

Roots of crosscap slides and crosscap transpositions

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Abstract Let N_g denote a closed nonorientable surface of genus g . For $g \geq 2$ the mapping class group $\mathcal{M}(N_g)$ is generated by Dehn twists and one crosscap slide (Y -homeomorphism) or by Dehn twists and a crosscap transposition. Margalit and Schleimer observed that Dehn twists on orientable surfaces have nontrivial roots. We give necessary and sufficient conditions for the existence of roots of crosscap slides and crosscap transpositions.

Keywords Mapping class group · Nonorientable surface · Punctured sphere · Elementary braid · Crosscap slide · Crosscap transposition

Mathematics Subject Classification 57N05 · 20F38 · 57M99

1 Introduction

Let $N_{g,s}^n$ be a connected nonorientable surface of genus g with s boundary components and n punctures, that is a surface obtained from a connected sum of g projective planes N_g by removing s open disks and specifying the set $\Sigma = \{p_1, \dots, p_n\}$ of n distinguished points in the interior of N_g . If s or/and n equals zero, we omit it from notation. The *mapping class group* $\mathcal{M}(N_{g,s}^n)$ consists of isotopy classes of self-homeomorphisms $h : N_{g,s}^n \rightarrow N_{g,s}^n$ fixing boundary components pointwise and such that $h(\Sigma) = \Sigma$. The mapping class group $\mathcal{M}(S_{g,s}^n)$ of an orientable surface is defined analogously, but we consider only orientation-preserving

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maps. If we allow orientation-reversing maps, we obtain the *extended mapping class group* $\mathcal{M}^\pm(S_{g,s}^n)$. By abuse of notation, we identify a homeomorphism with its isotopy class.

In the orientable case, the mapping class group $\mathcal{M}(S_g)$ is generated by Dehn twists [3]. As for nonorientable surfaces, Lickorish proved that Dehn twists alone do not generate $\mathcal{M}(N_g)$, $g \geq 2$. This group is generated by Dehn twists and one crosscap slide (Y -homeomorphism) [4].

A presentation for $\mathcal{M}(N_g)$ using these generators was obtained by Stukow [14]. This presentation was derived from the presentation given by Paris and Szepietowski [9], which used as generators Dehn twists and yet another homeomorphisms of nonorientable surfaces, so-called crosscap transpositions.

Margalit and Schleimer discovered a surprising property of Dehn twists: in the mapping class group of a closed orientable surface S_g of genus $g \geq 2$, every Dehn twist has a nontrivial root [5]. It is natural to ask if crosscap slides and crosscap transpositions also have a similar property. The main goal of this paper is to prove the following:

Main Theorem *In $\mathcal{M}(N_g)$ a nontrivial root of a crosscap transposition (resp. crosscap slide) exists if and only if $g \geq 5$ or $g = 4$ and the complement of the support of this crosscap transposition [or crosscap slide] is orientable.*

2 Preliminaries

2.1 Crosscap transpositions and crosscap slides

Let $N = N_g$ be a nonorientable surface of genus $g \geq 2$. Let α and μ be two simple closed curves on N intersecting in one point, such that α is two-sided and μ is one-sided. A regular neighborhood of $\mu \cup \alpha$ is homeomorphic to the Klein bottle with a hole denoted by K . A convenient model of K is a disk with 2 crosscaps, see Fig. 1. In this figure shaded disks represent crosscaps, thus the boundary points of these disks are identified by the antipodal map.

A *crosscap transposition* $U_{\mu,\alpha}$ specified by μ and α is a homeomorphism of K which interchanges two crosscaps keeping the boundary of K fixed [9]. It may be extended by the identity to a homeomorphism of N . If t_α is the Dehn twist about α (with the direction of the twist indicated by small arrows in Fig. 1), then $Y_{\mu,\alpha} = t_\alpha U_{\mu,\alpha}$ is a *crosscap slide* of μ along α , that is the effect of pushing μ once along α keeping the boundary of K fixed. Note that $U_{\mu,\alpha}^2 = Y_{\mu,\alpha}^2 = t_{\partial K}$.

Remark 2.1 If g is odd, then the complement of K in N_g is a nonorientable surface $N_{g-2,1}$, thus in this case every two crosscap slides [or crosscap transpositions] are conjugate. If g is even, then the complement of K in N_g can be either a nonorientable surface $N_{g-2,1}$ or an

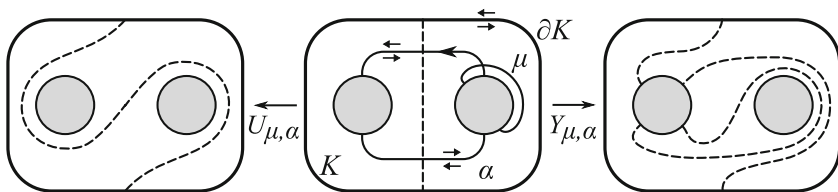


Fig. 1 A crosscap transposition and a crosscap slide

orientable surface $S_{\frac{g-2}{2}, 1}$, therefore in the mapping class group of a surface of even genus two conjugacy classes of crosscap slides and crosscap transpositions exist.

2.2 Notation

Represent N_g as a connected sum of g projective planes and let μ_1, \dots, μ_g be one-sided circles that correspond to crosscaps as indicated in Fig. 2. By abuse of notation, we identify μ_i with the corresponding crosscap.

If $\alpha_1, \dots, \alpha_{g-1}$ are two-sided circles indicated in the same figure, then for each $i = 1, \dots, g - 1$ by t_{α_i}, u_i, y_i we denote the Dehn twist about α_i , the crosscap transposition U_{μ_{i+1}, α_i} , and the crosscap slide Y_{μ_{i+1}, α_i} , respectively.

2.3 Relations in the mapping class group of a nonorientable surface

A full presentation for $\mathcal{M}(N_g)$ is given in [9, 14]. Among others, the following relations hold in $\mathcal{M}(N_g)$:

- (R1) $u_i u_j = u_j u_i$ for $i, j = 1, \dots, g - 1, |i - j| > 1,$
- (R2) $u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1}$ for $i = 1, \dots, g - 2,$
- (R3) $(u_1 u_2 \cdots u_{g-1})^g = 1,$
- (R4) $t_{\alpha_i} u_j = u_j t_{\alpha_i}$ and hence $y_i u_j = u_j y_i$ for $i, j = 1, \dots, g - 1, |i - j| > 1.$

It is straightforward to check that relations (R1)–(R3) imply

(R5) $(u_1^2 u_2 \cdots u_{g-1})^{g-1} = 1.$

Geometrically $u_1 u_2 \cdots u_{g-1}$ is a cyclic rotation of $\mu_1, \mu_2, \dots, \mu_g$ and $u_1^2 u_2 \cdots u_{g-1}$ is a cyclic rotation of $\mu_2, \mu_3, \dots, \mu_g$ around μ_1 . In particular,

(R6) $(u_1 u_2 \cdots u_{g-1})^g = (u_1^2 u_2 \cdots u_{g-1})^{g-1} = t_{\partial N_{g,1}}$ in $\mathcal{M}(N_{g,1}).$

We also have the following chain relation between Dehn twists (Proposition 4.12 of [1]): if $k \geq 2$ is even and c_1, \dots, c_k is a chain of simple closed curves on a surface S , such that the boundary of a closed regular neighborhood of their union is isotopic to d , then

(R7) $(t_{c_1} t_{c_2} \cdots t_{c_k})^{2k+2} = t_d.$

3 Proof of Main Theorem

Remark 3.1 Automorphisms of $H_1(N_g; \mathbb{R})$ induced by crosscap transpositions and crosscap slides have determinant equal to -1 , so if a root of a crosscap slide or a crosscap transposition exists, it must be of odd degree.

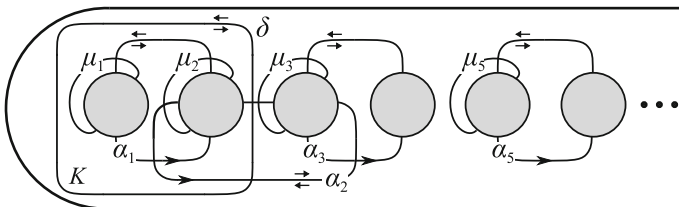


Fig. 2 A nonorientable surface N_g

Note that in order to prove Main Theorem, it is enough to prove it for some (arbitrary chosen) representatives of conjugacy classes of crosscap slides and crosscap transpositions (see Remark 2.1).

Let u and y be respectively a crosscap transposition and a crosscap slide supported in the same Klein bottle $K \subset N_g$ and let $\delta = \partial K$. In particular, $u^2 = y^2 = t_\delta$. If $N \setminus K$ is nonorientable, then we assume that $u = u_1, y = y_1$ and K is a Klein bottle which contains μ_1 and μ_2 (Fig. 2).

3.1 The case of $g \geq 5$ odd

Let $p, q \in \mathbb{Z}$ be such that $2p + q(g - 2) = 1$. By relations (R6) and (R1),

$$\begin{aligned} u_1^2 &= t_\delta = (u_3 u_4 \cdots u_{g-1})^{g-2} \\ u_1^{2p} &= (u_3 u_4 \cdots u_{g-1})^{p(g-2)} \\ u_1 &= ((u_3 u_4 \cdots u_{g-1})^p u_1^q)^{g-2}. \end{aligned}$$

Analogously, by relations (R6), (R1) and (R4), $y_1 = ((u_3 u_4 \cdots u_{g-1})^p y_1^q)^{g-2}$.

3.2 The case of $g \geq 6$ even and $N_g \setminus K$ nonorientable

Let $p, q \in \mathbb{Z}$ be such that $2p + q(g - 3) = 1$. By relations (R6) and (R1),

$$\begin{aligned} u_1^2 &= t_\delta = (u_3^2 u_4 \cdots u_{g-1})^{g-3} \\ u_1^{2p} &= (u_3^2 u_4 \cdots u_{g-1})^{p(g-3)} \\ u_1 &= ((u_3^2 u_4 \cdots u_{g-1})^p u_1^q)^{g-3}. \end{aligned}$$

Analogously, by relations (R6), (R1) and (R4), $y_1 = ((u_3^2 u_4 \cdots u_{g-1})^p y_1^q)^{g-3}$.

3.3 The case of $g \geq 4$ even and $N_g \setminus K$ orientable

Suppose now that $N_g \setminus K$ is orientable. If c_1, \dots, c_{g-2} is a chain of two-sided circles in $N_g \setminus K$, then by relation (R7),

$$\begin{aligned} u^2 &= t_{\partial K} = (t_{c_1} t_{c_2} \cdots t_{c_{g-2}})^{2g-2} \\ (u^2)^{\frac{g}{2}} &= ((t_{c_1} t_{c_2} \cdots t_{c_{g-2}})^{2g-2})^{\frac{g}{2}} \\ u &= ((t_{c_1} t_{c_2} \cdots t_{c_{g-2}})^g u^{-1})^{g-1}. \end{aligned}$$

Analogously, $y = ((t_{c_1} t_{c_2} \cdots t_{c_{g-2}})^g y^{-1})^{g-1}$.

3.4 The case of $g = 2$

Crosscap slides and crosscap transpositions are primitive in $\mathcal{M}(N_2)$ because [4]

$$\begin{aligned} \mathcal{M}(N_2) &\cong \langle t_{\alpha_1}, y_1 \mid t_{\alpha_1}^2 = y_1^2 = (t_{\alpha_1} y_1)^2 = 1 \rangle \\ &\cong \langle t_{\alpha_1}, u_1 \mid t_{\alpha_1}^2 = u_1^2 = (t_{\alpha_1} u_1)^2 = 1 \rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2. \end{aligned}$$

3.5 The case of $g = 3$

Remark 3.2 It is known that the mapping class group $\mathcal{M}(N_3)$ is hyperelliptic [15] and has the central element ϱ such that $\mathcal{M}(N_3)/\langle\varrho\rangle$ is the extended mapping class group $\mathcal{M}^\pm(S_0^{3,1})$ of a sphere with 4 punctures. Two upper subscripts mean that we have four punctures on the sphere, but one of them must be fixed. This implies [2] that the maximal finite order of an element in $\mathcal{M}^\pm(S_0^{3,1})$ is 3, and hence the maximal finite order of an element in $\mathcal{M}(N_3)$ is 6. Moreover, each of two rotations of order 3 in $\mathcal{M}^\pm(S_0^{3,1})$ are conjugate, which easily implies that each of two elements of order 6 in $\mathcal{M}(N_3)$ are conjugate. The details of the proof of the last statement are completely analogous to that used in [12], hence we skip them.

The same conclusion can also be obtained purely algebraically: it is known [11] that $\mathcal{M}(N_3) \cong \text{GL}(2, \mathbb{Z})$ and the maximal finite order of an element in $\text{GL}(2, \mathbb{Z})$ is 6. Moreover, there is only one conjugacy class of such elements in $\text{GL}(2, \mathbb{Z})$ —for details see for example Theorem 2 of [7].

We will show that crosscap transpositions do not have nontrivial roots in $\mathcal{M}(N_3)$. Suppose that $x \in \mathcal{M}(N_3)$ exists such that $x^{2k+1} = u_1$, where $k \geq 1$ (see Remark 3.1). Then

$$x^{2(2k+1)} = x^{4k+2} = u_1^2 = t_\delta = 1.$$

Note that the order of x can not be odd (because otherwise $u_1 = x^{2k+1} = 1$), it can not equal 2 (because otherwise $x = u_1$), and it is not divisible by 4. Hence, by Remark 3.2, x has order 6. Moreover, by relation (R7),

$$(t_{\alpha_1} t_{\alpha_2})^6 = t_d = 1,$$

where d is a boundary of a closed regular neighborhood of $\alpha_1 \cup \alpha_2$. Hence x is conjugate to $t_{\alpha_1} t_{\alpha_2}$ and this contradicts Remark 3.1, because Dehn twists induce automorphisms of $H_1(N_3; \mathbb{R})$ with determinant equal to 1 and $x^{2k+1} = u_1$.

In the case of a crosscap slide the argument is completely analogous, hence we skip the details.

3.6 The case of $g = 4$ and $N_4 \setminus K$ nonorientable.

If $N_4 \setminus K$ is nonorientable, then δ cuts N_4 into two Klein bottles with one boundary component: K and K_1 . Moreover, as was shown in Appendix A of [13],

$$\begin{aligned} \mathcal{M}(K) &= \langle t_{\alpha_1}, u_1 \mid u_1 t_{\alpha_1} = t_{\alpha_1}^{-1} u_1 \rangle \\ \mathcal{M}(K_1) &= \langle t_{\alpha_3}, u_3 \mid u_3 t_{\alpha_3} = t_{\alpha_3}^{-1} u_3 \rangle. \end{aligned}$$

By abuse of language, we say that $t_{\alpha_1}^k u_1^m \in \mathcal{M}(K)$ interchanges μ_1 and μ_2 if and only if m is odd. Equivalently, elements of $\mathcal{M}(K)$ which does not interchange μ_1 and μ_2 are precisely elements of the twist subgroup of $\mathcal{M}(K)$ (that is the subgroup of index 2 generated by Dehn twists). In the same way we define the notion of interchanging μ_3 and μ_4 for elements of $\mathcal{M}(K_1)$.

If $x \in \mathcal{M}(N_4)$ exists such that $x^{2k+1} = u_1$ and $k \geq 1$ (see Remark 3.1), then

$$x^{4k+2} = u_1^2 = t_\delta.$$

In particular, x commutes with t_δ and

$$t_\delta = x t_\delta x^{-1} = t_{x(\delta)}^\pm.$$

By Proposition 4.6 of [13], up to isotopy of N_4 , $x(\delta) = \delta$. Because u_1 does not interchange two sides of δ and does not reverse the orientation of δ , x has exactly the same properties. Therefore, we can assume that x is composed of maps of K and K_1 . Moreover $u_1 = x^{2k+1}$ interchanges μ_1 and μ_2 and does not interchange μ_3 and μ_4 , hence

$$\begin{aligned} x &= t_{\alpha_1}^{k_1} u_1^{2m_1+1} t_{\alpha_3}^{k_2} u_3^{2m_2} = t_{\alpha_1}^{k_1} u_1 t_{\alpha_3}^{k_2} t_{\delta}^{m_1+m_2} \\ x^2 &= t_{\alpha_3}^{2k_2} t_{\delta}^{2m_1+2m_2+1}. \end{aligned}$$

But then

$$t_{\delta} = (x^2)^{2k+1} = t_{\alpha_3}^{2k_2(2k+1)} t_{\delta}^{(2m_1+2m_2+1)(2k+1)}$$

and Proposition 4.4 of [13] implies that $k_2 = 0$ and $(2m_1 + 2m_2 + 1)(2k + 1) = 1$ which is a contradiction.

In the case of a crosscap slide the argument is completely analogous, hence we skip the details.

Remark 3.3 As was shown by McCullough, Rajeevsarathy [6] and independently by Monden [8], roots of Dehn twists constructed by Margalit and Schleimer [5] are of maximal possible degree. Since the geometric nature of our constructions of roots of crosscap slides and crosscap transpositions is quite similar to that of Margalit and Schleimer, it is natural to ask if the roots constructed in the proof of Main Theorem are also of maximal degree. We plan to answer this question in subsequent work, but it turns out that even in the case of Dehn twists on nonorientable surfaces, the study of possible degrees of their roots is significantly more complicated than in the orientable case—see for example Sect. 4 [10].

4 Roots of elementary braids in the mapping class group of n -punctured sphere

Margalit and Schleimer observed in [5] that if $g \geq 5$, then roots of elementary braids in $\mathcal{M}(S_0^g)$ exist. Main Theorem implies slightly stronger version of that result.

Corollary 4.1 *An elementary braid in the mapping class group $\mathcal{M}(S_0^n)$ or in the extended mapping class group $\mathcal{M}^{\pm}(S_0^n)$ of n -punctured sphere has a nontrivial root if and only if $n \geq 5$.*

Proof By Proposition 2.4 of [9], there is a monomorphism

$$\varphi: \mathcal{M}^{\pm}(S_0^g) \rightarrow \mathcal{M}(N_g)$$

which is induced by blowing up each puncture to a crosscap. In particular, this monomorphism sends elementary braids to crosscap transpositions. Moreover, all roots of crosscap transpositions constructed in the proof of Main Theorem are elements of $\varphi(\mathcal{M}(S_0^g))$. \square

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