes star-products in the case of a regular coadjoint orbit for $\mathfrak{so}_n$.

THEOREM 4.6. ([5] § 11 and [6])

- *Let $\mathfrak{g} := \mathfrak{so}_n$. Let $P_0 := 1$, $P_1(a) := a$, $\{P_2, P_3, \ldots\}$ be a complete set of homogeneous, irreducible, harmonic elements of the projection of $S(\mathfrak{g}')$ on a regular coadjoint orbit W, and $\{P_n(a, *)/n \in \mathbb{N}\}$ the corresponding set of symmetric star-polynomials. Then an associative, non degenerate, invariant star-product on W is given by an infinitesimal character $Z(U(\mathfrak{g})) \to \mathbb{C}$ and by the formulas*

$$P_n(a, *) := C_n P_n(a), \ C_n \in \mathbb{C} - \{0\}, \ C_0 = C_1 = 1,$$
$$a * b - b * a = \hbar[a, b], \ a, b \in \ \mathfrak{g}, \ n = 0, 1.$$
(10)

- *In the case when $\mathfrak{g} = \mathfrak{so}_3$, $P_n(a)$ is a solid Legendre polynomial and the polynomials $P_n(a, *)$ can be obtained from the recursion relation (found and solved in [6])*

$$(n+1)P_{n+1}(a, *) = (2n+1)a * P_n(a, *) - n\left(q + \frac{1 - n^2}{4}\hbar^2\right)|a|^2 P_{n-1}(a, *),$$

with $P_0(a, *) = 1$. The parameter q is the image of the Casimir operator $\sum_{i=1}^{3} L_i * L_i$ by the infinitesimal character.

Some finite-dimensional representations of $\mathfrak{g}$

An invariant star-product gives an action of $\mathfrak{g}$ on the star algebra, and on $\mathcal{C}^\infty(W)$, by the homomorphisms $\pi_l : a \mapsto a*$ and similarly by $\pi_r : a \mapsto *(-a)$, defined by

$$\pi_l(a)f = a * f \quad \text{and} \quad \pi_r(a)f = -f * a, \ \forall f \in \mathcal{C}^\infty(W).$$

Questions of domains of these left and right multiplicative operations have not yet been adequately discussed, to our knowledge.

In the case of compact Lie algebras we expect to find finite dimensional representations. How this actually comes about can be seen explicitly in the case of $\mathfrak{g} = \mathfrak{so}_3$.

PROPOSITION 4.7. *Let $\mathfrak{g} = \mathfrak{so}_3$. An invariant, associative star-product is of one of two types, both defined as in Equation (10).*

(1) *All $C_n \neq 0$; the action π_l or π_r generated by $a*$ or $*(-a)$, $a \in \mathfrak{g}$ is not semisimple. In this case the choice $C_n = 1 + o(\hbar)$ for $n > 1$ provides an equivariant deformation for every value q of the Casimir $\sum_{i=1}^{3} L_i * L_i$: we have a one-parameter family of star-products.*

(2) *If the image of the Casimir by the infinitesimal character takes the value $q = l(l + 1)$, with $2l \in \mathbb{N}$, then the algebra of star polynomials contains an ideal generated by $P_{2l+1}(a, *)$. The quotient is a finite dimensional $*$-algebra and is spanned by the projection of $\{P_n(a, *)/n = 0 \ldots 2l\}$. The action of $a*$ and $*(-a)$ is equivalent to the direct product of two copies of the irreducible representation of $\mathfrak{su}_2$, each with dimension $2l + 1$. The polynomial $P_{2l+1}(a, *)$ reduces in this case to*

$$P_{2l+1}(a, *) \propto \prod_{m=-l}^{l} (a * -m|a|), \quad a = \sum_{i=1}^{3} a_i L_i, \quad |a| := \sqrt{\sum_{i=1}^{3} a_i^2},$$

and every $P_n(a, *)$ with $n > 2l + 1$ contains $P_{2l+1}(a, *)$ as a factor.

The spectrum, the space of maximal ideals, has a finite number of disconnected components.

5. Minimal Orbits, Representations, and Joseph Ideals

5.1. ADJOINT AND COADJOINT ORBITS

In this section, we recall some results about adjoint and coadjoint orbits. See [17] for more details (and [54] for examples and other properties of coadjoint orbits). Let $\mathfrak{g}$ be a complex semi-simple Lie algebra, and G a connected semi-simple complex Lie group with Lie algebra $\mathfrak{g}$.

DEFINITION 5.1. • Let $X \in \mathfrak{g}$. We say that X is semi-simple (resp. nilpotent) if X is a semi-simple (resp. nilpotent) endomorphism of $\mathfrak{g}$. The adjoint orbit of X is the set $\mathcal{O}_X := \{Ad_g(X); g \in G\}$. This is said to be semi-simple (resp. nilpotent) if X is semi-simple (resp. nilpotent).
• Let $\varphi \in \mathfrak{g}'$ and set $\mathfrak{g}^\varphi := \{X \in \mathfrak{g}/ad_X^* \varphi = 0\}$. It is the Lie algebra of the isotropy group of φ for the coadjoint action. We say that φ is semi-simple (resp. nilpotent) if $\mathfrak{g}^\varphi$ is reductive in $\mathfrak{g}$ (resp. $\varphi|_{\mathfrak{g}^\varphi} = 0$). We denote by $\mathcal{S}$ (resp. $\mathcal{N}$) the set of semi-simple (resp. nilpotent) elements of $\mathfrak{g}'$. The coadjoint orbit of φ is the set $\mathcal{O}_\varphi := \{Ad_g^*(\varphi); \ g \in G\}$. It is said to be semi-simple (resp. nilpotent) if φ is semi-simple (resp. nilpotent).

For $\varphi \in \mathfrak{g}'$, set

$$\omega_\varphi : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}, \quad (X, Y) \mapsto \varphi([X, Y]).$$

Then ω_φ is a symplectic form on $\mathfrak{g}/\mathfrak{g}^\varphi \simeq T_\varphi \mathcal{O}_\varphi$, and ω_φ extends to a complex symplectic structure on $\mathcal{O}_\varphi$. See for example [64] for a proof of this result.

Since $\mathfrak{g}$ is semi-simple, the Killing form κ is non degenerate. Hence the isomorphism of $\mathfrak{g}$-spaces

$$i_\kappa : \mathfrak{g} \to \mathfrak{g}', \quad X \mapsto (Y \mapsto \kappa(X, Y)).$$

Then we have, because of the invariance of κ,

$$i_\kappa(\mathcal{O}_X) = \mathcal{O}_{i_\kappa(X)}.$$

Therefore there exists a bijective correspondence between the adjoint and the coadjoint orbits that preserves the properties of semi-simplicity and nilpotency.

In the sequel, we will sometimes identify adjoint and coadjoint orbits.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and W its Weyl group. Then we have the following classification of semi-simple orbits.

THEOREM 5.2. *The set of semi-simple orbits is in bijective correspondence with the set $\mathfrak{h}/W$. Thus this set is infinite.*

On the contrary, the set $\mathcal{NILP}$ of nilpotent orbits is finite. It is in bijection with the set of weighted Dynkin diagrams of $\mathfrak{g}$. The classification results from theorems of Jacobson–Morozov, Kostant, and Mal'cev. See [17] for these theorems and their proof. Moreover, there exists a partial order on the set $\mathcal{NILP}$, defined by

$$\forall \mathcal{O}, \mathcal{O}' \in \mathcal{NILP}, \ \left(\mathcal{O} \preceq \mathcal{O}' \Leftrightarrow \overline{\mathcal{O}} \subseteq \overline{\mathcal{O}'} \right),$$

where $\overline{\mathcal{O}}$ is the Zariski closure of $\mathcal{O}$.

The following theorems define the three canonical nilpotent orbits, the principal, the subregular, and the minimal orbits. We recall that the aim of this article is to study star-products on the minimal orbit. Here, $\mathfrak{g}$ is again assumed to be a complex semi-simple Lie algebra.

THEOREM 5.3. (Siebenthal, Dynkin, Kostant) *There exists a unique nilpotent orbit of maximal dimension, equal to $\dim \mathfrak{g} - \operatorname{rank} \mathfrak{g}$. This orbit, denoted by $\mathcal{O}_{\mathrm{princ}}$, is called the principal (or regular) orbit and is dense in $\mathcal{N}$.*

THEOREM 5.4. (Steinberg) *There exists a unique nilpotent orbit of dimension $\dim \mathfrak{g} - \operatorname{rank} \mathfrak{g} - 2$. This orbit, denoted by $\mathcal{O}_{\mathrm{subreg}}$, is called the subregular orbit and is dense in $\mathcal{N} \setminus \mathcal{O}_{\mathrm{princ}}$.*

THEOREM 5.5. *There exists a unique nonzero orbit of minimal dimension. This orbit, denoted by $\mathcal{O}_{\mathrm{min}}$, is called the minimal orbit and is contained in the closure of any nilpotent orbit. We have $\overline{\mathcal{O}_{\mathrm{min}}} = \mathcal{O}_{\mathrm{min}} \sqcup \{0\}$.*
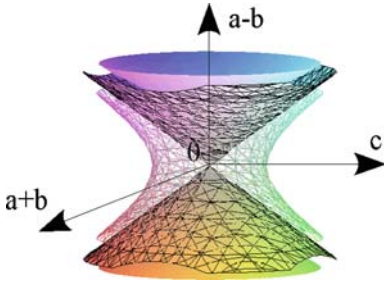
Figure 1. Coadjoint orbits of $\mathfrak{so}(2,1)$.

EXAMPLE 5.6. • The smallest $n \in \mathbb{N}^*$ for which the three canonical nilpotent orbits of $\mathfrak{sl}_n$ are distinct is $n = 4$. In this case, the nilpotent orbits of $\mathfrak{sl}_4$ are given in the following table.

Orbit	$\mathcal{O}_{[4]} = \mathcal{O}_{\text{princ}}$	$\mathcal{O}_{[3,1]} = \mathcal{O}_{\text{subreg}}$	$\mathcal{O}_{[2,2]}$	$\mathcal{O}_{[2,1,1]} = \mathcal{O}_{\text{min}}$	$\mathcal{O}_{[1,1,1,1]} = \{0\}$
Dimension	12	10	8	6	0

• Consider $\mathfrak{sl}_2 = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \ / \ (a, b, c) \in \mathbb{C}^3 \right\}$.

Then the semi-simple orbits are the sets $\{0\}$ and $\{(a, b, c) \in \mathbb{C}^3 / a^2 + bc = k\}$, for $k \in \mathbb{C}^*$.

$$\mathcal{O}_{\text{princ}} = \mathcal{O}_{\text{subreg}} = \mathcal{O}_{\text{min}} = \{(a, b, c) \in \mathbb{C}^3 / a^2 + bc = 0\} \setminus \{0\}$$

(cone without 0). For the convenience of the reader we represent here in Figure 1 the orbits in $\mathbb{R}^3$, i.e. the coadjoint orbits of the *real* Lie algebra $\mathfrak{so}(2, 1)$ considered in Section 5.3.

5.2. NILPOTENT ORBITS AND JOSEPH IDEALS

Let $\mathfrak{g}$ be a complex finite dimensional semi-simple Lie algebra, V a finite dimensional vector space, and $\pi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ an irreducible representation of $\mathfrak{g}$. This induces a unique representation $\widehat{\pi}$ of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. The two-sided ideal $I := \text{Ker}\,\widehat{\pi}$ is called the *primitive* ideal associated with π.
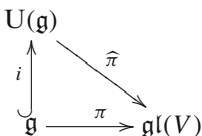
According to the Poincaré–Birkhoff–Witt theorem [18], there is an isomorphism of graded algebras $\mathrm{Gr}(\mathrm{U}(\mathfrak{g})) \simeq \mathrm{S}(\mathfrak{g})$, where $\mathrm{S}(\mathfrak{g})$ is the symmetric algebra of $\mathfrak{g}$, the algebra of polynomial functions on $\mathfrak{g}'$. And $J := \mathrm{Gr}(\mathrm{Ker}\,\widehat{\pi})$ is a graded ideal of $\mathrm{S}(\mathfrak{g})$. Let $\mathcal{V}(I)$ be the variety of common zeros of J in $\mathfrak{g}'$.

The two following theorems link the primitive ideals with the nilpotent orbits and allow us to define the Joseph ideal associated with the minimal orbit.

THEOREM 5.7. (Borho, Brylinski, Joseph [11,52]) *Let I be a primitive ideal of* U(𝔤). *Then $\mathcal{V}(I)$ is the closure $\overline{\mathcal{O}}$ of a nilpotent orbit $\mathcal{O}$ of $\mathfrak{g}$.*

But every coadjoint nilpotent orbit is determined by its closure, therefore we may say that $\mathcal{O}$ is *the* nilpotent orbit associated with the primitive ideal I.

THEOREM 5.8. (Joseph [49,50]) *Let $\mathfrak{g}$ be a complex simple Lie algebra, and $\mathcal{O}_{\min}$ its minimal orbit. Then there exists a unique completely prime two-sided ideal $J_{\min}$ in* U(𝔤) *such that $\mathcal{V}(J_{\min}) = \mathcal{O}_{\min} \sqcup \{0\}$. This ideal is primitive and called the Joseph ideal.*

The root space of the *highest root* of $\mathfrak{g}$ is contained in the minimal orbit. More precisely, let β be the highest root of $\mathfrak{g}$ (see [12,48]), and $\mathfrak{g}_\beta$ the associated root space,

$$\mathfrak{g}_\beta := \{X \in \mathfrak{g}; \ \forall h \in \mathfrak{h}, \ [h, X] = \beta(h)X\}.$$

Then we have $\mathfrak{g}_\beta \subset i_\kappa^{-1}(\mathcal{O}_{\min})$ (see [50]).

The following result, due to Kostant [58], is an important tool for our study of minimal orbits.

THEOREM 5.9. (Kostant) *The closure of the minimal orbit $(\mathcal{O}_{\min} \sqcup \{0\})$ is an algebraic variety*

$$\overline{\mathcal{O}_{\min}} = \{(x_1, \ldots, x_N) \in \mathbb{C}^N; \ \forall r \in R, \ r(x_1, \ldots, x_N) = 0\},$$

where N is the dimension of $\mathfrak{g}$ and $R := \{g_1, \ldots, g_K\}$ is a finite set of quadratic relations.

The set of regular functions on $\overline{\mathcal{O}_{\min}}$ is the set

$$\mathcal{F}\left(\overline{\mathcal{O}_{\min}}\right) \simeq \mathbb{C}[x_1, \ldots, x_N]/\langle R \rangle.$$

In the sequel, we will often call $\overline{\mathcal{O}_{\min}}$ the minimal orbit. Similarly, we will often talk about relations defining the minimal orbit, even if these relations define of course the *closure* of the orbit.

The last theorem of this section links the primitive ideals to the highest weight modules. See [19] and [51].

THEOREM 5.10. (Duflo, Joseph) *Every primitive ideal of* $U(\mathfrak{g})$ *is the annihilator of a simple highest weight module.*

5.3. COADJOINT ORBITS OF $\mathfrak{so}(2, 1)$

Consider the real Lie algebra $\mathfrak{so}(2, 1)$, and let (L_1, L_2, L_3) be its standard basis, with relations

$$[L_i, L_j] = \epsilon_{ijk} L_k$$

and Killing form κ such that, setting $L := a_1 L_1 + a_2 L_2 + a_3 L_3$, we have

$$\kappa(L, L) = g^{ij} a_i a_j = -(a_1)^2 - (a_2)^2 + (a_3)^2.$$

The moment map interprets L_1, L_2, L_3 as coordinates for the coadjoint space; in this role we write them $\mathbf{x} := (x_1, x_2, x_3)$. Set $g(\mathbf{x}) := g^{ij} x_i x_j$. Then the semi-simple orbits are the (real) algebraic varieties

$$\{\mathbf{x} \in \mathbb{R}^3; \ g(\mathbf{x}) = k\},$$

for $k \neq 0$, and the closure of the minimal nilpotent orbit is the (real) algebraic variety

$$\overline{\mathcal{O}_{\min}} = \{\mathbf{x} \in \mathbb{R}^3; \ g(\mathbf{x}) = 0\}.$$

This variety is a simple cone in the sense of Section 3.2.

The regular functions on $\overline{\mathcal{O}_{\min}}$ are the restrictions to $\overline{\mathcal{O}_{\min}}$ of polynomials in x_1, x_2, x_3. The set of regular functions is isomorphic to

$$\mathbb{R}[x_1, x_2, x_3]/\langle g \rangle.$$

The first order star-product $x_i * x_j = x_i x_j + \hbar C_1(x_i, x_j)$ is equivariant if $C_1^-(x_i, x_j) = \frac{1}{2}\epsilon_{ijk} x_k$ and $C_1^+(x_i, x_j)$ is proportional to g_{ij}. Since κ is non degenerate, there is a two parameter family of essential first order equivariant deformations, indexed by $\hbar$ and $C_1^+|_{x=0}$. Every first order equivariant deformation can be extended, for example by the method outlined above, to an invariant star-product.

5.4. MINIMAL ORBIT OF $\mathfrak{sl}_n$

The minimal orbit is defined by the relations

$$\boxed{U_a^b U_c^d - U_a^d U_c^b = 0, \ \forall a, b, c, d = 1 \ldots n} \tag{11}$$

We also have the relation $\sum_{j=1}^n U_j^j = 0$ (trace zero).

Note that if a matrix $U \in \mathfrak{sl}_n$ belongs to the minimal orbit, then $U^2 = 0$.

Remark 5.11. The relations (11) are solved by the factorization

$$U_a^b = p_a q^b, \quad \text{with } q \cdot p := q^a p_a = 0, \tag{12}$$

which defines an embedding of $\mathfrak{g}$ into the space of second order polynomials on $P^{2n-2} := \mathbb{R}^{2n}/(p \cdot q, \approx)$, where $\approx$ is the equivalence relation defined by

$$\forall \lambda > 0, \quad (\lambda p, \lambda^{-1} q) \approx (p, q).$$

A star-product can be defined on this space by introducing the Poisson bracket defined by

$$\{q^b, p_a\} = \delta_a^b, \quad \{q^b, q_d\} = \{p^a, p_c\} = 0,$$

and quantizing this in the manner of Weyl. See Sections 5.6 and 6.2.

5.5. MINIMAL ORBITS OF $\mathfrak{sp}_{2n}$, $\mathfrak{so}_n$ AND $\mathfrak{g}_2$

The Lie algebra $\mathfrak{sp}_{2n}$ is the algebra of traceless matrices that leave invariant a skew-symmetric, non degenerate 2-form η, $U_a^b \eta_{bc} =: L_{ac} = L_{ca}$.

   We choose the embedding $L_{ac} = \xi_a \xi_c$, with $\xi_1, \ldots, \xi_{2n} = q^1, \ldots, q^n, p_1, \ldots, p_n$, which incorporates all the relations that define the minimal orbit. In fact, consider the polynomial algebra $A := \mathbb{C}[q^1, \ldots, q^n, p_1, \ldots, p_n]$, endowed with the standard Poisson bracket

$$\{F, G\} := \nabla_p F \cdot \nabla_q G - \nabla_q F \cdot \nabla_p G.$$

The subspace $A(2)$ of the elements of degree 2 is a Lie algebra for $\{\cdot, \cdot\}$ that is isomorphic to the Lie algebra $\mathfrak{sp}_{2n}$. The relations that define the minimal orbit are

$$\boxed{L_{ab} L_{cd} = L_{ad} L_{cb}, \quad \forall a, b, c, d = 1 \ldots n}.$$

   The Lie algebra $\mathfrak{so}_n$ is the algebra of traceless matrices that leave invariant a symmetric, non degenerate 2-form η. Analogy with $\mathfrak{sp}_{2n}$ suggests using Grassmann variables, replacing the commutative affine algebra by a super-commutative super Lie algebra, as is done in field theories with fermions. More precisely, we may replace the polynomial algebra $A = S(E')$ (where $E = Vect(q^1, \ldots, q^n, p_1, \ldots, p_n)$) given for $\mathfrak{sp}_{2n}$ by the algebra $\Lambda^*(E')$, and the Poisson bracket $\{\cdot, \cdot\}$ by a convenient Poisson super-bracket. Then the subspace $\Lambda^2(E')$ is a Lie algebra isomorphic to the Lie algebra of the orthogonal group. For more information on these spaces and on the link with the orthosymplectic group, see [41, Chapter 9]. An alternative is the embedding $L_{ac} = q_a p_c - q_c p_a$ (with $q_a = q^b \eta_{ba}$). The relations that define the minimal orbit are

$$\boxed{\sum_{\text{cycl}(abc)} L_{ab} L_{cd} = 0, \quad \forall d = 1 \ldots n, \qquad \sum_{b,c=1 \ldots n} \eta^{bc} L_{ab} L_{cd} = 0, \quad \forall a, d = 1 \ldots n,} \tag{13}$$

where η^{ab} is the general term of the inverse of the matrix $(\eta_{ab})_{(a,b)\in\{1,\dots,n\}}$. The first relation is implied by the embedding and the second can be incorporated by restriction to $\eta^{ab}p_a p_b = \eta^{ab}q_a q_b = q^a p_a = 0$ and projecting on a quotient.

The Lie algebra $\mathfrak{g}_2$ is isomorphic to the Lie algebra of derivations of the non associative algebra $\mathbb{O}$ of complex Cayley octonions. This is a subalgebra of $\mathfrak{so}_7$ (see [34,48,65]). The minimal orbit can be parametrized as that of $\mathfrak{so}_7$ with the additional condition $p \times q = 0$.

5.6. ASSOCIATED REPRESENTATIONS AND JOSEPH IDEALS

5.6.1. Background   The ideals in the universal enveloping algebra of a compact simple Lie algebra are fixed by a central character. The non compact case is more interesting. An example is well known to physicists. This is the real Lie algebra $\mathfrak{so}(4, 2)$, with the usual basis $\{L_{ab};\ 1 \le a < b \le 6\}$ and relations

$$[L_{ab}, L_{cd}] = \eta_{bc}L_{ad} - \eta_{ac}L_{bd} - \eta_{bd}L_{ac} + \eta_{ad}L_{bc}, \tag{14}$$

where η is the pseudo Euclidean metric. The tensor $g^{bc}(L_{ab}L_{cd} + L_{db}L_{ca})$ in the universal enveloping algebra reduces, in a certain irreducible and unitarizable representation, to fixed numerical values, so that the relations

$$\eta^{bc}(L_{ab}L_{cd} + L_{db}L_{ca}) = -\hbar^2 \eta_{ad} \tag{15}$$

hold in the representation. This particular representation appears in the analysis of the Schrödinger theory of the hydrogen atom and in conformal field theory. The algebra also enters the description of Keplerian orbits on a six-dimensional phase space.

The orbit is all the more interesting since Kostant's method of geometric quantization encounters a difficulty: the non existence of an invariant polarization [59]. It was shown by Joseph that this is true for all minimal orbits except the case of $\mathfrak{sl}_n$ [50]. Although the corresponding quantum theory is known, an invariant Wigner-Weyl correspondence [5] is not. We know that an invariant star-product associated with such a correspondence exists, but it has not been constructed. Nevertheless, relation (15) suggests that there is an invariant star-product such that

$$\eta^{bc}(L_{ab} * L_{cd} + L_{db} * L_{ca}) = -\hbar^2 \eta_{ad}. \tag{16}$$

A question that motivated this work is whether such a deformation exists, and if it is a deformation in the direction of the Poisson bracket,

$$f * g = fg + \frac{\hbar}{2}\{f, g\} + \sum_{j=2}^{\infty} C_j(f, g)\hbar^j.$$

The undoubted presence of interesting homology on the (closure of) this orbit was expected to play a role in invariant quantization. A preliminary exploration of the associated representations will show us what to expect.

5.6.2. Associated representations of $\mathfrak{sl}_n$ Let V_N denote the vector space of polynomial functions on $\mathbb{C}^n$ spanned by the set of functions $x_1^{r_1}, \ldots, x_n^{r_n}$, with $r_j \in \mathbb{N}$, and $N = \sum_{i=1}^n r_i$ fixed. For $a, b = 1 \ldots n$, let $\widetilde{U}_a^b$ be the operator in V_N, given by

$$\widetilde{U}_a^b = \hbar \left(x_a \partial_b - \frac{N}{n} \delta_a^b \right),$$

where ∂_b stands for the partial derivation with respect to x_b. Then $[\widetilde{U}_a^b, \widetilde{U}_c^d] = \hbar(\delta_c^b \widetilde{U}_a^d - \delta_a^d \widetilde{U}_c^b)$, and the map

$$\mathfrak{sl}_n \to L(V_N), \quad E_{ab} \mapsto \frac{1}{\hbar} \widetilde{U}_a^b \text{ if } a \neq b, \quad E_{aa} - \frac{1}{n} I_n \mapsto \frac{1}{\hbar} \widetilde{U}_a^a$$

is a morphism of Lie algebras, a representation of $\mathfrak{sl}_n$.

To be more precise, consider the real form $\mathfrak{su}(n-1, 1)$, with the compact subalgebra $\mathfrak{su}_{n-1}$ generated by $\{\widetilde{U}_a^b/a, b = 1 \ldots n-1\}$. Taking $r_1, \ldots, r_{n-1}$ to run over the natural numbers we obtain, for a range of values of N, a unitarizable highest weight representation, with the highest weight reducing to zero on $\mathfrak{su}_{n-1}$.

The relations that define the minimal orbit are $U_a^b U_c^d - U_a^d U_c^b = 0$, while a direct calculation gives[5]

$$\widetilde{U}_a^b \widetilde{U}_c^d - \widetilde{U}_a^d \widetilde{U}_c^b = \hbar \delta_c^b \widetilde{U}_a^d - \hbar \frac{N}{n} (\delta_a^b \widetilde{U}_a^b \widetilde{U}_c^d + \delta_c^d \widetilde{U}_a^b) - \hbar^2 \frac{N}{n} \left(\frac{N}{n} + 1 \right) \delta_a^b \delta_c^d - (b \leftrightarrow d). \tag{17}$$

An invariant deformation in the direction of the Poisson bracket would have the following form

$$U_a^b * U_c^d = U_a^b U_c^d + \frac{\hbar}{2} \{U_a^b, U_c^d\} + \hbar^2 C^+ (U_a^b, U_c^d). \tag{18}$$

In this setting, because of the very strong relations that characterize the orbit, the most general equivariant, symmetric 2-form C^+ takes the form

$$\hbar C^+ (U_a^b, U_c^d) = \frac{k}{2} \left(\delta_c^b U_a^d + \delta_a^d U_c^b - \frac{2}{n} (\delta_a^b U_c^d + \delta_c^d U_a^b) \right) + k' \left(\delta_a^d \delta_c^b - \frac{1}{n} \delta_a^b \delta_c^d \right). \tag{19}$$

These relations have the same form as Equation (17) if the parameters k, k' are appropriately related to the degree N of the homogeneous functions in the vector space V_N, and if we have

$$k \left(1 + \frac{2}{n} \right) = -\hbar \left(1 + 2\frac{N}{n} \right), \quad k' \left(1 + \frac{1}{n} \right) = \hbar^2 \frac{N}{n} \left(\frac{N}{n} + 1 \right). \tag{20}$$

It turns out that such a deformation exists if and only if the parameters k and k' are related to each other, precisely as implied by Equation (20).

---

[5]The notation $\leftrightarrow$ has the following meaning: "Formula$(a, b) + (a \leftrightarrow b)$" means "Formula$(a, b) +$ Formula(b, a)". Similarly, "Formula$(a, b, c, d) - (a \leftrightarrow b) - (c \leftrightarrow d) + (a \leftrightarrow b, c \leftrightarrow d)$" means "Formula$(a, b, c, d) -$ Formula$(b, a, c, d) -$ Formula$(a, b, d, c) +$ Formula(b, a, d, c)".

5.6.3. Associated representations of $\mathfrak{so}_n$ and $\mathfrak{sp}_{2n}$

Study of $\mathfrak{so}_n$

Let $V_N = \langle x_1^{r_1}, \ldots, x_n^{r_n} \rangle_{\mathbb{C}}$ be the space defined in Section 5.6.2, and η a symmetric, non degenerate 2-form on $\mathbb{C}^n$. Let $\partial_a := \eta_{ab}\partial_b$ and $\widetilde{L}_{ab} := \hbar(x_a\partial_b - x_b\partial_a)$, which gives a formal representation of $\mathfrak{so}_n$ and the formulae

$$\widetilde{L}_{ab}\widetilde{L}_{cd} = \hbar^2 \left(x_a x_c \partial_b \partial_d + \eta_{bc} x_a \partial_d - (a \leftrightarrow b) - (c \leftrightarrow d) + (a \leftrightarrow b,\, c \leftrightarrow d) \right).$$

We want to simplify this representation as much as possible and therefore restrict the variables to the cone $\eta(x, x) = 0$ and the space V_N to the subspace of harmonic functions. (We verify that the choice $N = 2 - \frac{n}{2}$ of the degree of homogeneity gives to the Laplace operator a well defined action on functions defined on the cone.) The first term on the right now satisfies the constraints. This is not yet a model for a star-product since the first order operators on the right do not combine to $\widetilde{L}_{ab}$'s. But another way to write the last relations is

$$\widetilde{L}_{ab}\widetilde{L}_{cd} = \hbar^2 \left(x_a x_c \partial^b \partial^d + \frac{1}{2}\eta_{bc}(x_a\partial^d + x_d\partial^a) - (a \leftrightarrow b) - (c \leftrightarrow d) + (a \leftrightarrow b,\, c \leftrightarrow d) \right) +$$
$$+ \frac{\hbar}{2}\eta_{bc} \left(\widetilde{L}_{ad} - (a \leftrightarrow b) - (c \leftrightarrow d) + (a \leftrightarrow b,\, c \leftrightarrow d) \right).$$

The first line satisfies the constraints, and if we take this as a model for the classical part, then we are led to look for a star-product of the form

$$L_{ab} * L_{cd} = L_{ab}L_{cd} + \frac{\hbar}{2}\eta_{bc} \left(L_{ad} - (a \leftrightarrow b) - (c \leftrightarrow d) + (a \leftrightarrow b,\, c \leftrightarrow d) \right) + o(\hbar^2).$$

It turns out that such an invariant star-product exists if and only if the last term is precisely

$$\hbar^2 \left(2 - \frac{n}{2} \right) (1 - n)\, \kappa(L_{ab},\, L_{cd}),$$

where κ is the Killing form.

See [10] for a complete discussion of singular representations of $\mathfrak{so}(p, q)$.

EXAMPLE 5.12. In the case $n = 6$, $N = -1$, restricting to the real form $\mathfrak{so}(4, 2)$ and taking $r_1, \ldots, r_4 \in \mathbb{N}$, $\frac{r_5}{r_6} \in \mathbb{N}$, we recover the unitarizable representation that is realized on the space of solutions of a massless scalar field in 4 dimensions, in the form discovered by Dirac.

The same representation appears in the theory of the hydrogen atom, where it is realized by self-adjoint operators in $L^2(\mathbb{R}, \frac{d^3r}{r})$.

Study of $\mathfrak{sp}_{2n}$

Let $V_{2n} = \mathbb{C}[\xi_1, \ldots, \xi_{2n}]$ be the space of polynomials on a complex vector space of dimension $2n$ endowed with a skew-symmetric, non degenerate 2-form η and coordinates $\xi_1, \ldots, \xi_{2n}$.

A Poisson bracket is defined by $\{\xi_a, \xi_b\} = \eta_{ab}$. Let $\partial_a := \eta_{ab}\partial_b$ and $\widetilde{L}_{ab} :=$ $\hbar(\xi_a\partial_b + \xi_b\partial_a)$. Then a similar analysis suggests a star-product such that

$$L_{ab} * L_{cd} = L_{ab}L_{cd} + \frac{\hbar}{2}\eta_{bc}\left(L_{ad} + (a \leftrightarrow b) + (c \leftrightarrow d) + (a \leftrightarrow b,\ c \leftrightarrow d)\right) + \frac{\hbar^2}{2}\eta_{ab}\eta_{cd}.$$

This is of course the Moyal star-product, restricted to $\mathfrak{sp}_{2n}$.

6. $\mathfrak{g}$-Invariant Star-Products on the Minimal Orbit of a Simple Lie Algebra

6.1. COMPUTATION OF THE HOMOLOGY

Here we rely on Kostant's characterization (Theorem 5.9) of the minimal orbit. We need a generalization of the results of Section 3.2 to this case of multiple relations, up to the level of 3-chains and 3-cochains. It is clear that HH_2 is the space spanned by the following chains,

$$(Z_{2,1})_\alpha = g_\alpha^{ij} x_i \otimes x_j, \text{ for } \alpha \in \mathbb{N}, \quad \text{and} \quad (Z_{2,2})_{ij} = x_i \wedge x_j, \text{ for } i,\ j = 1 \ldots N.$$

Every closed 3-chain is homologous to a linear one,

$$a = x_i \otimes x_j \otimes x_k A^{ijk},$$

with $A^{ijk} \in \mathbb{C}$, and this chain is closed if and only if

$$x_i x_j A^{ijk} = 0 = x_j x_k A^{ijk}.$$

Hence

$$A^{ijk} + A^{jik} = g_\alpha^{ij} c_\alpha^k, \quad A^{ijk} + A^{ikj} = c_\alpha'^{\ i} g_\alpha^{jk},$$

with complex coefficients c_α, c_α'. This can be solved if and only if

$$\sum_{\text{cycl}} g_\alpha^{ij} c_\alpha^k = \sum_{\text{cycl}} c_\alpha'^{\ i} g_\alpha^{jk},$$

where the sums are over cyclic permutations, which implies that $c_\alpha'^{\ i} = c_\alpha^i + \rho_\alpha^i$, with the coefficients ρ_α^i subject to $\sum_{\text{cycl}} \rho_\alpha^i g_\alpha^{jk} = 0$, as follows,

$$6 A^{ijk} = \text{Alt} A^{ijk} + 3(g_\alpha^{ij} c_\alpha^k - g_\alpha^{ik} c_\alpha^j + g_\alpha^{jk} c_\alpha^i) + 2(\rho_\alpha^i g_\alpha^{jk} - \rho_\alpha^j g_\alpha^{ik}), \quad \sum_{\text{cycl}(ijk)} \rho_\alpha^i g_\alpha^{jk} = 0.$$

The three terms belong respectively to $Z_{3,3} = HH_{3,3}$, $Z_{3,2} = HH_{3,2}$ (they are of the form listed in Proposition 3.5) and $Z_{3,1} = HH_{3,1}$. This third space, empty in the case where there is only one relation, has not been determined. We only have the following conjecture:

CONJECTURE 6.1. *The space* $\mathsf{Z}_{3,1} = HH_{3,1}$ *is spanned by chains of the form*

$$e_3(1)g_\alpha^{ij}g_\beta^{kl}(x_i \otimes \{x_j, x_k\} \otimes x_l),$$

where $\{g_\alpha\}$ *is the full set of binary relations and* $e_3(1)$ *is the BGS idempotent.*

EXAMPLE 6.2. For $\mathfrak{sl}_n$, the two relations $U_a^b U_c^d - U_c^b U_a^d = 0$, $U_d^f U_g^h - U_d^h U_g^f = 0$ generate in this manner the closed chain

$$U_a^b \otimes U_c^f \otimes U_g^h - U_c^b \otimes U_a^f \otimes U_g^h - U_a^b \otimes U_c^h \otimes U_g^f + U_c^b \otimes U_a^h \otimes U_g^f.$$

The difficulty will be overcome with the help of the correspondence principle, an adaptation of Weyl's symmetric ordering. It is the subject of the following section.

6.2. CORRESPONDENCE PRINCIPLE FOR $\mathfrak{g}$-INVARIANT STAR-PRODUCTS

We take a fresh point of departure. Suppose that an invariant star-product has the following property

$$S(x_{i_1} * \cdots * x_{i_p}) = x_{i_1} \ldots x_{i_p} + \phi_{i_1 \ldots i_p}, \tag{21}$$

where S stands for the symmetrization, and the function $\phi_{i_1 \ldots i_p}$ belongs to the ideal $\langle \hbar \rangle$ of $\mathbb{C}[[\hbar]]$.

Remark 6.3. We shall postulate that, for each order in $\hbar$, $\phi_{i_1 \ldots i_p}$ is a polynomial of order less than p. In this way we guarantee an important property of the deformation: the Poincaré–Witt basis is preserved. Actually, in the present context this is a weak limitation, since equivariant 2-cochains of higher order are scarce by reason of the constraints.

EXAMPLE 6.4. If $\phi = 0$, then we get the quantization rule given by Abellanas and Martinez-Alonso in [1] and called by them Weyl quantization.

Invariance of the star-product requires that the map ϕ:

$$A \to A, \quad x_{i_1} \ldots x_{i_p} \mapsto \phi_{i_1 \ldots i_p}$$

be equivariant for the adjoint action. The only other requirement is the obvious fact that the correspondence must be consistent with the relations that define the variety. Applying these constraints to both sides of Equation (21), we obtain conditions on the map ϕ. We shall calculate these conditions explicitly for monomials of order 2 and 3. We shall show that these conditions are precisely the same as those implied by associativity, confirming the fact that associativity is not a separate concern. So long as the correspondence is consistent with the constraints

(now the only issue), detailed knowledge of the restricted homology spaces is not required. Cohomology was crucial for the demonstration of an extension to higher orders (Theorem 3.4), but it is not the best tool for establishing the basis at low orders.

Let the symbol S stand for symmetrization in the order of factors and set

$$S(x_i * x_j) = x_i x_j + \psi_{ij}, \quad S(x_i * x_j * x_k) = x_i x_j x_k + \phi_{ijk}, \tag{22}$$

and recall that

$$x_i * x_j - x_j * x_i = \hbar\{x_i, \, x_j\} = \hbar\epsilon_{ij}^m x_m, \tag{23}$$

The polynomials ϕ_{ijk} and ψ_{ij} are assumed to define equivariant maps as explained above.

PROPOSITION 6.5. *Assume that the equivariant polynomials ψ and ϕ have been chosen so that relations (22) are consistent with the constraints. Then there is an invariant associative star-product such that Equation (22) holds.*

Proof. It follows from Equation (22) that

$$(x_i x_j) * x_k = x_i * x_j * x_k - \frac{\hbar}{2}\epsilon_{ij}{}^m x_m * x_k - \psi_{ij} * x_k.$$

The symmetrized star product is

$$S(x_i * x_j * x_k) = \frac{1}{6}\sum_{\sigma \in S_3}(x_i * x_j * x_k) =$$

$$= \frac{1}{2}x_i * x_j * x_k + \frac{\hbar}{3}x_i * \{x_k, x_j\} + \frac{\hbar}{6}\{x_k, x_i\} * x_j + (i \leftrightarrow j),$$

and we deduce that

$$(x_i x_j) * x_k = x_i x_j x_k + \phi_{ijk} -$$
$$- \psi_{ij} * x_k - \frac{\hbar}{2}\left(\psi(\{x_k, x_i\}, x_j) + (i \leftrightarrow j)\right) -$$
$$- \frac{\hbar}{2}\left(\{x_k, x_i\}x_j + (i \leftrightarrow j)\right) - \frac{\hbar^2}{12}\left(\{x_i, \{x_k, x_j\}\} + (i \leftrightarrow j)\right).$$

Taking ψ to be equivariant leads to some cancelations,

$$(x_i x_j) * x_k = x_i x_j x_k + \phi_{ijk} - \psi_{ij}x_k - \psi(\psi_{ij}, x_k) -$$
$$- \frac{\hbar}{2}\left(\{x_k, x_i\}x_j + (i \leftrightarrow j)\right) - \frac{\hbar^2}{12}\left(\{x_i, \{x_k, x_j\}\} + (i \leftrightarrow j)\right). \tag{24}$$

Similarly

$$x_i * (x_j x_k) = x_i x_j x_k + \phi_{ijk} - x_i \psi_{jk} - \psi(x_i, \psi_{jk}) -$$
$$- \frac{\hbar}{2}\left(\{x_k, x_i\}x_j + (j \leftrightarrow k)\right) - \frac{\hbar^2}{12}\left(\{\{x_j, x_i\}, x_k\} + (j \leftrightarrow k)\right).$$

These equations yield explicit expressions for the values of the two-forms C_1 and C_2 defined by

$$(x_i x_j) * x_k = x_i x_j x_k + \hbar C_1(x_i x_j, x_k) + \hbar^2 C_2(x_i x_j, x_k),$$
$$x_i * (x_j x_k) = x_i x_j x_k + \hbar C_1(x_i, x_j x_k) + \hbar^2 C_2(x_i, x_j x_k).$$

These values solve the condition for associativity of the star product on linear chains. Theorem 6 then assures us that they can be satisfied in general. The proposition is proved.   □

In the present approach, associativity is satisfied trivially. What is far from trivial is the existence of a function ϕ_{ijk} that solves Equation (24). The obstructions are the constraints. Application of g_α^{ij} to the first and g_α^{jk} to the second gives

$$g_\alpha^{ij} \phi_{ijk} = (g_\alpha^{ij} \psi_{ij}) * x_k + \hbar g_\alpha^{ij} \epsilon_{ki}^m \psi_{mj} + \frac{\hbar^2}{6} g_\alpha^{ij} \epsilon_{ki}^m \epsilon_{mj}^p x_p, \tag{25}$$

$$g_\alpha^{jk} \phi_{ijk} = x_i * (g_\alpha^{jk} \psi_{jk}) + \hbar g_\alpha^{kj} \epsilon_{ki}^m \psi_{mj} + \frac{\hbar^2}{6} g_\alpha^{jk} \epsilon_{ki}^m \epsilon_{jm}^p x_p. \tag{26}$$

Since ϕ is symmetric both right hand expressions must agree,

$$(g_\alpha^{ij} \psi_{ij}) * x_k - x_k * (g_\alpha^{ij} \psi_{ij}) = 2\hbar g_\alpha^{ij} \epsilon_{ik}^m \psi_{mj}.$$

The only equivariant tensors available for the 2-chain are the Killing form and, in the case of $\mathfrak{sl}_n$, a term linear in the generators, as above. Both satisfy this last condition, so of the two Equations (25) and (26) it is enough to examine the first. The problem of consistency of Equation (21) is reduced to the existence of ϕ_{ijk} that solves Equation (25).

A complete determination of the restricted cohomology is not required; it is enough to know the relations that define the orbit.

6.3. CALCULATIONS FOR $\mathfrak{sl}_n$: JOSEPH IDEAL AND HIGHEST WEIGHT MODULES

6.3.1. Solving the constraints   We use the notations and the commutation relations of $\mathfrak{sl}_n$ given in Section 5.4. We replace the general variables x_i by the variables U_a^b of $\mathfrak{sl}_n$, and Equations (22) and (24) take the form

$$\frac{1}{2}(U_a^b * U_c^d + U_c^d * U_a^b) = U_a^b U_c^d + \psi_{ac}^{bd}, \tag{27}$$

and

$$(U_a^b U_c^d) * U_e^f = U_a^b U_c^d U_e^f + \phi_{ace}^{bdf} - \psi_{ac}^{bd} U_e^f - \psi(\psi_{ac}^{bd}, U_e^f) -$$
$$- \frac{\hbar}{2} \left(\{U_e^f, U_a^b\} U_c^d + \{U_e^f, U_c^d\} U_a^b \right) -$$
$$- \frac{\hbar^2}{12} \left(\{U_a^b, \{U_e^f, U_c^d\}\} + \{U_c^d, \{U_e^f, U_a^b\}\} \right). \tag{28}$$

Let us introduce a simpler notation to express the elements u of $\mathfrak{g}'$. We may write

$$u = \sum_{a,b=1\dots n} A_b^a U_a^b.$$

Set $U := (U_a^b)_{a,b} \in \mathbf{M}_n(\mathfrak{g}')$ and $A := (A_a^b)_{a,b} \in \mathbf{M}_n\mathbb{C}$. Then $u = (AU)$ is the trace of the matrix product AU. The coefficients A_b^a are coordinates for $\mathfrak{g}$, A ranges over the matrices of the adjoint representation of $\mathfrak{g}$. Equation (27) is linear in U_a^b and U_c^d, so we may replace U_a^b and U_c^d by the linear combination $u := (AU)$, and Equation (27) becomes

$$u * u = u^2 + \psi(u, u).$$

Equivariance restricts ψ:

$$\psi(u, u) = k(AAU) + k'(AA), \quad (k, k') \in \mathbb{C}^2, \tag{29}$$

where (AAU) and (AA) again stand for traces of products of matrices. Similarly,

$$u * u * u = u^3 + \phi(u, u, u),$$

with

$$\phi(u, u, u) = \phi_1(AAU)u + \phi_2(AAAU) + \phi_3(AA)u + \phi_4(AAA). \tag{30}$$

There are other invariants but their inclusion here is not allowed by the relations.

Let ψ be given in the form Equation (29) and look at Equation (28) as an equation to determine ϕ. Recall that the minimal orbit is defined by the relations

$$U_a^b U_c^d = U_a^d U_c^b, \quad \forall a, b, c, d = 1 \dots n.$$

PROPOSITION 6.6. *Equation* (28) *is consistent with the relations* $U_a^b U_c^d = U_a^d U_c^b$ *if and only if the parameters* k *and* k' *are related to each other as follows,*

$$4k'\left(1 + \frac{1}{n}\right) = k^2\left(1 + \frac{2}{n}\right)^2 - \hbar^2. \tag{31}$$

Proof. See Appendix.   □

6.3.2. Generators of the Joseph ideal

PROPOSITION 6.7. *The Joseph ideal for* $\mathfrak{sl}_n$ *is generated by the relations*

$$U_a^b * U_c^d - U_a^d * U_c^b = \frac{\hbar}{2}(\delta_c^b U_a^d - \delta_a^d U_c^b - \delta_c^d U_a^b + \delta_a^b U_c^d) + k'\left(1 + \frac{1}{n}\right)(\delta_a^d \delta_c^b - \delta_a^b \delta_c^d) +$$

$$+ \frac{k}{2}\left(1 + \frac{2}{n}\right)(\delta_a^d U_c^b + \delta_c^b U_a^d - \delta_a^b U_c^d - \delta_c^d U_a^b). \tag{32}$$

Proof. We find the relations of the deformed algebra by eliminating the original product from

$$U_a^b * U_c^d = U_a^b U_c^d + \frac{\hbar}{2}(\delta_c^b U_a^d - \delta_a^d U_c^b) + k' \left(\delta_a^d \delta_c^b - \frac{1}{n} \delta_a^b \delta_c^d \right) +$$

$$+ \frac{k}{2} \left(\delta_a^d U_c^b + \delta_c^b U_a^d - \frac{2}{n} (\delta_a^b U_c^d + \delta_c^d U_a^b) \right). \tag{33}$$

□

6.3.3. Highest weight module   For a Lie algebra $\mathfrak{g}$, we choose a Cartan sub-algebra $\mathfrak{h}$ consisting of diagonal matrices. Then *a highest weight* module of the deformed algebra is a module generated by a vector v such that

$$U_a^b * v = 0, \ a < b = 2 \dots n, \quad U_a^a * v = \lambda_a v, \ a = 1 \dots n, \tag{34}$$

with $\lambda \in \mathbb{C}^n$ and $\sum_{a=1}^n \lambda_a = 0$. The vector $\lambda := (\lambda_1, \dots, \lambda_n)$ is called the weight of the vector v.

PROPOSITION 6.8. *A highest weight module of the deformed algebra exists if and only if relation* (31) *holds. In that case there exists $m \in \{1, \dots, n\}$ such that the highest weight λ is given by*

$$\lambda_1 = \lambda_2 = \dots = \lambda_{m-1} = -\frac{\hbar}{2} - \gamma, \quad \lambda_{m+1} = \dots = \lambda_n = \frac{\hbar}{2} - \gamma, \tag{35}$$

where λ_m is determined by $\sum_{i=1}^n \lambda_i = 0$ and $\gamma = \frac{k}{2}\left(1 + \frac{2}{n}\right)$.

Proof. The only relations that are changed by the deformation are those where $\{a, c\}$ intersects $\{b, d\}$, so we may limit ourselves to the case $b = a$. Also, if a is equal to c or to d the relations are just the usual commutation relations. So take $c \neq a$ and $d \neq a$. Then

$$U_a^a * U_c^d - U_a^d * U_c^a = \frac{\hbar}{2}(-\delta_c^d U_a^a + U_c^d) - \frac{k}{2}\left(1 + \frac{2}{n}\right)(U_c^d + \delta_c^d U_a^a) - k'\left(1 + \frac{1}{n}\right)\delta_c^d.$$

In particular, for $c = d$,

$$U_a^a * U_c^c - U_a^c * U_c^a = \frac{\hbar}{2}(U_c^c - U_a^a) - \frac{k}{2}\left(1 + \frac{2}{n}\right)(U_c^c + U_a^a) - k'\left(1 + \frac{1}{n}\right).$$

Applying this to a highest weight vector v we obtain, for $a > c$,

$$\lambda_a \lambda_c = \frac{\hbar}{2}(\lambda_c - \lambda_a) - \gamma(\lambda_a + \lambda_c) - \gamma', \quad \gamma := \frac{k}{2}\left(1 + \frac{2}{n}\right), \quad \gamma' := k'\left(1 + \frac{1}{n}\right).$$

Thus, for $a > c$,

$$\left(\lambda_c + \frac{\hbar}{2} + \gamma\right)\lambda_a = \left(\frac{\hbar}{2} - \gamma\right)\lambda_c - \gamma'.$$

Suppose $n > 2$.

If $\gamma' \neq \gamma^2 - \left(\frac{\hbar}{2}\right)^2$, then $\lambda_c \neq -\frac{\hbar}{2} - \gamma$, hence

$$\lambda_a = \frac{\left(\frac{\hbar}{2} - \gamma\right)\lambda_c - \gamma'}{\lambda_c + \frac{\hbar}{2} + \gamma}, \quad a > c.$$

Then taking $c = 1$, we get $\lambda_2 = \cdots = \lambda_n$, and taking $a = n$, we find $\lambda_1 = \cdots = \lambda_{n-1}$. Since $\sum_{i=1}^{n} \lambda_i = 0$, we have $\lambda_i = 0$ for every i, which is not interesting.

We conclude that

$$\gamma' = \gamma^2 - \left(\frac{\hbar}{2}\right)^2, \tag{36}$$

which is the same as Equation (31). Then there is an integer m, $1 \leq m \leq n$ such that λ_m differs from its neighbors, and the statement of the proposition follows immediately. $\qquad\square$

6.3.4. Abelian deformations

When $\hbar = 0$ the deformed algebra is commutative, so a representation is just a character, or a maximal ideal. The group acts on the maximal ideals, and among the maximal ideals there are upper triangular ones. This is possible only if either $n = 2$ or if Equation (31) holds,

$$4k'\left(1 + \frac{1}{n}\right) = k^2\left(1 + \frac{2}{n}\right)^2.$$

In that case the deformed variety is the space of traceless matrices with all but one of the eigenvalues equal to $-\frac{k}{2}(1 + \frac{2}{n})$.

6.4. CALCULATIONS FOR $\mathfrak{so}_n$: JOSEPH IDEAL AND HIGHEST WEIGHT MODULES

6.4.1. Solving the constraints

We use the notations and the commutation relations of $\mathfrak{so}_n$ given in Sections 5.6.3. So, we now take for the general variables x_i the variables L_{ab} of $\mathfrak{so}_n$, and Equation (22) takes the form

$$\frac{1}{2}(L_{ab} * L_{cd} + L_{cd} * L_{ab}) = L_{ab}L_{cd} + \psi_{ab,cd}.$$

Then Equation (24) now reads

$$\begin{aligned}
(L_{ab}L_{cd}) * L_{ef} = {} & L_{ab}L_{cd}L_{ef} + \phi_{ab,cd,ef} - \psi_{ab,cd}L_{ef} - {} \\
& - \frac{\hbar}{2}\left(\{L_{ef}, L_{ab}\}L_{cd} + \{L_{ef}, L_{cd}\}L_{ab}\right) - {} \\
& - \frac{\hbar^2}{12}\left(\{L_{ab}, \{L_{ef}, L_{cd}\}\} + \{L_{cd}, \{L_{ef}, L_{ab}\}\}\right).
\end{aligned} \tag{37}$$

As for $\mathfrak{sl}_n$, a simpler notation is to write $u \in \mathfrak{g}'$ as $u = (AL) = A^{ab}L_{ab}$. The coefficients $A^{ab} = -A^{ba}$ are coordinates for $\mathfrak{g}$. Then the two preceding equations become

$$u * u = u^2 + \psi(u, u), \tag{38}$$

and

$$u^2 * v = u^2 v + \phi(A, A, B) - \psi(A, A)v - \hbar[v, u] - \frac{\hbar^2}{6}\{u, \{v, u\}\}. \tag{39}$$

Equivariance restricts ψ and ϕ,

$$\psi_{A,A} = k(AA),$$

and

$$\phi(AAA) = \phi_1(AAAL) + \phi_2(AA)u,$$

where (AAU) and (AA) denote traces of products of matrices. In this context we take the form η defined by $\eta_{ab} = 0$ if $a \neq b$ and $\eta_{aa} = 1$ for $a = 1 \ldots n$, so that the two-forms become matrices without any fuss.

We fix the parameter k and look at Equation (37) as an equation to determine the coefficients ϕ_1, ϕ_2. Then we have the following proposition.

PROPOSITION 6.9. *Equation* (37) *is consistent with the relations that define the minimal orbit if and only if the parameter k takes the value $\hbar \frac{n-4}{n-1}$. In this case the parameters ϕ_1 and ϕ_2 are fixed by Equation* (37).

Proof. See Appendix. $\qquad\square$

6.4.2. Generators of the Joseph ideal

PROPOSITION 6.10. *The Joseph ideal for $\mathfrak{so}_n$ is generated by* (*the commutation relations and*) *the relations*

$$S(\eta^{bc}L_{ab} * L_{cd}) + \frac{\hbar^2}{2}(n-4)\eta_{ad} = 0, \qquad \sum_{\text{cycl}(abc)} (L_{ab} * L_{cd} - \hbar\eta_{ad}L_{bc}) = 0. \tag{40}$$

These relations were derived by Binegar and Zierau [10], who also determined the highest weight module and the associated unitary representations of $\mathbf{SO}(p, q)$. We are interested, nevertheless, in deriving these results with the help of the star-product. Equation (40) is a direct consequence of Equation (38), that reads in full

$$L_{ab} * L_{cd} = L_{ab}L_{cd} + \frac{\hbar}{2}\{L_{ab}, L_{cd}\} + \frac{\ell}{2}(\eta_{ac}\delta_{bd} - \eta_{bc}\delta_{ad}). \tag{38a}$$

6.4.3. Highest weight module for $\mathfrak{so}_n$ Let $\{E_{ab}; a, b = 1 \ldots n\}$ denote the set of unit matrices of $\mathbf{M}_n\mathbb{C}$ as earlier, and take

$$\eta := \sum_{a=1}^{n} \delta_{a,a'}, \quad \text{with } a' := n+1-a.$$

A basis for $\mathfrak{so}_n$ in the natural representation is the set $\{L_{ab} \ a < b \in \{1, \ldots, n\}\}$ of matrices

$$L_{ab} := (E_{ab} - E_{ba})\eta = E_{ab'} - E_{ba'}.$$

The Cartan subalgebra $\mathfrak{h}$ that we choose has for basis vectors

$$H_a = L_{aa'} - L_{a'a}, \quad a = 1 \ldots l.$$

The set of positive root vectors is the collection $\{L_{ab}; \ a + b \leq n, \ a < b\}$.
Our calculations are insensitive to the parity of n. Nevertheless we note the following facts:

▷ When $n = 2l + 1$, the simple root vectors are $L_{1,n-1}, L_{2,n-2}, \ldots, L_{l,l+1}$, and the associated simple roots are $\alpha_i = (0, \ldots, 0, 1, -1, 0, \ldots, 0)$, $i = 1 \ldots l - 1$ with 1 in the i'th place and $\alpha_l = (0, \ldots, 0, 1)$.

▷ When $n = 2l$, the simple root vectors are $L_{1,n-1}, L_{2,n-2}, \ldots, L_{l-1,l+1}, L_{l,l}$, and the associated simple roots are $\alpha_i = (0, \ldots, 0, 1, -1, 0, \ldots, 0)$, $i = 1 \ldots l - 1$ with 1 in the i'th place and $\alpha_l = (0, \ldots, 0, 1, 1)$.

All this is in [12] (see also [34]).

Having fixed the root system and the Cartan subalgebra, we define a highest weight module as a module generated by a character $H_a \mapsto \lambda_a \in \mathbb{C}$ and a (highest weight) vector v_0 with the property

$$\forall a + b \leq n, \ L_{ab}v_0 = 0,$$
$$\forall a = 1 \ldots l, \ H_a v_0 = \hbar \lambda_a v_0.$$

PROPOSITION 6.11. *A highest weight module of the deformed algebra exists if and only if the parameter ℓ takes the value $\hbar^2 \frac{n-4}{n-1}$. In that case, there is $k \in \{1, \ldots, l\}$ such that the highest weight λ is given by*

$$\lambda_1 = \cdots = \lambda_{k-1} = -1, \quad \lambda_k = k + 1 - \frac{n}{2}, \quad \lambda_{k+1} = \cdots = \lambda_l = 0.$$

Proof. See Appendix. Joseph's choice is $k = l - 1$. □

In the formulation of Proposition 6.11 and of Equation (38a), in order to avoid confusion with the integer index k, we denoted by ℓ the complex parameter denoted by k throughout this Section and the Appendix. In all other instances the context makes it clear which k is meant.

When $n = 2l + 1$, we have $\mathfrak{so}_n = \mathfrak{so}_{2l+1} = B_l$, and

$$\lambda + \rho = \omega_1 + \cdots + \omega_{l-3} + \frac{1}{2}(\omega_{l-2} + \omega_{l-1}) + \omega_l =$$
$$= \left(l - \frac{3}{2}, l - \frac{1}{2}, \ldots, \frac{5}{2}, \frac{3}{2}, 1, \frac{1}{2}\right),$$
$$\lambda = \left(-1, \ldots, -1, -\frac{1}{2}, 0\right).$$

EXAMPLE 6.12. The case $n=5$ corresponds to the Bose singleton (a.k.a. the Rac). Another choice is $\lambda=(-1,\ldots,-1,\frac{1}{2})$; in the case $n=5$ it is the Fermi singleton (a.k.a. the Di). Both are unitarizable (after taking a quotient) representations of $\mathfrak{so}(3,2)$. Binegar and Zierau take $k=1$.

When $n=2l$, we have $\mathfrak{so}_n=\mathfrak{so}_{2l}=D_l$, and Joseph again takes $k=l-1$ and

$$\lambda+\rho=\omega_1+\cdots+\omega_{l-3}+\omega_{l-1}+\omega_l=$$
$$=(l-3,\,l-2,\ldots,\,1,\,1,\,0)\,,$$
$$\lambda=(-1,\,-1,\ldots,\,-1,\,0,\,0)\,.$$

EXAMPLE 6.13. When $n=6$, this is the highest weight of the representation of the conformal group by a scalar massless field in 4 dimensions, the same representation that appears in Schrödinger's hydrogen atom. Binegar and Zierau take $k=1$.

6.5. UNIFORM CALCULATIONS FOR THE EXCEPTIONAL SIMPLE LIE ALGEBRAS

6.5.1. Minimal orbits of the exceptional simple Lie algebras
Here we use some results of [63]. Let κ denote the Killing form,

$$\kappa(x_i,\,x_j)=\kappa_{ij}=-\text{tr}(x_ix_j)=-\epsilon_{mi}{}^n\epsilon_{nj}{}^m$$

and κ^{ij} the matrix elements of the inverse matrix. The reduction of the symmetric part of the adjoint representation is governed by the operator $L:\mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}\otimes\mathfrak{g}$ defined by

$$L:x_i\otimes x_j\mapsto\kappa^{mn}\epsilon_{im}{}^s\epsilon_{jn}{}^t x_s\otimes x_t=\kappa^{mn}[x_i,\,x_m]\otimes[x_j,\,x_n]. \tag{39}$$

The symmetric part of the product $ad\otimes ad$ of the adjoint representation by itself decomposes into a direct sum of three representations, $I\oplus R_2\oplus R_3$. The last representation, that contains the extremal weights, is the minimal nilpotent orbit of $\mathfrak{g}$. The relations that define this orbit are the projections on the first two representations. The one-dimensional representation expresses the condition

$$\kappa^{ij}x_ix_j=0.$$

The following table (see [63]) gives the eigenvalues $l_i=l_1,\,l_2,\,l_3$ associated with the representations I, R_2, R_3. In all cases, we have $l_1=1$, and $l_2+l_3=-\frac{1}{6}$.

$\mathfrak{g}$	G_2	F_4	E_6	E_7	E_8
l_3	$\frac{1}{4}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{30}$
$D=\dim\mathfrak{g}$	27	52	78	133	248

We turn to Equation (24). Actually, the exceptional simple Lie algebras are the easiest to deal with, and all five can be done uniformly. The immediate reason for this is the non existence of an irreducible invariant fourth order polynomial. The only equivariant symmetric tensors are (see Proposition 6.5)

$$\psi_{ij} = k\kappa_{ij}, \quad \phi_{ijk} = \frac{k'}{3} \sum_{\text{cycl}(ijk)} \kappa_{ij}x_k, \tag{40}$$

where k and k' are determined by the relations. So Equation (24) simplifies:

$$(x_i x_j) * x_k - x_i x_j x_k + \frac{\hbar}{2} \left(\{x_k, x_i\}x_j + (i \leftrightarrow j) \right) =$$
$$= \phi_{ijk} - \psi_{ij}x_k - \frac{\hbar^2}{12} \left(\{x_i, \{x_k, x_j\}\} + (i \leftrightarrow j) \right). \tag{41}$$

6.5.2. Main result

THEOREM 6.14. *Let $\mathfrak{g}$ be one of the five exceptional simple Lie algebras, and κ the Killing form. There exists a unique, invariant star-product on the minimal nilpotent orbit, such that*

$$S(x_i * x_j) = x_i x_j + k\kappa_{ij}, \quad k \in \mathbb{C},$$

for one and only one choice of k, $k' \in \mathbb{C}$, namely $k = \frac{l_2-1}{D}\frac{\hbar^2}{6} + l_2\frac{\hbar^2}{12}$, $k' = \frac{\hbar^2}{4}$.

Proof. It has already been pointed out that the only possible forms of the deformed products

$$S(x_i * x_j) = x_i x_j + o(\hbar), \quad S(x_i * x_j * x_k) = x_i x_j x_k + o(\hbar)$$

are

$$S(x_i * x_j) = x_i x_j + \psi_{ij}, \quad S(x_i * x_j * x_k) = x_i x_j x_k + \phi_{ijk},$$

with the equivariant maps ψ, ϕ as in Equation (40). The left hand side of Equation (41) satisfies the constraints. It remains to know if the right hand side does. Fix the index k and define vectors X, Y in $\mathfrak{g} \otimes \mathfrak{g}$ with components

$$X_{ij} = \kappa_{ik}x_j + \kappa_{jk}x_i, \quad Y_{ij} = \kappa_{ij}x_k.$$

Let L_s denote the projection of the operator L on the symmetric part of $\mathfrak{g} \otimes \mathfrak{g}$. Then the right hand side of Equation (41) is the vector

$$\frac{k'}{3}(X + Y) - kY - \frac{\hbar^2}{12}L_s X. \tag{42}$$

The operator L_s satisfies the characteristic equation $(L_s - 1)(L_s - l_2)(L_s - l_3) = 0$, and since $L_s Y = Y$ there is a constant c such that

$$(L_s - l_2)(L_s - l_3)X = cY.$$

To determine the constant c, we contract this equation with κ and find that

$$c = \frac{2}{D}(1 - l_2)(1 - l_3).$$

With this, the vector (42) reduces to

$$Z := c^{-1}\left(\frac{k'}{3} - k\right)(L_s - l_2)(L_s - l_3)X + \frac{k'}{3}X - \frac{\hbar^2}{12}L_s X.$$

Projecting on the trivial representation we get

$$0 = \frac{k'}{3}(D + 2) - kD - \frac{\hbar^2}{6} = 0. \tag{43}$$

All the constraints are expressed by $(L_s - l_3)Z = 0$, which yields the three conditions:

$$
\begin{aligned}
(1 - l_3)c^{-1}\left(\frac{k'}{3} - k\right) - \frac{\hbar^2}{12} &= 0, \\
(1 - l_3)c^{-1}\left(\frac{k'}{3} - k\right)(l_2 + l_3) + \frac{k'}{3} &= l_3\frac{\hbar^2}{12}, \\
(1 - l_3)c^{-1}\left(\frac{k'}{3} - k\right)l_2 l_3 &= l_3\frac{k'}{3}.
\end{aligned}
\tag{44}
$$

All four Equations (43)–(44) agree on the unique values of k and k'

$$k' = l_2\frac{\hbar^2}{4}, \quad k - \frac{k'}{3} = \frac{l_2 - 1}{D}\frac{\hbar^2}{6}.$$

The proposition is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

6.5.3. Generators of the Joseph ideal The Joseph ideal is of course generated by the relations

$$x_i * x_j - x_j * x_i = \hbar\{x_i,\, x_j\},$$

and

$$(L_s - l_3)(x_i * x_j - k\kappa_{ij}) = 0.$$

7. Appendix

Proof of Proposition 6.6. The lack of symmetry in Equation (28) forces us to introduce a second variable, denoted by $v = (BU)$. Then the equation can be written as

$$u^2 * v = u^2 v + \phi(u, u, v) - \psi(u, u)v - \psi(\psi(u, u), v) - \hbar\{v, u\}u - \frac{\hbar^2}{6}\{u, \{v, u\}\},$$

with ϕ given by Equation (30),

$$\phi(u, u, v) = \frac{\phi_1}{3}((AAU)v + 2(ABU)u) + \frac{\phi_2}{3}((AABU) + (ABAU) + (BAAU)) +$$
$$+ \frac{\phi_3}{3}((AA)v + 2(AB)u) + \phi_4(AAB).$$

Exchange of the upper indices on the two A's is a type of Fierz transformation. The effect of this exchange is the following:

$$(ABAU) \mapsto (AB)u - \frac{1}{n}(AABU) - \frac{1}{n}(AAUB) + \frac{1}{n^2}(AA)v,$$

$$(AABU) \mapsto -\frac{2}{n}(AABU) + \frac{1}{n^2}(AA)v,$$

$$(ABU)u \mapsto (ABU)u - \frac{1}{n}(AAU)v,$$

$$(BAU)u \mapsto (BAU)u - \frac{1}{n}(AAU)v,$$

$$(AAU) \mapsto -\frac{2}{n}(AAU),$$

$$(AAB) \mapsto -\frac{2}{n}(AAB),$$

$$(AB)u \mapsto (AUAB),$$

$$(AA) \mapsto -\frac{1}{n}(AA).$$

For example, consider $(AB)u = (\sum_{a,b} A_a^b B_b^a)(\sum_{c,d} A_c^d U_d^c)$. After the exchange of the upper indices on the two A's, this expression becomes $\sum_{a,b,c,d} A_a^d U_d^c A_c^b B_b^a = (AUAB)$.

In the same way,

$$\phi(u, u, v) \mapsto \frac{\phi_1}{3}\left(-\frac{2}{n}(AAU)v + 2(ABU)u - \frac{2}{n}(AAU)v\right) +$$
$$+ \frac{\phi_2}{3}\left(-\frac{3}{n}(AABU + AAUB) + \frac{3}{n^2}(AA)v + (AB)u\right) +$$
$$+ \frac{\phi_3}{3}\left(-\frac{1}{n}(AA)v + 2(ABAU)\right) + \phi_4\left(\frac{-2}{n}(AAB)\right).$$

Furthermore, we have

$$-\psi(u,u)v = -k(AAU)v - k'(AA)v \mapsto \frac{2k}{n}(AAU)v + \frac{k'}{n}(AA)v,$$

$$-\psi(\psi(u,u),v) = -\frac{k^2}{2}\left((AABU+AAUB) - \frac{2}{n}(AA)v\right) - kk'(AAB) \mapsto$$

$$\mapsto \frac{k^2}{n}\left((AABU+AAUB) - \frac{2}{n}(AA)v\right) -$$

$$-\frac{k^2}{n}(AA)v + \frac{kk'}{n}(AAB),$$

$$-\frac{\hbar^2}{6}\{u,\{v,u\}\} = -\frac{\hbar^2}{3}(ABAU) + \frac{\hbar^2}{6}(AABU+AAUB) \mapsto$$

$$\mapsto \frac{\hbar^2}{3}\left(-(AB)u + \frac{1}{n}(AABU+AAUB) - \frac{1}{n^2}(AA)v\right) -$$

$$-\frac{\hbar^2}{6}\left(-\frac{2}{n}(AABU+AAUB) + \frac{2}{n^2}(AA)v\right).$$

Applying the constraint $U_a^b U_c^d = U_a^d U_c^d$ to Equation (28), we get the following equations

$$\frac{1}{3}\phi_1\left(1+\frac{2}{n}\right) - k\left(1+\frac{2}{n}\right) = 0,$$

$$2\phi_2\left(1+\frac{3}{n}\right) - 3k^2\left(1+\frac{2}{n}\right) + \hbar^2 = 0,$$

$$\phi_2 - 2\phi_3 - \hbar^2 = 0,$$

$$\phi_3\left(1+\frac{1}{n}\right) - 3k'\left(1+\frac{1}{n}\right) - \frac{3}{n^2}\phi_2 + \frac{3}{n}k^2\left(1+\frac{2}{n}\right) = 0,$$

$$\phi_4\left(1+\frac{2}{n}\right) - kk'\left(1+\frac{2}{n}\right) = 0.$$

We eliminate the parameters ϕ_i's and we find the relation

$$4k'\left(1+\frac{1}{n}\right) = k^2\left(1+\frac{2}{n}\right)^2 - \hbar^2.$$

The proposition is proved.   □

Proof of Proposition 6.9. Recall that the relations that define the minimal nilpotent orbit are

$$\sum_{\text{cycl}(abc)} L_{ab}L_{cd} = 0, \quad \eta^{bc}L_{ab}L_{cd} = 0.$$

By applying the first relation (contraction on b, c) to Equation (37), we get

$$0 = \eta^{bc}\phi_{ab,cd,ef} - \frac{k}{2}(n-1)\eta_{ad}L_{ef} - \frac{\hbar^2}{12}((n-4)(\eta_{fd}L_{ae} + (a \leftrightarrow d) -$$
$$-(e \leftrightarrow f) - (a \leftrightarrow d, \ e \leftrightarrow f)) - 4\eta_{ad}L_{ef}),$$

and

$$\eta^{bc}\phi_{ab,cd,ef} = \left(\frac{\phi_1}{24}(n-1) - \frac{\phi_2}{6}\right)(\eta_{fd}L_{ae} + (a \leftrightarrow d) - (e \leftrightarrow f) - (a \leftrightarrow d, \ e \leftrightarrow f)) +$$
$$+ \left(-\frac{\phi_1}{6} + \frac{\phi_2}{6}(n-1)\right)\eta_{ad}L_{ef}.$$

The second relation (a kind of Fierz transformation) gives

$$0 = \sum_{\text{cycl}}\phi_{ab,cd,ef} + \frac{\hbar^2}{3}\eta_{fa}(\eta_{de}L_{bc} + \eta_{ec}L_{db}) - (e \leftrightarrow f)$$

and

$$\sum_{\text{cycl}}\phi_{ab,cd,ef} = \left(\frac{\phi_1}{12} + \frac{\phi_2}{6}\right)\sum_{\text{cycl}}\eta_{fa}(\eta_{de}L_{cb} + \eta_{ce}L_{bd}) - (e \leftrightarrow f).$$

Hence the equations:

$$\frac{\phi_1}{24}(n-1) - \frac{\phi_2}{6} - \frac{\hbar^2}{12}(n-4) = 0,$$

$$-\frac{\phi_1}{6} + \frac{\phi_2}{6}(n-1) - \frac{k}{2}(n-1) + \frac{\hbar^2}{3} = 0,$$

$$\frac{\phi_1}{12} + \frac{\phi_2}{6} + \frac{\hbar^2}{3} = 0.$$

Finally, we obtain

$$\phi_1(n+1) = 2\hbar^2(n-8)$$
$$\phi_2(1+n) = -3\hbar^2(n-2)$$
$$k(n-1) = \hbar^2(n-4).$$

The proposition is proved.   □

Proof of Proposition 6.11. All Cartan subalgebras are isomorphic and any two systems of simple roots are related by a transformation of the Weyl group. The ideal determines only the infinitesimal character $\chi(\lambda) := \rho + \lambda$, where ρ is half the sum of the positive roots and λ is the highest weight, up to a Weyl reflection. It does not distinguish between weights that are related by a Weyl transformation of the infinitesimal character. In the case of $B_l = \mathfrak{so}_{2l+1}$ the formula is $\rho = (l - \frac{1}{2}, l - \frac{3}{2}, \ldots, \frac{1}{2})$.

The problem is to determine the possible values of the infinitesimal character. We begin with the relation

$$\sum_b L_{db} * L_{b'a} - \frac{\hbar}{2}(n-2)L_{da} + \frac{k}{2}(n-1)\eta_{da} = 0,$$

in the case $a+d=n+1$, $a<d$. Applied to the highest vector v_0, it gives

$$\left(\sum_{b\leq a} L_{db} * L_{b'a} + \frac{\hbar}{2}(n-2)\lambda_a + \frac{k}{2}(n-1)\right)v_0 = 0.$$

If $b>a$, then $b'<a'$ and $a+b'<n+1$, so $L_{b'a}v_0 = 0$. Thus

$$\left(\sum_{b<a}\{L_{db}, L_{b'a}\} + \lambda_a 2 + \frac{\hbar}{2}(n-2)\lambda_a + \frac{k}{2}(n-1)\right)v_0 = 0.$$

Finally, as the bracket is equal to $\hbar^2(\lambda_b - \lambda_a)$, for $b=1\ldots a-1$, we have

$$(1-a)\lambda_a + (\lambda_1 + \cdots + \lambda_{a-1}) + \lambda_a^2 + \frac{1}{2}(n-2)\lambda_a + \frac{1}{2}(n-4).$$

In particular

$$\lambda_1^2 + \frac{1}{2}(n-2)\lambda_1 + \frac{1}{2}(n-4) = 0,$$
$$(\lambda_1 + 1)\left(\lambda_1 + \frac{n-4}{2}\right) = 0.$$

Thus

$$(1-a)\lambda_a + (\lambda_1 + \cdots + \lambda_{a-1}) + \lambda_a^2 - \lambda_1^2 + \frac{n-2}{2}(\lambda_a - \lambda_1),$$

and

$$(\lambda_a - \lambda_{a-1})\left(\lambda_a + \lambda_{a-1} + \frac{1}{2}(n-a)\right) = 0. \tag{45}$$

We return to the relation

$$\sum_b L_{db} * L_{b'a} - \frac{\hbar}{2}(n-2)L_{da} + \frac{k}{2}(n-1)\eta_{da} = 0,$$

now in the case $a+d=n+2$, $2\leq a<d$. Applying this relation to the highest weight vector v_0, we have

$$\left(\sum_{b\leq a} L_{db} * L_{b'a} - \frac{\hbar}{2}(n-2)L_{da}\right)v_0 = 0,$$

$$\left(\sum_{b\leq a-2} L_{db} * L_{b'a} + L_{da} * L_{a'a} + L_{dd'}L_{da} - \frac{\hbar}{2}(n-2)L_{da}\right)v_0 = 0.$$

The first term can be replaced by the bracket $\sum_{a \leq a-2}[L_{db}, L_{b'a}] = \hbar \sum_{b \leq a-2} L_{da}$. So, for $a = 2 \ldots l$, we have the relation

$$\left((a-1)L_{da} - \lambda_a L_{da} - \lambda_{a-1} L_{da} - \frac{1}{2}(n-2)L_{da} \right) v_0 = 0.$$

Hence

$$\left(\lambda_a + \lambda_{a-1} + \frac{1}{2}(n-2a) \right) L_{da} v_0 = 0, \quad \forall a = 2 \ldots l.$$

Note that $\{L_{da}, L_{a'd'}\} = \hbar^2(\lambda_a - \lambda_{a-1})v_0$. The information contained in this last result is therefore precisely the same as in Equation (45), with $\lambda_2 = \cdots = \lambda_n = 0$. Then $L_{ab}v_0 = 0$, $a, b > 1$.

Next, the other relation,

$$\sum_{\text{cycl}(bcd)} (L_{ab} - \hbar\eta_{ab}) * L_{cd} = 0,$$

applied to v_0 in the case $a + b = c + d = n + 1$, $a < b$, $c < d$, $a \neq c$, gives

$$((\lambda_a - 1)\lambda_c + L_{ac} * L_{db} + L_{ad} * L_{bc}) v_0 = 0.$$

By evaluating this in two cases, $a > b$, $a < c$, we find that $\lambda_<$ is -1 or $\lambda_>$ is 0. This completes the proof. $\qquad\square$

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