AN INFINITESIMAL RATTRAY THEOREM

V. V. Makeev*

UDC 514.172

Let X be a continuous vector field on the unit Euclidean sphere $S^{n-1} \subset \mathbb{R}^n$ centered at the origin. It is proved that if $X(-a) = -X(a)$ holds true for each vector a , then there is an orthonormal basis such that for any two vectors a and b in the basis one has $X(a) \cdot b + a \cdot X(b) = 0$. Bibliography: 1 title.

It is well known that if two unit orthogonal vectors a and b in $\mathbb{R}^n$ smoothly depend on time, then we have the relation

$$a' \cdot b + a \cdot b' = 0.$$

Theorem. Let X be a continuous vector field on the unit Euclidean sphere S^{n-1} in $\mathbb{R}^n$ centered at the origin. Assume that we have $X(-a) = -X(a)$. Then there is an orthonormal basis in $\mathbb{R}^n$ such that for any two vectors a and b in the basis we have $X(a) \cdot b + a \cdot X(b) = 0$.

Proof. Shift along trajectories of the vector field X determines an antipodal continuous mapping of the sphere: $S^{n-1} \rightarrow S^{n-1}$. By the Rattray Theorem [1], the mapping maps a certain orthonormal basis to another orthonormal basis. Since for small times t the shift of a point a of the sphere is approximately equal to $X(a)t$, it follows that passing to the limit as $t \rightarrow 0$ we obtain the required assertion. $\square$

Remark. We have actually proved a theorem about tangent vector fields on the projective space because the considered vector fields X on the sphere S^{n-1} induce vector fields on the projective space, and vice versa.

Anti-symmetrizing an arbitrary vector field on the sphere, we obtain the following corollary.

Corollary. Let X be a continuous vector field X on S^{n-1} . Then there is an orthonormal basis in $\mathbb{R}^n$ such that for any two basis vectors a and b we have the relation

$$(X(a) - X(-a)) \cdot b + a \cdot (X(b) - X(-b)) = 0.$$

Translated by V. V. Makeev.

REFERENCES

1. R. Rattray, "An antipodal point, orthogonal point theorem," *Ann. Math.*, **80**, 502–512 (1954).

*St. Petersburg State University, St. Petersburg, Russia, e-mail: mvv57@inbox.ru.