DISTRIBUTION OF EIGENVALUES OF NUCLEAR OPERATORS AND GROTHENDIECK–LIDSKII TYPE FORMULAS

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We study spectral properties of quasinormed operator ideals $N_{r,p}$, $r \in (0, 1]$, $p \in [1, 2]$, and the corresponding approximation properties $AP_{r,p}$. We prove that the ideal $N_{r,p}$ is of spectral type s , where $1/s = 1/r - 1/p + 1/2$, and obtain new trace formulas similar to the classical Grothendieck–Lidskii formulas. We obtain a negative answer to the following question: for $0 < s < 1$ and $T \in L(X, X)$ whether an operator T is s -nuclear if its adjoint T^ is s -nuclear? Bibliography: 19 titles.*

Introduction

This paper is a continuation of [1, 2], where the classical trace formulas due to Grothendieck [3] and Lidskii [4] were generalized (cf. also [5]).

Theorem A (cf. [2]). *Let Y be a subspace of the quotient space of L_p for some $1 \leq p \leq \infty$, $1/s = 1 + |1/2 - 1/p|$. If an operator $T : Y \rightarrow Y$ admits the representation*

$$T = \sum_i \lambda_i x'_i \otimes x_i,$$

where $(\lambda_i) \in l_s$ and $(x'_i), (x_i)$ are bounded sequences, then the following assertions hold:

1) *the trace of T is fully defined,*

2) $\sum_{n=1}^{\infty} |\lambda_n(T)| < \infty$, *where $\{\lambda_n(T)\}$ is the family of all eigenvalues (with the algebraic*

multiplicity taken into account) of T and trace $T = \sum_{n=1}^{\infty} \lambda_n(T)$.

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Theorem B (cf. [1]). *Suppose that $r \in (0, 1]$, $1 \leq p \leq 2$, and $u \in X^* \widehat{\otimes} X$ admits the representation*

$$u = \sum_i \lambda_i x'_i \otimes x_i,$$

where $(\lambda_i) \in l_r$, (x'_i) is a bounded sequence, and $(x_i) \in l_{p'}^w(X)$. If $1/r + 1/2 - 1/p = 1$, then the family (μ_k) of all eigenvalues (with the algebraic multiplicity taken into account) of the corresponding operator $\tilde{u}$ is absolutely summable and $\text{trace } u = \sum_k \mu_k$.

The Crothendieck theorem and the Lidskii theorem about traces of nuclear operators in Hilbert spaces are particular cases of Theorem A, and the Pietsch theorem [6] about the ideal $N_{1,2}$ is a particular case of Theorem B.

The paper is organized as follows. Section 1 contains the notation and preliminary facts. In Section 2, we introduce the class of Banach spaces W possessing the so-called property $(2; p)$. $L_q(\mu)$ -spaces can serve as examples of such spaces. In Section 3, we prove assertions about traces and determinants of operators in operator ideals. Based on these assertions, we are able to strengthen the results on distribution of eigenvalues of operators in subspaces of the L_q -space, obtained earlier by the authors. In the same section, we prove the Crothendieck–Lidskii type trace formula. For this purpose we use, in fact, only the density of finite-dimensional operators in the ideal $N_{r,p}$. In Section 4, we study spectral properties of $N_{r,p}$ and prove the main theorem about spectral types of $N_{r,p}$. As a consequence, we obtain one more Crothendieck–Lidskii type trace formula. We also introduce important approximation properties $AP_{r,p}$ of Banach spaces and show that, under certain relations between r and s , any Banach space possesses such an approximation property. Three examples in Section 5 show that all the results of Section 4 are sharp in a certain sense. As a colloateral result of our consideration, we answer Pietsch’ question: for $s \in (0, 1)$ and $T : X \rightarrow X$ is it true that the operator T is s -nuclear if its adjoint T^* is s -nuclear? In Section 6, we prove that any Banach space possesses property $AP_{r,p}$ (under certain relations between r and p); moreover, only the results of Section 3 are used in the proof.

1 Preliminaries

All terms and facts given below without explanations can be found in [3] and [6]–[12]. Let X and Y be Banach spaces. We denote by X^* the adjoint of X . If $x \in X$ and $x' \in X^*$, then we write $\langle x', x \rangle$ for $x'(x)$. We denote by $X^* \widehat{\otimes} Y$ the completion of the tensor product $X^* \otimes Y$ considered as a linear space of all finite-dimensional operators from X to Y relative to the norm (cf., for example, [3, 6])

$$\|w\| := \inf \left\{ \left(\sum_{k=1}^N \|x'_k\| \|y_k\| \right) : w = \sum_{k=1}^N x'_k \otimes y_k \right\}.$$

In the case $X = Y$, the natural continuous linear trace functional on $X^* \otimes X$ admits a unique continuous extension to the space $X^* \widehat{\otimes} X$, also called the “trace.”

We denote by $N(X, Y)$ the image of the tensor product $X^* \widehat{\otimes} Y$ in the space $L(X, Y)$ of all bounded linear mappings under the canonical quotient mapping $X^* \widehat{\otimes} Y \rightarrow N(X, Y) \subset L(X, Y)$. We consider the (Grothendieck) space $N(X, Y)$ of all nuclear operators from X to Y , equipped with the natural norm induced from $X^* \widehat{\otimes} Y$. For a tensor element $u \in X^* \widehat{\otimes} Y$ we denote by $\tilde{u}$

the corresponding nuclear operator from X to Y . Sometimes, the norm of the projective tensor product is denoted by π and the injective norm (i.e., the norm induced by the usual operator norm) is denoted by ε . For example, $X^* \widehat{\otimes}_\pi Y = X^* \widehat{\otimes} Y$ and $X^* \widehat{\otimes}_\varepsilon Y$ is the closure of the set of finite-dimensional operators in $L(X, Y)$.

For $q \in (0, +\infty]$ we denote by $l_q^w(X)$ the space of all weakly q -summable sequences $(x_i) \subset X$ (cf., for example, [6, 10]), equipped with the quasinorm

$$\varepsilon_q((x_i)) := \sup \left\{ \left(\sum_i |\langle x', x_i \rangle|^q \right)^{1/q} : x' \in X^*, \|x'\| \leq 1 \right\}$$

(for $q = \infty$ we assume that (x_i) is bounded, i.e., $\varepsilon_\infty((x_i)) = \sup_i \|x_i\|$).

We introduce the standard notation: $l_p(X)$ is the space of absolutely summable sequences in X , $L(X) := L(X, X)$, Π_p is the ideal of absolutely p -summable operators, and N_s , $s \in (0, 1]$, is the quasinormed ideal of s -nuclear operators. If α is the tensor norm, then $X^* \widehat{\otimes}_\alpha Y$ denotes the completion of $X^* \otimes Y$ relative to the norm α . The norm in a Banach space X is denoted by $\|\cdot\|$ or $\|\cdot\|_X$. Sequences of elements are denoted by (x_k) , $(x_k)_k$, $(x_k)_{k=1}^\infty$, $\{x_k\}$, and so on.

We note that the Fredholm determinant $\det(1 - zu)$ of an element $u \in X^* \widehat{\otimes} X$ is an entire function with zeros $\mu_k^{-1}(\tilde{u})$ (with multiplicity taken into account) which are inverse to nonzero eigenvalues (with multiplicity taken into account) of the operator $\tilde{u}$.

2 Banach Spaces with Property $(2; p)$

We say that a Banach space W possesses property $(2; p)$, where $p \geq 2$, if

$$\{T \in L(l_2, W), T^* \in \Pi_2(W^*, l_2)\} \implies \{T \in \Pi_p(l_2, W)\}.$$

Remark 2.1. If a space W possesses property $(2; p)$, then each subspace of W also possesses this property (because the ideal Π_q is injective).

The following assertion provides us with important examples.

Proposition 2.1. *If $p \geq 2$, then any space $L_p(\mu)$ constructed on some measurable space with positive measure possesses property $(2; p)$.*

Proof. For the sake of brevity we write L_u instead of $L_u(\mu)$. For $p \geq 2$ we consider an operator $S : l_2 \rightarrow L_p$. Let $S^* \in \Pi_2(L_{p'}, l_2)$. Since $\Pi_2 = I_2 \subset I_p$, the operator S^* is a p -integral operator from $L_{p'}$ to l_2 , i.e., S^* can be factorized as

$$S^* : L_{p'} \xrightarrow{A} C(K) \xrightarrow{j} L_p(\nu) \xrightarrow{B} l_2,$$

where A and B are continuous mappings, K is a separable compact space, ν is the Radon probability measure on K , and j is the identity mapping. Consequently, the operator $(jA)^*$ is order bounded. The same is true for $(BjA)^*$ and S . However, any order bounded operator taking values in L_p is p -integral and, in particular, absolutely p -summable. $\square$

Lemma 2.1. *Suppose that $s \in (0, 1]$, $p \in [2, \infty]$, and $1/s = 1/q + |1/2 - 1/p|$. Let Y be a subspace of the quotient space of a Banach space W possessing property $(2; p)$. Then the family of all eigenvalues (with the algebraic multiplicity taken into account) of any operator $U \in N_s(Y, Y)$ belongs to the space l_q .*

Proof. Suppose that an operator $U \in N_s(Y, Y)$ admits the s -nuclear representation

$$U = \sum_{k=1}^{\infty} \mu_k y'_k \otimes y_k,$$

where

$$\|y'_k\|, \|y_k\| = 1, \quad \mu_k \geq 0, \quad \sum_{k=1}^{\infty} \mu_k^s < \infty.$$

The operator U can be factorized as

$$U : Y \xrightarrow{A} l_{\infty} \xrightarrow{\Delta_{1-s}} l_r \xrightarrow{j} c_0 \xrightarrow{\Delta_s} l_1 \xrightarrow{B} Y,$$

where A and B are linear bounded mappings, j is the natural embedding, $\Delta_s \sim (\mu_k^s)_k$ and $\Delta_{1-s} \sim (\mu_k^{1-s})_k$ are natural diagonal operators from c_0 to l_1 and from l_{∞} to l_r respectively. Here, r is found from the relations $1/s = 1/q - 1/p + 1/2$ and $\sum_k \mu_k^s < \infty$. It is required that

$$\sum_k \mu_k^{(1-s)r} < \infty.$$

For this purpose we can choose r such that $(1-s)r = s$. Therefore, we set $1/r = 1/s - 1$ or $1/r = 1/q - 1 - 1/p + 1/2$.

We note that if $s = 1$ and $q = 1$, then $r = \infty$, $p = 2$, and $j\Delta_{1-s} \equiv j$, whereas if $s = 2/3$ and $q = 1$, then $r = 2$, $p = \infty$, and $\Delta_{1-s} \sim (\mu_k^{1/3})_k \in l_2$. We represent the diagonal operator Δ_s as

$$\Delta_s = \Delta_2 \Delta_1 : c_0 \xrightarrow{\Delta_1} l_2 \xrightarrow{\Delta_2} l_1,$$

where Δ_1 belongs to $\Pi_2(c_0, l_2)$ and Δ_2^* belongs to $\Pi_2(l_{\infty}, l_2)$ respectively.

Case (i) Y is a subspace of W . We denote by $l : Y \hookrightarrow W$ an isomorphic embedding of Y into W . Then

$$\Delta_2^* B^* l^* : W^* \xrightarrow{l^*} Y^* \xrightarrow{B^*} l_{\infty} \xrightarrow{\Delta_2^*} l_2$$

is a mapping from Π_2 and, consequently, $lB\Delta_2\Pi_p(l_2, W)$ (by the assumptions on W). Thus,

$$B\Delta_2 : l_2 \xrightarrow{\Delta_2} l_1 \xrightarrow{B} Y$$

is an absolutely p -sumnable operator.

Case (ii) Y is the quotient space of W . Denoting by $q : W \rightarrow Y$ the quotient mapping from W to Y and considering the lifting $Q : l_1 \rightarrow W$ of B such that $B = qQ$, we conclude that the mapping

$$\Delta_2^* Q^* : W^* \xrightarrow{Q^*} l_{\infty} \xrightarrow{\Delta_2^*} l_2$$

belongs to Π_2 and, consequently, its preadjoint

$$Q\Delta_2 : l_2 \xrightarrow{\Delta_2} l_1 \xrightarrow{Q} L_p$$

is absolutely p -sumnable. Hence

$$B\Delta_2 : l_2 \xrightarrow{\Delta_2} l_1 \xrightarrow{Q} L_p \xrightarrow{q} Y$$

is an absolutely p -summable mapping. The aforesaid implies that, in all the cases, the operator $U : Y \rightarrow Y$ can be represented as the superposition of three operators: $T = U_1 U_2 U_3$, where $U_3 \in \Pi_r$, $U_2 \in \Pi_2$, $U_1 \in \Pi_p$; moreover, all the exponents are not less than 2. Finally, $1/r + 1/2 + 1/p = (1/q - 1 - 1/p + 1/2) + 1/2 + 1/p = 1/q$, which completes the proof of the lemma. $\square$

Lemma 2.2. *Let W be a Banach space. Any subspace of the quotient space of W is isomorphic to some quotient space of some subspace of W . The converse assertion is also valid.*

Proof. 1. Suppose that $V \subset Y \subset W$, $Z := Y/V$, $j : Y \rightarrow W$ is the identity embedding, and $\hat{j} : Y/V \rightarrow \hat{Z} := W/V$ is the induced embedding. Then Z is a subspace of the quotient space (namely, $\hat{Z}$).

2. Suppose that $S \subset W$, $P \subset R := W/S$, and $\psi : W \rightarrow R$ is the quotient mapping (i.e., P is a subspace of the quotient space). We set $W_0 := \psi^{-1}(P)$. Then $W_0 \subset W$ and $\psi|_{W_0} : W_0 \rightarrow P$ is the quotient mapping from a subspace of W to P (i.e., P is the quotient space of a subspace). $\square$

Lemma 2.3. *Suppose that $s \in (0, 1]$, $p \in [2, \infty]$, and $1/s = 1/q + |1/2 - 1/p|$. Let Y be a subspace of a Banach space W with property $(2; p)$, and let Z be the quotient space of Y . Then the family of all eigenvalues (with the algebraic multiplicity taken into account) of any operator $U \in N_s(Z, Z)$ belongs to the space l_q .*

Proof. Let $U \in N_s(Z, Z)$. We consider the factorization:

$$U : Z \xrightarrow{A} l_\infty \xrightarrow{\Delta_{1-s}} l_r \xrightarrow{j} c_0 \xrightarrow{\Delta_s} l_1 \xrightarrow{B} Z,$$

where A and B are linear bounded mappings, j is the natural embedding, Δ_s and Δ_{1-s} are natural diagonal operators with diagonal elements in l_1 and l_r respectively. Denoting by ψ the quotient mapping from Y onto Z , we take the lifting $\varphi : l_1 \rightarrow Y$ of B such that $B = \psi\varphi$. The mapping $U_0 := \varphi\Delta_s j\Delta_{1-s}A\psi$ acts from a subspace Y of W into itself belongs to the ideal N_r and has the same set of eigenvalues (with multiplicity taken into account) as the operator U . Using Lemma 2.1, we complete the proof. $\square$

Lemmas 2.2 and 2.3 imply the following assertion.

Lemma 2.4. *Suppose that $s \in (0, 1]$, $p \in [2, \infty]$, and $1/s = 1/q + |1/2 - 1/p|$. Let Z be a subspace of the quotient space of a Banach space W with property $(2; p)$. Then the family of all eigenvalues (with the algebraic multiplicity taken into account) of any operator $U \in N_s(Z, Z)$ belongs to the space l_q .*

The above lemmas lead to the following assertion.

Proposition 2.2. *Suppose that $s \in (0, 1]$, $p \in [2, \infty]$, and $1/s = 1/q + |1/2 - 1/p|$. Let W be a Banach space with property $(2; p)$. Then for the quotient space Y of a subspace (or, which is equivalent, for any subspace Y of the quotient space) of W the family of all eigenvalues (with the algebraic multiplicity taken into account) of any operator $U \in N_s(Y, Y)$ belongs to the space l_q .*

Based on examples following from Proposition 2.1, we arrive at the following assertion.

Proposition 2.3. *Suppose that $s \in (0, 1]$, $p \in [2, \infty]$, and $1/s = 1/q + |1/2 - 1/p|$. If Y is a subspace of the quotient space (or the quotient space of a subspace) of $L_p(\mu)$ constructed on some measurable space with positive measure, then the family of all eigenvalues (with the algebraic multiplicity taken into account) of any operator $U \in N_s(Y, Y)$ belongs to the space l_q .*

Using the duality between $L_u(\mu)$ type spaces, we obtain the following important consequence of the above-mentioned results.

Theorem 2.1. *Suppose that Let $s \in (0, 1]$, $p \in [1, \infty]$ and $1/s = 1/q + |1/2 - 1/p|$. If Y is a subspace of the quotient space (or the quotient space of a subspace) of $L_p(\mu)$ constructed on some measurable space with positive measure, then the family of all eigenvalues (with the algebraic multiplicity taken into account) of any operator $U \in N_s(Y, Y)$ belongs to the space l_q .*

Theorem 2.1 implies the lemma in [2] which is formulated below in a more general form.

Corollary 2.1. *Suppose that $p \in [1, \infty]$ and $1 = 1/q + |1/2 - 1/p|$. If Y is a subspace of the quotient space (or the quotient space of a subspace) of $L_p(\mu)$, then the family of all eigenvalues (with the algebraic multiplicity taken into account) of any operator $U \in N_s(Y, Y)$ is absolutely summable.*

3 Application to Trace Formulas

This section contains some necessary facts of the theory of traces and determinants.

Proposition 3.1. *Let A be an s -normed operator ideal ($s \in (0, 1]$), and let X be a Banach space such that the set of finite-dimensional operators is dense in $A(X)$. Assume that the standard functional trace is bounded on the subspace of all finite-dimensional operators in $A(X)$. Then the corresponding Fredholm determinant is uniformly continuous in the A -quasinorm on some A -ball in the subspace of all finite-dimensional operators in $A(X)$. Furthermore, there exist constants $r_0 \in (0, 1)$ and $c_0 > 0$ such that if $u, v \in A(X)$, $\|u\|_A \leq r_0$, and $\|v\|_A \leq r_0$, then*

$$|\det(1 - u) - \det(1 - v)| \leq c_0 \|u - v\|_A.$$

Proof. We denote by b a constant such that $|\text{trace } R| \leq b\|R\|_A$ for any finite-dimensional operator R in A . Let u and v be two finite-dimensional operators in A such that $\|u\|_A^s \leq r$ and $\|v\|_A^s \leq r$, where $r > 0$ is a small number. Then (cf., for example, [7, Theorem I.3.3] and [3]) for $|z| \leq 1$

$$\det(1 - zu) = \exp \left(- \sum_{n=1}^{\infty} \frac{1}{n} \text{trace}(u^n) z^n \right)$$

which implies

$$\begin{aligned} |\det(1 - u) - \det(1 - v)| &= \left| \exp \left(\sum_{n=1}^{\infty} \left(-\frac{1}{n} \right) \text{trace}(u^n) - \sum_{n=1}^{\infty} \left(-\frac{1}{n} \right) \text{trace}(v^n) \right) \right| \\ &\leq c_1 \sum_{n=1}^{\infty} \frac{1}{n} |\text{trace}(u^n) - \text{trace}(v^n)| \leq c_1 b \sum_{n=1}^{\infty} \frac{1}{n} \|u^n - v^n\|_A \end{aligned}$$

for small $r > 0$, where c_1 is a constant. If $q := \max\{\|u\|_A; \|v\|_A\}$, then

$$\begin{aligned} \|u^n - v^n\|_A^s &\leq \|(u^{n-1} - v^{n-1})u\|_A^s + \|v^{n-1}(u - v)\|_A^s \\ &\leq \|u\|_A^s [\|(u^{n-2} - v^{n-2})u\|_A^s + \|v^{n-2}(u - v)\|_A^s] + \|v^{n-1}\|_A^s \|u - v\|_A^s. \end{aligned}$$

We used the inequality $\|HK\|_A \leq \|H\|_A \|K\|_A$ for any H and K in A , which is valid because $\|K\|_L \leq \|K\|_A$. Continuing the inequality, we obtain

$$\begin{aligned}
\|u^n - v^n\|_A^s &\leq (q^{(n-1)s} \|u - v\|_A^s + q^s \cdot q^{s(n-2)} \|u - v\|_A^s) + q^{2s} \|u^{n-2} - v^{n-2}\|_A^s \\
&\leq 2q^{s(n-1)} \|u - v\|_A^s + q^{2s} [\|(u^{n-3} - v^{n-3})u\|_A^s + \|v^{n-3}(u - v)\|_A^s] \\
&\leq 3q^{s(n-1)} \|u - v\|_A^s + q^{3s} \|u^{n-3} - v^{n-3}\|_A^s \leq \dots \\
&\leq (n-1)q^{s(n-1)} \|u - v\|_A^s + q^{s(n-1)} \|u - v\|_A^s = nq^{s(n-1)} \|u - v\|_A^s.
\end{aligned}$$

Thus,

$$\|u^n - v^n\|_A \leq n^{1/s} q^{(n-1)} \|u - v\|_A.$$

Therefore, if $r \in (0, 1)$ is sufficiently small and $\|u\|_A^s \leq r$, $\|v\|_A^s \leq r$, then

$$\begin{aligned}
|\det(1 - u) - \det(1 - v)| &\leq c_1 b \sum_{n=1}^{\infty} \frac{1}{n} \|u^n - v^n\|_A \\
&\leq c_1 b \sum_{n=1}^{\infty} n^{1/s-1} r^{n-1} \|u - v\|_A = c_0 \|u - v\|_A. \quad \square
\end{aligned}$$

Corollary 3.1. *Under the assumptions of Proposition 3.1, the function $\det(1 - u)$ admits a continuous extension (in the A -quasinorm) from the subspace of all finite-dimensional operators in $A(X)$ to the entire space $A(X)$.*

This assertion follows from the uniform continuity (in the A -quasinorm) on some A -ball in the subspace of all finite-dimensional operators in $A(X)$. We omit the proof (cf. [7, p. 28]).

Proposition 3.2. *Let A be an s -normed operator ideal ($s \in (0, 1]$), and let X be a Banach space such that the set of finite-dimensional operators is dense in $A(X)$. Assume that the standard functional $\det(1 + u)$ admits a continuous extension from the subspace of all finite-dimensional operators in $A(X)$ to the entire space $A(X)$ relative to the $A(X)$ -quasinorm. Then the corresponding functional trace is bounded in the A -quasinorm in the subspace of all finite-dimensional operators in $A(X)$ and thereby is uniquely extended by continuity to the entire space $A(X)$.*

Remark 3.1. To prove Proposition 3.2, it suffices to assume that the determinant is continuous at zero.

Proof of Proposition 3.2. For a finite-dimensional operator $u \in A(X)$

$$\det(1 + zu) = 1 + z \operatorname{trace} u + \sum_{n=1}^m a_n z^n.$$

By the residue theorem,

$$\operatorname{trace} u = \frac{1}{2\pi i} \int_{|z|=1} \frac{\det(1 + zu) - 1}{z^2} dz.$$

Since $\det(1 + zu)$ continuous at the point $u = 0$ (in the A -quasinorm), there exists $\delta > 0$ such that $|\det(1 + zu) - 1| < 1$ for $\|u\|_A < \delta$ and $|z| \leq 1$. Therefore,

$$|\operatorname{trace} u| \leq \frac{1}{\pi} \int_{|z|=1} \left\| \frac{\det(1 + zu) - 1}{z^2} \right\| |dz| \leq 1. \quad \square$$

Now, we apply the above results to particular situations. We say that a Banach space X possesses *property* AP_s , $s \in (0, 1]$, if the identity mapping $X^* \widehat{\otimes}_s X \rightarrow N_s(X)$ is one-to-one.

For example, any subspace of an L_p type space possesses property AP_s for $1/s = 1 + |1/p - 1/2|$. The same is valid for the quotient space of a subspace of an L_p type space (cf., for example, [2, 13, 14]).

We denote by $\mathfrak{F}_{AP_s}$ the quasiideal of operators that are factorized via Banach spaces with property AP_s , equipped with the natural quasinorm. The completeness of the ideal is proved in a standard way by using the completeness criterion in [6]. We need to use only the following simple lemma. The proof of the lemma is left to the reader.

Lemma 3.1. *If $\{X_n\}$ is a sequence of Banach spaces with property AP_s , $0 < s \leq 1$, and $1 \leq p < \infty$, then the space $\left(\sum_{n=1}^{\infty} X_n\right)_{l_p}$ possesses property AP_s .*

The same assertion is valid for sums of type c_0 .

Proposition 3.3. *Let $s \in (0, 1]$, and let $\mathcal{F}$ be a family of Banach spaces with property AP_s , closed with respect to taking countable l_p -sums ($1 \leq p < \infty$). If for $X \in \mathcal{F}$ the space $N_s(X)$ is of spectral type l_t , where $t > 0$, then there exists a constant $C > 0$ such that for any $X \in \mathcal{F}$*

$$\|\{\mu_k(T)\}\|_{l_t} \leq C \|T\|_{N_s},$$

where $\{\mu_k(T)\}$ is the complete family of eigenvalues of the operator T .

Proof. Assume the contrary. Then for every n there is a Banach space $X_n \in \mathcal{F}$ and an operator $T_n \in N_s(X_n)$ such that $\|\{\mu_k(T_n)\}\|_{l_t} \geq n$ and $\|T_n\|_{N_s} \leq (2\nu_s)$, where ν_s is the constant in the “triangle inequality” for the quasinorm in N_s . We set

$$X := \left(\sum_{n=1}^{\infty} X_n\right)_{l_p}.$$

Let $j_n : X \rightarrow X_n$ and $i_n : X_n \rightarrow X$ be natural projections and embeddings. Then

$$\left\| \sum_{n=m+1}^{m+l} j_n T_n i_n \right\|_{N_s} \leq \sum_{k=1}^{\infty} \nu_c^k \|T_{m+k}\|_{N_s} \leq (2\nu_s)^{-m}, \quad l > 0.$$

Therefore,

$$T := \sum_{n=1}^{\infty} j_n T_n i_n \in N_s(X).$$

Since $T_n = j_n T i_n$, all eigenvalues of the operator T_n are contained in the family $\{\mu_k(T)\}$. Hence

$$\|\{\mu_k(T)\}\|_{l_t} \geq \|\{\mu_k(T_n)\}\|_{l_t} \geq n, \quad n = 1, 2, \dots$$

We obtain a contradiction, which proves the required assertion. $\square$

Proposition 3.4. *Let $s \in (0, 1]$, and let $\mathcal{F}$ be a family of Banach spaces with property AP_s , closed with respect to taking countable l_p -sums ($1 \leq p < \infty$). If for any $X \in \mathcal{F}$ the space $N_s(X)$ is of spectral type l_1 , then for any space $X \in \mathcal{F}$ and any operator $T \in N_s(X)$ the nuclear trace T is fully defined and coincides with its spectral trace, i.e.,*

$$\text{trace } T = \sum_{k=1}^{\infty} \mu_k(T),$$

where $\{\mu_k(T)\}$ is the complete family of eigenvalues (with multiplicity taken into account) of T .

Proof. Let $T \in N_s(X)$, where $X \in \mathcal{F}$. Since $X \in AP_s$, we have $N_s(X) = X^* \widehat{\otimes}_s X$, which guarantees the existence of a unique continuous trace on $N(X)$. This trace is a continuous extension of the usual trace functional from the subspace of all finite-dimensional operators in X . By Corollary 3.1, on $N_s(X)$ there is a unique continuous (Fredholm) determinant $\det(1 - zT)$. We consider a sequence $\{T_n\}$ of finite-dimensional operators that converges in $N_s(X)$ to an operator T . The space $N_s(X)$ is of spectral type l_1 . Therefore, by Proposition 3.3, there exists a constant $C > 0$ such that

$$\|\{\mu_k(T)\}\|_{l_1} \leq C \|T\|_{N_s}$$

for any operator $T \in N_s(X)$. In particular, this inequality is valid for all operators under consideration. For a finite-dimensional operator $U \in N_s(X)$ the determinant has the form

$$\det(1 - zU) = \prod_{i=1}^M (1 - \mu_i(U)z).$$

Hence for any T_n

$$|\det(1 - zT_n)| \leq \exp \left\{ \sum_k |\mu_k(T_n)| |z| \right\} \leq e^{C \|T_n\|_{N_s} |z|}.$$

By the continuity of determinant,

$$|\det(1 - zT)| \leq e^{C \|T\|_{N_s} |z|}.$$

However, $\{\mu_k(T)\} \in l_1$ and, by the Lindelöf theorem [6, Subsection 8.7.9], the entire function $\det(1 - zT_n)$ can be represented as

$$\det(1 - zT) = \prod_{i=1}^{\infty} (1 - \mu_i(T)z) = 1 - \text{trace } T z + \dots$$

Consequently, $\text{trace } T = \sum_{i=1}^{\infty} \mu_i(T)$. □

Using Propositions 3.3 and 3.4, it is possible to strengthen Theorem 2.1 and Corollary 2.1 (special case of L_q). Furthermore, we can prove Theorem A by a method different from that used in [2], where the White theorem [15] was applied.

Theorem 3.1. *Suppose that $s \in (0, 1]$, $p \in [1, \infty]$, and $1/s = 1/q + |1/2 - 1/p|$. There exists a constant $C_p > 0$ such that if Y is a subspace of the quotient space (or the quotient space of a subspace) of $L_p(\mu)$ constructed on some measurable space with positive measure, then for any operator $T \in N_s(Y)$*

$$\|\{\mu_k(T)\}\|_{l_q} \leq C_p \|T\|_{N_s},$$

where $\{\mu_k(T)\}$ is the complete family of eigenvalues of the operator T .

As an important consequence of Theorem 3.1 we obtain the result of [2], which is formulated below in a more general form. We note that the proof in [2] is considerably more complicated since it is based on the strong result of [15].

Corollary 3.2. *Suppose that $p \in [1, \infty]$ and $1/s = 1 + |1/2 - 1/p|$. If Y is a subspace of the quotient space (or the quotient space of a subspace) of $L_p(\mu)$, then for any operator $T \in N_s(Y)$*

$$\|\{\mu_k(T)\}\|_{l_1} \leq C_p \|T\|_{N_s},$$

where $\{\mu_k(T)\}$ is the complete family of eigenvalues of the operator T ; moreover, the nuclear trace T is fully defined and coincides with the spectral trace, i.e.,

$$\text{trace } T = \sum_{k=1}^{\infty} \mu_k(T).$$

4 The Operator Ideal $N_{r,p}$

For arbitrary Banach spaces X and Y we consider the linear space $X^* \otimes Y$ of finite-dimensional operators from X to Y and equip it with the quasinorm $\|\cdot\|_{N_{r,p}}$ defined by

$$\|u\|_{N_{r,p}} := \inf \left\{ \|(x'_i)_{i=1}^n\|_{\ell_r(X^*)} \cdot \|(y_i)_{i=1}^n\|_{\ell_{p'}^w(X)} : u = \sum_{i=1}^n x'_i \otimes y_i \right\}.$$

Denote by $X^* \widehat{\otimes}_{r,p} Y$ the completion of the spaces $(X^* \otimes Y, \|\cdot\|_{N_{r,p}})$. There is a natural continuous embedding $j_{r,p} : X^* \widehat{\otimes}_{r,p} Y \rightarrow X^* \widehat{\otimes}_{\pi} Y$, where $\|j_{r,p}\| \leq 1$. A typical element $u \in X^* \widehat{\otimes}_{r,p} Y$ can be represented as

$$u = \sum_{i=1}^{\infty} x'_i \otimes y_i,$$

where $(x'_i)_{i=1}^{\infty} \in \ell_r(X^*)$ and $(y_i)_{i=1}^{\infty} \in \ell_{p'}^w(X)$, and also as

$$u = \sum_{i=1}^{\infty} \lambda_i x'_i \otimes x_i,$$

where $(\lambda_i)_{i=1}^{\infty} \in \ell_r$, $(x'_i)_{i=1}^{\infty} \in \ell_{\infty}(X^*)$, and $(x_i)_{i=1}^{\infty} \in \ell_{p'}^w(X)$. If necessary, we will assume that $\lambda_i \searrow 0$, $\|(x'_i)\|_{\ell_{\infty}(X^*)} = 1$, and $\|(x_i)_{i=1}^{\infty}\|_{\ell_{p'}^w(X)} = 1$.

With each tensor element $u \in X^* \widehat{\otimes}_{r,p} Y$ we associate the continuous operator

$$\tilde{u}(x) = \sum_{i=1}^{\infty} \langle x'_i, x \rangle y_i \quad \forall x \in X.$$

More exactly, we use the natural composition $\tilde{j}_{r,p} := \tilde{j} \circ j_{r,p}$ in order to associate a tensor element with the corresponding operator in a continuous way:

$$X^* \widehat{\otimes}_{r,p} Y \xrightarrow{j_{r,p}} X^* \widehat{\otimes}_{\pi} Y \xrightarrow{\tilde{j}} L(X, Y).$$

In general, the operator $\tilde{j} : X \widehat{\otimes}_\pi Y \rightarrow L(X, Y)$ is not injective. Consequently, the injectivity of $\tilde{j}_{r,p} : X^* \widehat{\otimes}_{r,p} Y \rightarrow L(X, Y)$ depends on the Banach spaces under consideration. We equip the quotient space of the space $\tilde{j}_{r,p}(X^* \widehat{\otimes}_{r,p} Y)$ with the quasinorm and denote by $N_{r,p}(X, Y)$ the obtained (complete) quasinormed space. Then we obtain an approximate quasinormed ideal, i.e., finite-dimensional operators are dense in every $N_{r,p}(X, Y)$.

We recall (cf., for example, [3]) that there exists a unique continuous determinant on $X^* \widehat{\otimes} X$ and, consequently, on $X^* \widehat{\otimes}_{r,p} X$ for any Banach space X . Here, we describe the spectral type of the operator ideal $N_{r,p}$ with the help of the notion of tensor stability.

Proposition 4.1. *Let $1 \leq p \leq 2$, $0 < s \leq 1$, and r and be such that $1/r = 1/s + |1/2 - 1/p|$. Then for any $u \in X^* \widehat{\otimes}_{r,p} X$ there exists a constant $M \geq 0$ such that*

$$|\det(1 - zu)| \leq \exp(M|z|^s) \quad \forall z \in \mathbb{C}. \quad (4.1)$$

Proof. We represent u as

$$u = \sum_{i=1}^{\infty} \lambda_i x'_i \otimes x_i,$$

where $\|(\lambda_i)_{i=1}^{\infty}\|_{\ell_r} \leq 2\|u\|_{N_{r,p}}$, $\|(x'_i)_{i=1}^{\infty}\|_{\ell_{\infty}(X^*)} = 1$, $\|(x_i)_{i=1}^{\infty}\|_{\ell_p^w(X)} = 1$. We have

$$\det(1 - zu) = 1 - \text{trace}(u)z + \alpha_2(u)z^2 - \alpha_3(u)z^3 + \dots,$$

where

$$\alpha_n(u) = \sum_{i_1 < \dots < i_n} \lambda_{i_1} \cdots \lambda_{i_n} \det(\langle x'_{i_\alpha}, x_{i_\beta} \rangle)_{1 \leq \alpha, \beta \leq n}.$$

Let us estimate $\alpha_n(u)$ from above:

$$\begin{aligned} |\alpha_n(u)| &\leq \sum_{i_1 < \dots < i_n} \lambda_{i_1} \cdots \lambda_{i_n} |\det(\langle x'_{i_{\alpha}}, x_{i_{\beta}} \rangle)| \leq \sum_{i_1 < \dots < i_n} \lambda_{i_1} \cdots \lambda_{i_n} \prod_{\alpha=1}^n \left(\sum_{\beta=1}^n |\langle x'_{i_\alpha}, x_{i_\beta} \rangle|^2 \right)^{\frac{1}{2}} \\ &\leq \sum_{i_1 < \dots < i_n} \lambda_{i_1} \cdots \lambda_{i_n} \prod_{\alpha=1}^n n^{\frac{1}{p}-\frac{1}{2}} \left(\sum_{i=1}^n |\langle x'_i, x_i \rangle|^{p'} \right)^{\frac{1}{p'}} \leq \sum_{i_1 < \dots < i_n} \lambda_{i_1} \cdots \lambda_{i_n} n^{n(\frac{1}{p}-\frac{1}{2})}. \end{aligned}$$

Using these estimates, we find

$$\begin{aligned} |\det(1 - zu)| &\leq 1 + (\lambda_1 + \lambda_2 + \dots)|z| + (\lambda_1 \lambda_2 + \dots)|z|^2 + \dots \\ &\leq \prod_{n=1}^{\infty} (1 + \lambda_n n^{\frac{1}{p}-\frac{1}{2}}|z|) \leq \exp \left(\sum_{n=1}^{\infty} \frac{1}{s} \lambda_n^s n^{s(\frac{1}{p}-\frac{1}{2})} |z|^s \right) \\ &\leq \exp \left(\frac{1}{s} \sum_{n=1}^{\infty} \lambda_n^{s-r} n^{\frac{s}{r}-1} \lambda_n^r |z|^s \right) \leq \exp \left(\frac{1}{s} \sum_{n=1}^{\infty} n \lambda_n^{\frac{s}{r}-1} \lambda_n^r |z|^s \right) \\ &\leq \exp \left(\frac{1}{s} c \sum_{n=1}^{\infty} \lambda_n^r |z|^s \right) \quad (\text{where } c = \sup(n \lambda_n^{\frac{s}{r}-1}) < +\infty) \\ &\leq \exp \left(\frac{1}{s} c 2^r \|u\|_{N_{r,p}}^r |z|^s \right) = \exp(M|z|^s), \end{aligned}$$

where $M = \frac{1}{s} c 2^r \|u\|_{N_{r,p}}^r$. □

Proposition 4.2. *Let $1 \leq p \leq 2$, $0 < s \leq 1$, and r be such that $1/r = 1/s + 1/p - 1/2$. Then the operator ideal $N_{r,p}$ is stable with respect to the tensor norm ε . Furthermore, for any Banach spaces X , X_0 , Y , and Y_0 and any $S \in N_{r,p}(X, X_0)$ and $T \in N_{r,p}(Y, Y_0)$*

$$\|S \widehat{\otimes}_\varepsilon T : X \widehat{\otimes}_\varepsilon Y \rightarrow X_0 \widehat{\otimes}_\varepsilon Y_0\|_{N_{r,p}} \leq \|S\|_{N_{r,p}} \|T\|_{N_{r,p}}. \quad (4.2)$$

Proof. For any Banach spaces X , X_0 , Y , and Y_0 and any operators $S \in N_{r,p}(X, X_0)$ and $T \in N_{r,p}(Y, Y_0)$ we consider the representations

$$S = \sum_{i=1}^{\infty} x'_i \otimes x_i \quad T = \sum_{i=1}^{\infty} y'_i \otimes y_i,$$

where $(x'_i)_{i=1}^{\infty} \in \ell_r(X^*)$, $(x_i)_{i=1}^{\infty} \in \ell_{p'}^w(X_0)$, $(y'_i)_{i=1}^{\infty} \in \ell_r(Y^*)$, $(y_i)_{i=1}^{\infty} \in \ell_{p'}^w(Y_0)$ are such that

$$\|(x'_i)_{i=1}^{\infty}\|_{\ell_r(X^*)} \leq \|S\|_{N_{r,p}}, \quad \|(x_i)_{i=1}^{\infty}\|_{\ell_{p'}^w(X_0)} \leq 1 + \delta,$$

$$\|(y'_i)_{i=1}^{\infty}\|_{\ell_r(Y^*)} \leq \|T\|_{N_{r,p}}, \quad \|(y_i)_{i=1}^{\infty}\|_{\ell_{p'}^w(Y_0)} \leq 1 + \delta.$$

The operator $S \widehat{\otimes}_\varepsilon T : X \widehat{\otimes}_\varepsilon Y \rightarrow X_0 \widehat{\otimes}_\varepsilon Y_0$ is represented as

$$S \widehat{\otimes}_\varepsilon T = \sum_{i,j=1}^{\infty} (x'_i \otimes y'_j) \otimes (x_i \otimes y_j),$$

where

$$\begin{aligned} \|(x'_i \otimes y'_j)_{i,j=1}^{\infty}\|_{\ell_r((X \widehat{\otimes}_{r,p} Y)^*)} &= \left(\sum_{i,j=1}^{\infty} \pi(x'_i \otimes y'_j)^r \right)^{\frac{1}{r}} = \left(\sum_{i,j=1}^{\infty} \|x'_i\|^r \|y'_j\|^r \right)^{\frac{1}{r}} \\ &= \left(\sum_{i=1}^{\infty} \|x'_i\|^r \right)^{\frac{1}{r}} \cdot \left(\sum_{j=1}^{\infty} \|y'_j\|^r \right)^{\frac{1}{r}} \leq \|S\|_{N_{r,p}} \cdot \|T\|_{N_{r,p}}. \end{aligned}$$

For any sequence $(c_{ij})_{i,j=1}^{\infty} \in \ell_p(\mathbb{N} \times \mathbb{N})$, where $1 < p < \infty$,

$$\begin{aligned} \varepsilon \left(\sum_{i,j=1}^{\infty} c_{ij} x_i \otimes y_j \right) &= \sup \left\{ \left| \sum_{i,j=1}^{\infty} c_{ij} \langle x', x_i \rangle \langle y', y_j \rangle \right| : x' \in B_{X_0^*}, y' \in B_{Y_0^*} \right\} \\ &\leq \left(\sum_{i,j}^{\infty} |c_{ij}|^p \right)^{\frac{1}{p}} \cdot \|(x_i)_{i=1}^{\infty}\|_{\ell_r^w(X_0)} \cdot \|(y_i)_{i=1}^{\infty}\|_{\ell_r^w(Y_0)} \leq (1 + \delta)^2 \left(\sum_{i,j}^{\infty} |c_{ij}|^p \right)^{\frac{1}{p}}. \end{aligned}$$

This shows that the family $(x_i \otimes y_j)_{i,j=1}^{\infty}$ is weakly p' -summable and

$$\|(x_i \otimes y_j)_{i,j=1}^{\infty}\|_{\ell_{p'}^w(X_0 \widehat{\otimes}_\varepsilon Y_0)} \leq (1 + \delta)^2. \quad (4.3)$$

From the last two inequalities it follows that $S \widehat{\otimes}_\varepsilon T \in N_{r,p}(X \widehat{\otimes}_\varepsilon Y, X_0 \widehat{\otimes}_\varepsilon Y_0)$; moreover,

$$\|(x'_i \otimes y'_j)_{i,j=1}^{\infty}\|_{\ell_r((X \widehat{\otimes}_\varepsilon Y)^*)} \cdot \|(x_i \otimes y_j)_{i,j=1}^{\infty}\|_{\ell_r^w(X_0 \widehat{\otimes}_\varepsilon Y_0)} \leq \|S\|_{N_{r,p}} \cdot \|T\|_{N_{r,p}} (1 + \delta)^2.$$

Taking the infimum over all representations $S \widehat{\otimes}_\varepsilon T$ and letting $\delta \rightarrow 0$, we find

$$\|S \widehat{\otimes}_\varepsilon T\|_{N_{r,p}} \leq \|S\|_{N_{r,p}} \cdot \|T\|_{N_{r,p}},$$

which completes the proof. $\square$

Theorem 4.1. *Let $1 \leq p \leq 2$, $0 < s \leq 1$, and r be such that $1/r = 1/s + 1/p - 1/2$. Then the operator ideal $N_{r,p}$ has the spectral type ℓ_s :*

$$\|(\mu_n)_{n=1}^\infty\|_{\ell_s} \leq \|T\|_{N_{r,p}}.$$

Proof. For any Banach space X and any operator $T \in N_{r,p}(X)$ we consider one of the representations

$$T(\cdot) = \sum_{i=1}^\infty \lambda_i \langle x'_i, \cdot \rangle x_i$$

provided by

$$u = \sum_{i=1}^\infty \lambda_i x'_i \otimes x_i.$$

Since the family $(\mu_i(T))_{i=1}^\infty$ of nonzero eigenvalues (with the algebraic multiplicity taken into account) of the operator T is connected with the family of zeros $(z_i)_{i=1}^\infty$ of the function $\det(1 - zu)$ by the equality $\mu_i = 1/z_i$, from Proposition 4.1 (by the classical results of Hadamard and Jensen [6, Theorem 4.8,6]) it follows that $(\mu_i)_{i=1}^\infty \in \ell_{s,\infty}$. Thus, $(\mu_n)_{n=1}^\infty \in \ell_{s+\varepsilon}$ for all $\varepsilon > 0$. Hence we can conclude that $(\mu_i)_{i=1}^\infty \in \ell_s$ and

$$\|(\mu_n)_{n=1}^\infty\|_{\ell_s} \leq \|T\|_{N_{r,p}}.$$

The operator ideal $N_{r,p}$ is stable with respect to the tensor norm ε with the corresponding constant $c = 1$; moreover, $N_{r,p}$ is of spectral type $\ell_{s+\delta}$ for all $\delta > 0$. By the Pietsch theorem [6], the ideal $N_{r,p}$ is of spectral type ℓ_s . $\square$

We formulate some consequences of Theorem 4.1. We recall that a Banach space possesses the approximation property $AP_{r,p}$ of order (r, p) if for any Banach space Y the canonical mapping $Y^* \widehat{\otimes}_{r,p} X \rightarrow L(Y, X)$ is one-to-one.

Corollary 4.1. *Let $1 \leq p \leq 2$, $0 < s \leq 1$, and r be such that $1/r = 1/s + 1/p - 1/2$. Every Banach space possesses the approximation property $AP_{r,p}$ of order (r, p) .*

Proof. Let $u \in Y^* \widehat{\otimes}_{r,p} X$ be such that $\tilde{u} = 0$. If $u \neq 0$, then $j_{r,p}(u) \neq 0$ and, by the Hahn–Banach theorem, there exists an operator $T \in L(X, Y^{**})$ such that $\text{trace}(T \circ u) = 1$, where

$$T \circ u = \sum_{i=1}^\infty y'_i \otimes T(x_i) \in Y^* \widehat{\otimes}_{r,p} Y^{**} \subset Y^{***} \widehat{\otimes}_{r,p} Y^{**}.$$

On the other hand, by the results of [2],

$$\text{trace}(T \circ u) = \sum_{i=1}^\infty \mu_i,$$

where $(\mu_i)_{i=1}^\infty$ is the family of eigenvalues (with multiplicity taken into account) of the associated operator $\widetilde{T \circ u} = T \circ \tilde{u}$. However, all eigenvalues μ_i of the operator $\widetilde{T \circ u}$ are zeros. Since this operator identically vanishes on Y , we arrive at a contradiction. Thus, $u \neq 0$ and, consequently, the mapping $\tilde{j}_{r,p}$ is injective. $\square$

Corollary 4.2. *Under the assumptions of Theorem 4.1 (or Corollary 4.1), a unique continuous unique functional trace is fully defined on the ideal $N_{r,p}$ of (r, p) -nuclear operators.*

Corollary 4.3. *Under the assumptions of Theorem 4.1 (or Corollary 4.2), a unique continuous determinant is fully defined on the ideal $N_{r,p}$ of (r,p) -nuclear operators.*

Now, we prove the following trace theorem with the help the White theorem [15].

Theorem 4.2. *Suppose that $1 \leq p \leq 2$, $0 < s \leq 1$, and $r \in (0, 1]$ are such that $1/r = 1/s + 1/p - 1/2$. Then for any Banach space X and any operator $T \in N_{r,p}(X)$ the functional trace (T) is fully defined and*

$$\text{trace}(T) = \sum_{i=1}^{\infty} \mu_i,$$

where $(\mu_i)_{i=1}^{\infty}$ is the family of eigenvalues of the operator T .

Proof. As was already mentioned, $N_{r,p}$ is the operator ideal. By Theorem 4.1, $N_{r,p}$ is of spectral type ℓ_s , and, consequently, of type ℓ_1 (cf. [6]). By [15, Theorem 2.2], the function $\tau(T) = \sum_{i=1}^{\infty} \mu_i(T)$ defines a continuous trace on $N_{r,p}$. However, there exists a unique continuous trace on $N_{r,p}$. Therefore, $\text{trace}(T) = \sum_{i=1}^{\infty} \mu_i(T)$. $\square$

5 Three Examples

Example 5.1. Suppose that $r \in (2/3, 1]$, $p \in (1, 2]$, and $1/r - 1/p = 1/2$. There are Banach spaces E , V and $z_0 \in E^* \widehat{\otimes} V$, $S \in L(V, E)$ such that for every $p_0 \in [1, p)$

- 1) $z_0 \in E^* \widehat{\otimes}_{r,1} V$,
- 2) V satisfies the Crothendieck metric approximation condition,
- 3) V is a space of type p_0 and cotype 2,
- 4) $S \circ z_0 \in E^* \widehat{\otimes}_{r,p_0} E$,
- 5) $\text{trace } S \circ z_0 = 1$,
- 6) $\widetilde{S \circ z_0}$ is equal zero identically and, consequently, has no nonzero eigenvalues.

Remark 5.1. In the case $r = 1$ and $p = 2$, we see that the result of Pietsch [6] is sharp. The case $p = 1$ is excluded from the consideration.

To prove the assertions in Example 5.1, we consider a Banach space X such that there exists an element $z \in X^* \widehat{\otimes} X$ with nonzero trace that generates the zero operator and is represented as

$$z = \sum_{N=1}^{\infty} \sum_{n=1}^{3 \cdot 2^N} N^{1/2} 2^{-3N/2} x'_{nN} \otimes x_{nN}, \quad (5.1)$$

where the sequences (x'_{nN}) and (x_{nN}) are bounded in the norm by 1 (cf. [10, Subsection 10.4.5] or [16]). We fix $r \in (2/3, 1]$ and set $1/q = 3/2 - 1/r$ (hence $p = q'$). Assume that $\{\varepsilon_N\}_1^{\infty}$ is a sequence such that $\sum N^{-1-\varepsilon_N} < +\infty$, $\gamma_N = 2 + 3\varepsilon_N/2$, and q_N is such that $1/q - 1/q_N = N^{-1} \log_2 N^{\gamma_N}$ (consequently, $q_N > q$). We set

$$Y = \left(\sum_N l_{q_N}^{3 \cdot 2^N} \right)_{l_q}.$$

Since $l_q \subset Y \subset l_{q+\delta}$ for any positive δ (or $l_{(q+\delta)'} \subset Y^* \subset l_{q'}$; moreover, $q \in [2, \infty]$), the sequence of unit vectors in Y^* is weakly $(q+\delta)$ -summable. We denote by e_{nN} and e'_{nN} ($N = 1, 2, \dots; n = 1, 2, \dots, 3 \cdot 2^N$) the unit vectors in Y and Y^* respectively. We set

$$z_1 = \sum_N \sum_n 2^{-N/r} N^{-(1+\varepsilon_N)/r} e'_{nN} \otimes x_{nN},$$

$$T = \sum_N \sum_n 2^{-(3/2-1/r)N} N^{(1/2+1/r+e_N/r)} x'_{nN} \otimes e_{nN}.$$

We show that $z_1 \in Y^* \widehat{\otimes}_r X$ and $T \in L(X, Y)$ (then it is clear that $z = z_1 \circ T$). The first inclusion is obvious by the choice of ε_N . If $\|x\| \leq 1$, then

$$\|Tx\| \leq 3 \left(\sum_N (N^{1/2+1/r+\varepsilon_N/r} / 2^{(3/2-1/r-1/q_N)N})^q \right)^{1/q}.$$

Since $3/2 - 1/r - 1/q_N = 1/q - 1/q_N = N^{-1} \log_2 N^{\gamma_N}$, we have

$$\|T\| \leq 3 \left(\sum_N N^{(1/2+1/r+\varepsilon_N/r-\gamma_N)q} \right)^{1/q} = 3 \left(\sum_N N^{-1-\varepsilon_N} \right)^{1/q} < \infty.$$

Thus, $z = z_1 \circ T$, $\tilde{z} : X \xrightarrow{T} Y \xrightarrow{\tilde{z}_1} X$ and $\tilde{z}_1^* \in N_r(X^*, Y^*)$, $T^* : Y^* \rightarrow X^*$; moreover,

$$T^* = \sum_{N,n} e_{nN} \otimes \bar{x}'_{nN},$$

where $\{\bar{x}'_{nN}\}$ is a bounded sequence in X^* and thereby it is weakly $(q+\delta)$ -summable. In other words, $\tilde{z}^* = T^* \circ \tilde{z}_1^* \in N_{r,(q+\delta)'}(X^*, X^*)$ for any $\delta > 0$. Respectively, $z^t = T^* \circ z_1^t \in X^{**} \widehat{\otimes}_{r,(q+\delta)'} X^*$ for all positive δ . It remains to set $E := X^*$, $V := Y^*$, $S := T^*$, and $z_0 := z_1^t$.

The aforesaid shows that $S \circ z_0 \in X \widehat{\otimes}_{r,(q+\delta)'} X^*$ for all positive δ , where X is the Enflo space.

Example 5.2. Suppose that $r \in (2/3, 1)$, $p \in (1, 2]$, and $1/r - 1/p = 1/2$. There exist Banach spaces E , V and $z_0 \in E^* \widehat{\otimes} V$, $S \in L(V, E)$ such that for every $\epsilon > 0$

- 1) $z_0 \in E^* \widehat{\otimes}_{r+\epsilon, 1} V$,
- 2) V satisfies the Crothendieck metric approximation condition,
- 3) $S \circ z_0 \in E^* \widehat{\otimes}_{r+\epsilon, p} E$,
- 4) $\text{trace } S \circ z_0 = 1$,
- 5) the operator $\widetilde{S \circ z_0}$ is equal to zero identically and, consequently, has no nonzero eigenvalues.

Remark 5.2. For $r = 2/3$ and $p = 1$ Example 5.2 exactly coincides with the example due to Enflo [16]. We confirm the sharpness of the classical result due to Crothendieck

To prove the assertion of Example 5.2, we set $q := p'$ (since $r < 1$, we have $q > 2$) and argue in the same way as in Example 5.1. Let X be a Banach space such that there exists an element $z \in X^* \widehat{\otimes} X$ with nonzero trace that generates the zero operator and is represented as

$$z = \sum_{N=1}^{\infty} \sum_{n=1}^{3 \cdot 2^N} N^{1/2} 2^{-3N/2} x'_{nN} \otimes x_{nN}, \quad (5.2)$$

where the sequences (x'_{nN}) and (x_{nN}) are bounded in the norm by 1 (cf. [16] or [10, Subsection 10.4.5]). We fix $r \in [2/3, 1)$. Now, $1/q = 3/2 - 1/r$, $\varepsilon_N = 0$, and $\gamma_N = 2$ for all N , where q_N are equal, say, to q for all N . We fix some $a > 0$ which will be specified below. We set $Y = l_q$ or, which is the same,

$$Y = \left(\sum_N l_q^{3 \cdot 2^N} \right)_{l_q}.$$

We denote by e_{nN} and e'_{nN} ($N = 1, 2, \dots; n = 1, 2, \dots, 3 \cdot 2^N$) the unit vectors in Y and Y^* respectively and set

$$z_1 = \sum_N \sum_n 2^{-N/r} N^a e'_{nN} \otimes x_{nN},$$

$$T = \sum_N \sum_n 2^{-(3/2-1/r)N} N^{(1/2-a)} x'_{nN} \otimes e_{nN}.$$

We show that $z_1 \in Y^* \widehat{\otimes}_{r+\varepsilon} X$ for any $\varepsilon > 0$. Indeed,

$$\sum_N \sum_n [2^{-N/r} N^a]^{r+\varepsilon} = \sum_N 3 \cdot 2^N \cdot 2^{-N(r+\varepsilon)/r} N^{a(r+\varepsilon)} = \sum_N 3 \cdot 2^{-\varepsilon_0 N} N^{a(r+\varepsilon)} < \infty,$$

where $\varepsilon_0 = (r + \varepsilon)/r > 0$.

We show that $T \in L(X, Y)$ (then it is obvious that $z = z_1 \circ T$). Indeed, if $\|x\| \leq 1$, then

$$\|Tx\|_{l_q}^q \leq \sum_N \sum_n (2^{-N/q} N^{(1/2-a)} |\langle x'_{nN}, x \rangle|)^q \leq \sum_N 3 \cdot 2^{N-(N/q)q} N^{(1/2-a)q} = \sum_N 3 \cdot 1 \cdot N^\alpha,$$

where $\alpha = (1/2-a)q$. Now, we choose $a > 0$ such that $\alpha = -2$. Thus, $z = z_1 \circ T$, $\tilde{z} : X \xrightarrow{T} Y \xrightarrow{\tilde{z}_1} X$ and $\tilde{z}_1^* \in N_{r+\varepsilon}(X^*, Y^*)$, $T^* : Y^* \rightarrow X^*$; moreover,

$$T^* = \sum_{N,n} e_{nN} \otimes \bar{x}'_{nN},$$

where $\{\bar{x}'_{nN}\}$ is a bounded sequence in X^* and thereby it is weakly q -summable. In other words, $\tilde{z}^* = T^* \circ \tilde{z}_1^* \in N_{r+\varepsilon,p}(X^*, X^*)$ for any $\varepsilon > 0$. Respectively, $z^t = T^* \circ z_1^t \in X^{**} \widehat{\otimes}_{r+\varepsilon,p} X^*$ for all positive ε . It remains to set $E := X^*$, $V := Y^*$, $S := T^*$, and $z_0 := z_1^t$.

The following example yields a negative answer to Problem 10.1 in [17]. Suppose that $T \in L(X, Y)$, $T^* \in N_s(Y^*, X^*)$, and $s \in (0, 1)$. Is it true that the operator T is s -nuclear?

Example 5.3. Suppose that $r \in (2/3, 1]$, $p \in (1, 2]$, and $1/r - 1/p = 1/2$. There exist two separable Banach spaces X and Z such that

- 1) Z^{**} is separable and has basis,
- 2) there exists $V \in X^* \widehat{\otimes} Z^{**}$ such that $V = \sum_{k=1}^{\infty} x'_k \otimes z''_k(x'_k)$ weakly p'_0 -summable for any $p_0 \in [1, p)$, where $(z''_k) \in l_r(Z^{**})$,

- 3) $V(X) \subset Z$ and V , regarded as a mapping from X to Z , is not nuclear.

Furthermore, there exists an operator $U : Z^{**} \rightarrow Z$ such that

- 1) $\pi_Z U \in N^{r,p_0}(Z^{**}, Z^{**}) = Z^{***} \widehat{\otimes}^{r,p_0} Z^{**}$ for all $p_0 \in [1, p)$,
- 2) U , regarded as a mapping from Z^{**} to Z , is not nuclear,
- 3) $\text{trace } \pi_Z U = 1$,
- 4) $\pi_Z U : Z^{**} \rightarrow Z^{**}$ has no nonzero eigenvalues.

To prove the assertions of Example 5.3, we suppose that $r \in (2/3, 1]$, $p \in (1, 2]$, $1/r - 1/p = 1/2$ and consider X , $z = z_1 \circ T \in X^* \widehat{\otimes} X$ as above with the following properties:

- (i) trace $z = 1$ and $\tilde{z} = 0$,
- (ii) a tensor element $z = z_1 \circ T \in X^* \widehat{\otimes} X$ can be represented as $z = \sum x'_k \otimes x_k$, where (x'_k) is a weakly p'_0 -summable for every $p_0 \in [1, p)$ and $(x_k) \in l_r(Y)$.

Then we consider (cf. [18] or [9]) a separable Banach space Z such that Z^{**} is separable, has basis and there exists the quotient mapping $\varphi : Z^{**} \rightarrow X$ with kernel $\varphi^{-1}(0) = \pi_Z Z$ such that $\varphi^*(X^*)$ is completable in Z^{***} . A tensor element z can be lifted to a tensor element $V \in X^* \widehat{\otimes} Z^{**}$ in such a way that $V = \sum_{k=1}^{\infty} x'_k \otimes z''_k$, where (x'_k) is weakly p'_0 -summable for every $p_0 \in [1, p)$ and $(z''_k) \in l_r(Z^{**})$. Since $\varphi V = \tilde{v} = 0$, we have $V(X) \subset \pi_Z Z = Z$. We denote by V_0 the operator from X to Z induced by the operator V . Then $\varphi \pi_Z V_0 = \varphi V = 0$. We assume that $V_0 \in N_1(X, Z)$. Let $V_0 = \sum f'_k \otimes z_k$ be the nuclear representation of the operator V_0 . Then $V_0 \neq 0$, Z possesses an approximation property, and trace $\varphi \pi_Z V_0 = 1$. However, trace $\varphi \pi_Z V_0 = \sum \langle f'_k, \varphi \pi_Z z_k \rangle = \sum 0 = 0$, since $\varphi|_{\pi_Z Z} = 0$. Consequently, V_0 , regarded as a mapping from X to Z , cannot be nuclear. On the other hand, it is obvious that V_0^* is r -nuclear. To prove the second assertion, it suffices to set $U := V_0 \varphi : Z^{**} \rightarrow X \rightarrow Z$.

Corollary 5.1. *For any $r \in (2/3, 1]$ there exists a number $q \in (1, 2]$, Banach spaces X , Z , and an operator $V_0 : X \rightarrow Z$ such that Z^{**} is separable and has basis, $V_0^* \in N_{r,q}$, but V_0 is not even nuclear.*

On the other hand, it is clear that the adjoint W^* of $W \in N^{r,p}$ belongs to $N_{r,p}$.

6 Remarks

Using a new approach, we prove that any Banach space X possesses the approximation property AP of order (r, p) . Assume the contrary: $X \notin AP_{r,p}$, where $1/r - 1/p = 1/2$. Let $z \in X^* \widehat{\otimes}_{r,p} X$ be such that trace $z = 1$, $\tilde{z} = 0$. Since $z = \sum x'_k \otimes x_k$, where $(x'_k) \in l_r(X^*)$ and (x_k) is weakly p' -summable, $\tilde{z}$ is represented as $\tilde{z} : X \xrightarrow{A} l_{\infty} \xrightarrow{\Delta} l_1 \xrightarrow{j} l_p \xrightarrow{V} X$, where all the mappings are continuous, j is an embedding, Δ is the diagonal operator with diagonal elements in l_r . Since $\tilde{z} = 0$, we have $V|_{j\Delta A(X)} = 0$. We consider $S := j\Delta AV : l_p \rightarrow l_p$. It is obvious that $S^2 = 0$ and trace $S = \text{trace } z = 1$. Since $S \in N_r(l_p, l_p)$, the nuclear trace is equal to the sum of all eigenvalues of S (cf. [3, Theorem 2.b.13] and [19]). Since $S^2 = 0$, we arrive at a contradiction.

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