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On the freeness of Böröczky line arrangements

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Abstract
In the present note, we focus on the freeness and some combinatorial properties of line arrangements in the projective plane having only double and triple points. The main result shows that for this class of line arrangements the freeness property is combinatorially determined. As a corollary, we show that Böröczky line arrangements in the sense of Füredi and Palásti (Proc Am Math Soc 92(4):561–566, 1984), except exactly three cases, are not free.

Keywords Line arrangements · Freeness · Hirzebruch inequality · Böröczky arrangements

Mathematics Subject Classification 52C35 · 51D20 · 13C10

1 Introduction

In the present note we focus on certain line arrangements in the projective plane over the real numbers. Our aim here is to show that almost whole family of Böröczky line arrangements is not free. Our result seems to be unraveled explicitly in the literature (according to our best knowledge), and we would like to fill this gap. Our result is actually much stronger since it allows to provide a complete characterization of free line arrangements having only double and triple points. Before we proceed to the core of the note, let us emphasize that there are some discrepancies in the literature around the notion of Böröczky arrangements of lines. For us Böröczky arrangements of lines are arrangements $\mathcal{B}_n$ defined in [3, Example 2] and they provide the maximal possible number of triple intersection points for arrangements defined over the real numbers. However, in [4, Proposition 2.1] the authors define four different classes of Böröczky arrangements of lines and for those Anzis and Tohăneanu proved their supersolvability and thus the freeness. Fortunately, there is no contradiction in our claim since at the end of the day we consider different objects. Now we present key ingredients, we will follow mostly [10,11].

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an essential and central hyperplane arrangement in $\mathbb{C}^k$, it means that $H_i = V(\ell_i)$ for homogeneous ℓ_i and $V(\ell_1, \ldots, \ell_n) = 0 \in \mathbb{C}^k$. The central condition means here that $\mathcal{A}$ also defines an arrangement in $\mathbb{P}^{k-1}(\mathbb{C})$. The main combinatorial object associated to $\mathcal{A}$ is the intersection lattice $L_{\mathcal{A}}$, which consists of the intersections of elements

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of $\mathcal{A}$, ordered by reverse inclusion. $\mathbb{C}^k$ is the lattice element $\widetilde{0}$ and the rank one elements of $L_{\mathcal{A}}$ are the hyperplanes. We recall the following definition and we refer to [7] (p. 15, 32–44) for details.

Definition 1.1 The Möbius function $\mu : L_{\mathcal{A}} \to \mathbb{Z}$ is defined by

$$\begin{cases} \mu(\widetilde{0}) = 1, \\ \mu(t) = -\sum_{s<t} \mu(s), \quad \text{if } \widetilde{0} < t. \end{cases}$$

Definition 1.2 The Poincaré and characteristic polynomials of $\mathcal{A}$ are defined as

$$\pi(\mathcal{A}, t) = \sum_{x \in L_{\mathcal{A}}} \mu(x) \cdot (-t)^{\operatorname{rank}(x)}, \text{ and } \chi(\mathcal{A}, t) = t^{\operatorname{rank}(\mathcal{A})} \pi\left(\mathcal{A}, \frac{-1}{t}\right).$$

Here $rank(x)$ is the codimension of element $x \in L_{\mathcal{A}}$, and $rank(\mathcal{A}) = max\{rank(x) : x \in L_{\mathcal{A}}\}$.

Let $\mathcal{A} = \bigcup_{i=1}^{n} H_i \subset \mathbb{C}^k$ be a central arrangement and for each $i \in \{1, \ldots, n\}$ we fix $V(\ell_i) = H_i \in \mathcal{A}$, and define $Q_{\mathcal{A}} = \prod_{i=1}^{n} \ell_i \in S = \mathbb{C}[x_1, \ldots, x_k]$.

Definition 1.3 The module of $\mathcal{A}$-derivations is the submodule of $\operatorname{Der}_{\mathbb{C}}(S)$ consisting of vector fields tangent to $\mathcal{A}$:

$$D(\mathcal{A}) = \{\theta \in \operatorname{Der}_{\mathbb{C}}(S) | \theta(\ell_i) \in \langle \ell_i \rangle \text{ for all } \ell_i \text{ with } V(\ell_i) \in \mathcal{A}\}.$$

Definition 1.4 An arrangement is free when $D(\mathcal{A})$ is a free S-module. In this case, the degrees of the generators of $D(\mathcal{A})$ are called the exponents of the arrangement.

In order to verify whether a certain arrangement is free, we can use the following famous result due to Saito [9, Theorem 1.8, statement ii)].

Theorem 1.5 *$\mathcal{A}$ is free iff there exist k elements*

$$\theta_i = \sum_{j=1}^{k} f_{ij} \frac{\partial}{\partial x_j} \in D(\mathcal{A})$$

such that $\det([f_{ij}]) = c \cdot Q_{\mathcal{A}}$ *for some* $c \neq 0$.

The famous open problem, due to Terao, tells us that the freeness is combinatorial in nature. For this conjecture, we can assume that the ground field is of characteristic zero.

Conjecture 1.6 *Freeness of $D(\mathcal{A})$ depends only on the intersection lattice $L_{\mathcal{A}}$.*

In the note we are going to use the following famous Terao Factorization Theorem [12].

Theorem 1.7 *If $D(\mathcal{A})$ is free, then*

$$\pi(\mathcal{A}, t) = \prod (1 + a_i t),$$

where a_i's are the exponents.

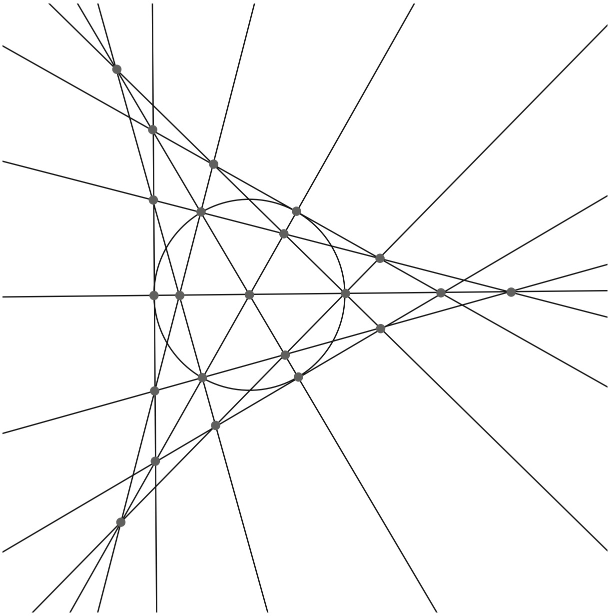

Fig. 1 Böröczky arrangement of 12 lines

2 Böröczky's construction

Let us recall briefly the main construction of Böröczky line arrangements [3].

In general, if we want to obtain the arrangement $\mathcal{B}_n$, which consists of n lines, we have to start with a regular $2n$-gon inscribed in a circle O. Then we denote the vertices of this $2n$-gon traced in clockwise order by $P_0, \ldots, P_{2n-1}$. In order to define the lines, we should note that we consider indices modulo $2n$. Let $\alpha \in \{0, 1, \ldots, n-1\}$. We get our n lines if in every α-step we join $P_{n-4\alpha}$ with $P_{2\alpha}$. Of course, it may happen that $P_{n-4\alpha}$ with $P_{2\alpha}$ coincide – then we draw the tangent line to O at $P_{2\alpha}$. As a result we obtain the arrangement $\mathcal{B}_n$ which consists of n lines, $n-3+\varepsilon$ double points, and $1+\left\lfloor \frac{n(n-3)}{6} \right\rfloor$ triple points, where ε is equal to 0 if $n = 0 \operatorname{mod}(3)$, or 2, otherwise. In Fig. 1. we depicted the case with $n = 12$.

3 The Poincaré polynomials of line arrangements with only double and triple points

Let us restrict our attention to the following setting: we assume that our line arrangements have only double and triple points as the intersections and these are defined over the complex numbers.

This class of arrangements plays an important role in different branches of algebraic geometry and combinatorics. To mention just a few of the appearance, these can be found in Hirzebruch's construction of surfaces which are ball-quotients [6], or around the so-called orchard problem and Dirac–Motzkin Conjecture [4].

Let us recall that for this class of line arrangements we have the following combinatorial equality, see [6, p. 114]

$$\binom{|\mathcal{A}|}{2} = t_2(\mathcal{A}) + 3t_3(\mathcal{A}).$$

Here $|\mathcal{A}|$ denotes the number of lines, and $t_i(\mathcal{A})$ denotes the number of i-fold points, i.e., points where i-lines meet, hence $t_i = 0$ for $i \geqslant 4$. In our setting, it is easy to observe that $\pi(\mathcal{A}, t)$ has the following form

$$\pi(A, t) = 1 + |\mathcal{A}|t + (t_2(\mathcal{A}) + 2t_3(\mathcal{A}))t^2 + (t_2(\mathcal{A}) + 2t_3(\mathcal{A}) + 1 - |\mathcal{A}|)t^3.$$

4 Results

The first one tells us that for our particular class of line arrangements the Poincaré polynomial characterizes the combinatorics, which is not the case in general.

Proposition 4.1 *Let $\mathcal{A}$, $\mathcal{B}$ be two line arrangements having only double and triple points as the intersections. Suppose that $\pi(\mathcal{A}, t) = \pi(\mathcal{B}, t)$, then $t_2(\mathcal{A}) = t_2(\mathcal{B})$ and $t_3(\mathcal{A}) = t_3(\mathcal{B})$.*

At the beginning, let us emphasize here that we can obtain an analogous result replacing triple points by arbitrary k-fold points with $k \geqslant 3$.

Proof Suppose that $\pi(\mathcal{A}, t) = \pi(\mathcal{B}, t)$. This implies in particular that

$$|\mathcal{A}| = |\mathcal{B}|$$

and

$$t_2(\mathcal{A}) + 2t_3(\mathcal{A}) = t_2(\mathcal{B}) + 2t_3(\mathcal{B}),$$

which gives

$$t_2(\mathcal{A}) - t_2(\mathcal{B}) = 2(t_3(\mathcal{B}) - t_3(\mathcal{A})).$$

Observe that

$$t_2(\mathcal{A}) + 3t_3(\mathcal{A}) = \binom{|\mathcal{A}|}{2} = \binom{|\mathcal{B}|}{2} = t_2(\mathcal{B}) + 3t_3(\mathcal{B}),$$

and this gives

$$t_2(\mathcal{A}) - t_2(\mathcal{B}) = 3(t_3(\mathcal{B}) - t_3(\mathcal{A})).$$

Since

$$3(t_3(\mathcal{B}) - t_3(\mathcal{A})) = t_2(\mathcal{A}) - t_2(\mathcal{B}) = 2(t_3(\mathcal{B}) - t_3(\mathcal{A})),$$

then this implies $t_3(\mathcal{A}) = t_3(\mathcal{B})$ and we also have $t_2(\mathcal{A}) = t_2(\mathcal{B})$, which completes the proof. □

The main result of the note is the following classification result.

Theorem 4.2 *Let $\mathcal{A}$ be a line arrangement having only double and triple points as the intersections. Suppose that $\mathcal{A}$ is free, then $3 \leqslant |\mathcal{A}| \leqslant 9$.*

Proof Suppose that $\mathcal{A}$ is free. Then by Terao's result, the Poincaré polynomial $\pi(\mathcal{A}, t)$ splits into linear factors over the integers. Observe that

$$\pi(\mathcal{A}, t) = (1+t) \cdot \left(1 + (|\mathcal{A}| - 1)t + (t_2(\mathcal{A}) + 2t_3(\mathcal{A}) + 1 - \mathcal{A})t^2\right),$$

and the quadratic factor also splits into linear factors. This condition implies that

$$(|\mathcal{A}| - 1)^2 - 4t_2(\mathcal{A}) - 8t_3(\mathcal{A}) + 4|\mathcal{A}| - 4 \geqslant 0.$$

Using the combinatorial equality

$$|\mathcal{A}|(|\mathcal{A}| - 1) = 2t_2(\mathcal{A}) + 6t_3(\mathcal{A})$$

one gets

$$3|\mathcal{A}| - 3 \geqslant 2t_2(\mathcal{A}) + 2t_3(\mathcal{A}) \geqslant 2|\mathcal{A}|,$$

where the last right-hand side inequality comes from Hirzebruch's inequality [6, p.132, Theorem]. This implies, in particular, that $|\mathcal{A}| \geqslant 3$. Again, using the combinatorial equality one obtains

$$|\mathcal{A}|(|\mathcal{A}| - 1) = 2t_2(\mathcal{A}) + 6t_3(\mathcal{A}) \leqslant 3(2t_2(\mathcal{A}) + 2t_3(\mathcal{A})) \leqslant 9|\mathcal{A}| - 9.$$

This provides the condition $|\mathcal{A}| \leqslant 9$, and finally we obtain $3 \leqslant |\mathcal{A}| \leqslant 9$, which completes the proof. □

Corollary 4.3 *Except the cases* $n = 4, 5, 6$*, Böröczky configurations of* n *lines are not free.*

Proof By the above result, it is enough to check the cases $n \in \{4, 5, 6, 7, 8, 9\}$. Since

$$\pi(\mathcal{A}, t) = 1 + |\mathcal{A}|t + (t_2(\mathcal{A}) + 2t_3(\mathcal{A}))t^2 + (t_2(\mathcal{A}) + 2t_3(\mathcal{A}) + 1 - |\mathcal{A}|)t^3,$$

we obtain the following:

$n = \|\mathcal{A}\|$	$t_2(\mathcal{A})$	$t_3(\mathcal{A})$	$\pi(\mathcal{A}, t)$
4	3	1	$1 + 4t + 5t^2 + 2t^3 = (t+1)^2(2t+1)$
5	4	2	$1 + 5t + 8t^2 + 4t^3 = (t+1)(2t+1)^2$
6	3	4	$1 + 6t + 11t^2 + 6t^3 = (t+1)(2t+1)(3t+1)$
7	6	5	$1 + 7t + 16t^2 + 10t^3 = (t+1)(10t^2 + 6t + 1)$
8	6	5	$1 + 8t + 21t^2 + 14t^3 = (t+1)(14t^2 + 7t + 1)$
9	6	10	$1 + 9t + 26t^2 + 18t^3 = (t+1)(18t^2 + 8t + 1)$

It is easy to see that for $n \in \{7, 8, 9\}$ we cannot factorize Poincaré polynomials into linear factors over the integers, so in these cases Böröczky arrangements are not free. When $n = 6$, then this arrangement is projectively equivalent to the well-known arrangement $\mathcal{A}_1(6)$ (see [5, p. 9]), which is known to be free. Now we focus on the remaining cases $n \in \{4, 5\}$ showing for them freeness explicitly. We will follow [2] and we compute the minimal free resolutions of the Milnor algebras. The formula (1.1) in [2] is crucial:

$$0 \to \oplus_{i=1,2} S(-a_i - (n-1)) \to S^3(-n+1) \to S.$$

If $n=4$, then the defining equation has the form $Q_4(x,y,z)=xy(y-x+z)(y+x-z)$. Denote by $J_{\mathcal{B}_4}$ the Jacobian ideal generated by the partials of Q_4, and by $S/J_{\mathcal{B}_4}$ the Milnor algebra. Then the resolution of $S/J_{\mathcal{B}_4}$ has the following form:

$$0 \to S(-4)\oplus S(-5) \to S^3(-3) \to S,$$

with the following relations:

$$\begin{aligned}
&3x\cdot\partial_x Q_4 - y\cdot\partial_y Q_4 + (4x-z)\cdot\partial_z Q_4 = 0,\\
&(4x^2-7xz)\cdot\partial_x Q_4 + (13yz-12xy)\cdot\partial_y Q_4 + (16y^2-3z^2)\cdot\partial_z Q_4 = 0.
\end{aligned}$$

This allows us to conclude that $\mathcal{B}_4$ is free.

Now consider the case $n=5$. As the coordinates of $P_0,\ldots,P_9$ (vertices of a regular 10-gon) in this case satisfy the condition $\{P_0,\ldots,P_9\}\subseteq\{(\pm(\frac{1}{4}\sqrt{5}\pm\frac{1}{4}),\pm\frac{1}{4}\sqrt{10\pm2\sqrt{5}})\}$, it is more convenient to change the coordinates and consider the following equivalent arrangement of lines given by

$$Q_5(x,y,z)=y(2y-x)(2y+3x)(x+2y-4z)(3x-2y-12z).$$

The minimal resolution of $S/J_{\mathcal{B}_5}$ has the following form:

$$0 \to S(-6)\oplus S(-6) \to S^3(-4) \to S,$$

with the following syzygies:

$$\begin{aligned}
&(3x^2-20xy-20y^2+24xz)\cdot\partial_x Q_5 + (24yz-12xy)\cdot\partial_y Q_5 + (18xz+20yz-10xy\\
&\quad -36z^2)\cdot\partial_z Q_5 = 0,\\
&(176xy+160y^2-288xz)\cdot\partial_x Q_5 + (120xy+16y^2-288yz)\cdot\partial_y Q_5 + (15x^2+100xy\\
&\quad -20y^2-240xz-224yz+432z^2)\cdot\partial_z Q_5 = 0,
\end{aligned}$$

which tells us that $\mathcal{B}_5$ is also free. □

Remark 4.4 Observe that our simple criterion allows us, for instance, to conclude automatically that Rybnikov's line arrangements [8] are not free.

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