ON THE LAGRANGE PROBLEM ON THE OPTIMAL FORM FOR CIRCULAR COLUMNS

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ABSTRACT. A new approach to the classical Lagrange problem on the form of the strongest circular column of fixed volume and height clamped at both ends is proposed. The existence of the optimal column is proved and an algorithm for finding its design is given.

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1. Introduction

The problem of design of the strongest elastic column of fixed volume with pinned ends was stated by Lagrange in 1773 on the basis of some works of Euler and Bernoulli (see [7]). This problem was solved in [1] by Clausen.

The recent interest in this problem was initiated by Keller and Tadjbakhsh [5, 6]. They found the optimal columns with clamped-free and clamped-hinged ends. They also obtained a unimodal optimal solution for the clamped-clamped case, but Olhoff and Rasmussen showed numerically that it is wrong. An analytical proof of this fact was given by Seyranian [11].

Despite numerous efforts and publications (see [2, 3, 8, 10–12] and the references therein), the existence of the optimal column has not yet been proved, although in 1977 Olhoff and Rasmussen found the bimodal optimal numerical solution (see [9]). Their results were later confirmed by Seyranian [10, 12] and Masur [8] by using different numerical methods.

In this paper, we propose a new approach leading to the existence theorem and give a new algorithm for finding this optimal form. Note that the equations for the function r in Theorem 1 below are formally close to those found by Seyranian and Gura [10, 12] and Masur [8], but the meaning of these equations here is very different; they come from solving an auxiliary problem of minimizing a nonlinear functional. We study the stationary points of this functional and show that one of them is connected with the Lagrange problem. Theorem 1 can serve as a mathematical base for some results obtained earlier by Olhoff and Rasmussen, Seyranian, and Masur. The proposed method can also be applied to the study of columns with elastic ends considered in [11]. Our approach allows one to study other similar unsolved problems, for example, the problem on the optimal form of circular columns with thin walls.

Now we state the problem using the Pearson formulation. Consider a vertical column whose horizontal sections are disks centered at the points of the x -axis for $0 \leq x \leq h$. (Actually, the horizontal sections can have other forms; they must be homothetic to a fixed plane figure with centers of homotheties lying on the x -axis.) Let V be the volume of the column and E be the Young modulus of it. If $z(x)$ is the

lateral deflection of the column at a point x , then the potential energy is

$$T = \int_0^h EI(x)z''(x)^2 dx - \lambda \int_0^h z'(x)^2 dx,$$

where $I(x)$ is the second moment of area of the cross section of the column and λ is the magnitude of the axial load. We neglect the weight of the column. Since $z(0) = 0$ and $z'(0) = 0$, the potential energy is positive for small values of λ for all $z \neq 0$. The buckling load λ_0 of the column is the supremum of the values λ such that $T \geq 0$ for any z . The problem considered is to find the form of the column, i.e., the function $I(x)$ for which the value λ_0 is maximal and

$$\int_0^h I(x)^{1/2} dx = V.$$

After rescaling and the passage from z to $y = z'$, the mathematical problem takes the following form.

Problem L. Find a positive function $Q(x) \in C[0, 1]$ such that

$$\int_0^1 Q(x)^{1/2} dx = 1 \quad (1)$$

and for which the minimal value λ of the functional

$$L_1[Q, y] \equiv \frac{\int_0^1 Q(x)y'(x)^2 dx}{\int_0^1 y^2(x) dx} \quad (2)$$

in the class of functions $y \in C^1(0, 1)$ satisfying the conditions

$$y(0) = 0, \quad y(1) = 0, \quad \int_0^1 y(x) dx = 0 \quad (3)$$

is maximal.

We propose a new approach based on two-dimensional variation of the functional L_1 , which allows one to prove the existence of solutions and to find the optimal form of the column using the properties of the related nonlinear functional. We also propose an algorithm allowing one to find the shape of the optimal column.

Note that Cox and Overton proved in [3] the existence theorem under the supplementary conditions $0 < a \leq Q(x) \leq b < \infty$. This does not imply the existence theorem for the Lagrange problem.

We begin with some heuristic considerations, which will not be used later. First, we assume that the optimal function Q is symmetric, i.e., $Q(x) = Q(1 - x)$. It is rather natural, and later we shall show that it is true. If $y_0(x)$ is an optimal function giving the minimal value to L_1 , then the function $y_0(1 - x)$ is also optimal. The Euler–Lagrange equation has the form

$$(Qy')' + \lambda y = C, \quad y(0) = y(1) = 0, \quad \int_0^1 y(x) dx = 0. \quad (4)$$

The functions $y_0(x)$ and $y_0(1 - x)$ satisfy (4). Moreover, the odd and even parts of y_0 with respect to the point $x = 1/2$ do so also:

$$(Qu')' + \lambda u = 0, \quad (Qv')' + \lambda v = C,$$

where

$$u(x) = (y_0(x) - y_0(1 - x))/2, \quad v(x) = (y_0(x) + y_0(1 - x))/2,$$

and

$$u(0) = u(1) = 0, \quad v(0) = v(1) = 0, \quad \int_0^1 u(x)dx = \int_0^1 v(x)dx = 0.$$

We can consider the boundary-value problem on the half-interval $[0, 1/2]$ and use the boundary conditions of the form

$$u(0) = u(1/2) = 0, \quad v(0) = 0, \quad \int_0^{1/2} v(x)dx = 0. \quad (5)$$

Notation. $\mathcal{A}$ is the set of positive continuous functions Q satisfying (1).

S is the set of functions y of class $C^1(0, 1)$ satisfying (3).

S_{odd} and S_{even} are the sets of odd and even (about $x = 1/2$) functions of class S .

S_1 is the set of functions u of class S_{odd} , which are positive in $]0, 1/2[$ and whose derivatives monotonically decrease on $[0, 1/2]$.

S_2 is the set of functions v of class S_{even} , which are positive on some interval $0 < x < t_p$, negative if $t_p < x < 1/2$, and whose derivatives v' monotonically decrease on $[0, t'_p]$ and increase on $[t'_p, 1/2]$.

S_0 is the set of pairs of functions u, v of class S such that

$$u'(0) > 0, \quad v'(0) > 0, \quad u'(1) > 0, \quad v'(1) < 0,$$

the function $\theta(x) = \arctan(v'(x)/u'(x))$ monotonically decreases, $\theta(0) - \theta(1) \leq 3\pi$, and the function $r(x) = u'(x)^2 + v'(x)^2$ has in $]0, 1[$ no more than three extremum points and is monotone on the intervals between these points and the points $x = 0$ and $x = 1$.

We have

$$\lambda = \frac{\int_0^{1/2} Q(x)u'(x)^2 dx}{\int_0^{1/2} u(x)^2 dx} = \frac{\int_0^{1/2} Q(x)v'(x)^2 dx}{\int_0^{1/2} v(x)^2 dx} = \frac{\int_0^{1/2} Q(x)(u'(x)^2 + v'(x)^2) dx}{\int_0^{1/2} (u(x)^2 + v(x)^2) dx}.$$

This leads to the following problem, equivalent to Problem L, as will be shown later.

Problem L1. Find a positive function $Q \in \mathcal{A}$ such that the functional

$$\lambda(Q) = \inf_{\substack{u \in S_{\text{odd}} \\ v \in S_{\text{even}}}} L[Q, u, v], \quad L[Q, u, v] \equiv \frac{\int_0^1 Q(x)(u'(x)^2 + v'(x)^2) dx}{\int_0^1 (u(x)^2 + v(x)^2) dx},$$

takes its maximal value.

We set

$$M = \max_{Q \in \mathcal{A}} \lambda(Q) = \max_{Q \in \mathcal{A}} \min_{\substack{u \in S_{\text{odd}} \\ v \in S_{\text{even}}}} L[Q, u, v].$$

If we change the order of max and min in this equality (which is not legitimate in general), we can easily verify that $\max_{Q \in \mathcal{A}} L[Q, u, v]$ with fixed u and v is attained if

$$Q(x) = (u'(x)^2 + v'(x)^2)^{-2}. \quad (6)$$

Using the equalities

$$\int_0^1 (u'(x)^2 + v'(x)^2)^{-1} dx = \int_0^1 Q(x)^{1/2} dx = 1,$$

we obtain a related problem: find the extremal values of the functional

$$F[u, v] \equiv \int_0^1 (u'(x)^2 + v'(x)^2)^{-1} dx \int_0^1 (u(x)^2 + v(x)^2) dx \quad u \in S_{\text{odd}}, \quad v \in S_{\text{even}}.$$

It is easy to see that

$$F[ku, kv] = F[u, v] \quad \text{for all } k \neq 0,$$

the minimal value of this functional is 0, and the maximal value is ∞ . Thus, this problem is ill-posed at least if we do not set some supplementary restrictions on u and v .

The principal idea of our method is to find the minimal value of the functional $F[u, v]$ in $S_1 \times S_2$ and to show that the optimal functions u and v define, by (6), the optimal shape of the most solid column. The functional F has an infinite set of points of local minimum (u_k, v_k) in the space $S \times S$ and $F[u_k, v_k] \rightarrow 0$ as $k \rightarrow \infty$. The first point (u_1, v_1) such that $F[u_1, v_1] = \max F[u_k, v_k]$ is the most interesting to us. However, the other points are also interesting for the Lagrange problem. These points are characterized by the number of rotations of the vector $(u'_k(t), v'_k(t))$ when t increases from 0 to 1.

Our main result is as follows.

Theorem 1. *There exists a solution of Problem L. The optimal function Q_0 can be found from the relation*

$$Q_0(x) = (u'(x)^2 + v'(x)^2)^{-2},$$

where $u(x)$ and $v(x)$ are linearly independent solutions of the equations

$$m \left(\frac{u'(x)}{(u'(x)^2 + v'(x)^2)^2} \right)' + u = 0, \quad u(0) = u(1) = 0, \quad \int_0^1 u(x) dx = 0, \quad (7)$$

$$m \left(\frac{v'(x)}{(u'(x)^2 + v'(x)^2)^2} \right)' + v = C, \quad v(0) = v(1) = 0, \quad \int_0^1 v(x) dx = 0, \quad (8)$$

$$u(x) = -u(1-x), \quad v(x) = v(1-x).$$

The function Q_0 is symmetric, $Q_0(x) = Q_0(1-x)$, and can also be found as

$$Q_0(x) = \frac{r^2(x)}{9m^2},$$

where r is the solution of the problem

$$r^4(x)r'(x)^2 = 12m[c_1r(x)^3 - r(x)^4 - c_2(3m)^3], \quad r(0) = 4m,$$

which is not constant on any subinterval, and

$$c_1 = 4m + \left(\frac{32}{9}mk \right)^2, \quad c_2 = \left(\frac{512}{81}mk \right)^2 k^2 \left(\frac{3}{4} - k^2 \right), \quad k = v'(0).$$

If $P(r) = (c_1 - r)r^3 - c_2(3m)^3$ and r_1 and r_2 are real roots of P , then $0 < r_1 < 4m < r_2 < c_1$ and

$$2 \int_{r_1}^{4m} \frac{r^2 dr}{\sqrt{P(r)}} + \int_{4m}^{r_2} \frac{r^2 dr}{\sqrt{P(r)}} = \sqrt{3m},$$

$$2 \int_{r_1}^{4m} \frac{dr}{(c_1 - r)\sqrt{P(r)}} + \int_{4m}^{r_2} \frac{dr}{(c_1 - r)\sqrt{P(r)}} = \frac{2\pi}{c_2 3m \sqrt{3m}}.$$

The latter system of two equations for k and m (as well as system (7), (8)) can be solved numerically. Similar equations were found earlier by Seyranian and Gura [10, 12] and Masur [8], but the content of these equations here is very different, as is shown below.

The calculation shows that

$$m = 0.019100, \quad 0.0519 \leq Q(t) \leq 1.7824, \quad k = 0.27321.$$

The critical load $M = 1/m = 52.3562$ is close to the value found in [9] by Olhoff and Rasmussen, but our calculations are simpler.

The optimal column with circular sections is formed by rotation of the curve

$$R(x) = \frac{Q_0(x)^{1/4}}{\sqrt{\pi}} = \sqrt{\frac{r(x)}{3\pi m}}.$$

We have $R(0) = R(1) = 0.65140$. The minimal value of R is 0.2694; it is attained at $x = 0.2449$ and $x = 0.7551$. The maximal value of R is 0.6519; it is attained at $x = 1/2$.

Remark. The case of a clamped-hinged column where (3) is replaced by the conditions

$$y(0) = 0, \quad Q(1)y'(1) = 0, \quad \int_0^1 y(x)dx = 0$$

can be studied similarly. In this case, $c_2 = 0$ and $Q_0(x) = r^2(x)/9m^2$, where r is the solution of the problem

$$r'(x)^2 = 12m \left(\frac{c_1}{r(x)} - 1 \right), \quad r(0) = 4m,$$

which can be solved explicitly and gives the formulas obtained by Keller and Tadjbakhsh [6].

2. Auxiliary Constructions

A. We consider the following auxiliary problem. Let ε be a positive number.

Problem F_ε . Find the minimal value of the functional

$$F_\varepsilon[u, v] = \int_0^1 [u(x)^2 + v(x)^2 + \varepsilon(u'(x)^2 + v'(x)^2)]dx \int_0^1 [u'(x)^2 + v'(x)^2]^{-1}dx$$

in the class $S_0 \times S_0$.

Lemma 1. There exist two functions $u_0, v_0 \in S_0 \times S_0$ such that

$$m_\varepsilon \equiv \inf_{\substack{u \in S_0 \\ v \in S_0}} F_\varepsilon[u, v] = F_\varepsilon[u_0, v_0].$$

Proof. Let $u_k, v_k \in S_0 \times S_0$ be such that

$$\int_0^1 [u'_k(x)^2 + v'_k(x)^2]^{-1} dx = 1, \quad \int_0^1 [(u_k(x)^2 + v_k(x)^2) + \varepsilon(u'_k(x)^2 + v'_k(x)^2)] dx \rightarrow m_\varepsilon.$$

We can assume that

$$\int_0^1 [(u_k(x)^2 + v_k(x)^2) + \varepsilon(u'_k(x)^2 + v'_k(x)^2)] dx \leq m_\varepsilon + \varepsilon.$$

Since

$$\int_0^1 [u'_k(x)^2 + v'_k(x)^2] dx \leq \frac{m_\varepsilon}{\varepsilon} + 1,$$

there exist subsequences u_{n_k} and v_{n_k} uniformly converging in I to u_0 and v_0 such that u'_{n_k} and v'_{n_k} weakly converge in $L_2(I)$ to u'_0 and v'_0 . The function $r_k(x) = u'_k(x)^2 + v'_k(x)^2$ is monotone when $a_{j,k} < x < a_{j+1,k}$, where $0 = a_{0,k} < a_{1,k} < \dots < a_{L,k} = 1$ and $L \leq 4$. We can assume that $a_{j,k} \rightarrow a_j$ as $k \rightarrow \infty$. Let $\delta > 0$ and I_δ be the set of the points of $I = [0, 1]$ such that $|x - a_j| > \delta$. If $x \in I_\delta$ and $1 - 8\delta > 1/2$, then

$$\delta \leq u'_k(x)^2 + v'_k(x)^2 \leq 2 \frac{m_\varepsilon + \varepsilon}{\varepsilon}.$$

The functions $\theta_k(x) = \arctan(v'_k(x)/u'_k(x))$ are monotone, $0 \leq \theta_k(0) \leq \pi/2$, $-2\pi - \pi/2 \leq \theta_k(1) \leq -2\pi$. By the Helly theorem, there exists a subsequence $n_k \rightarrow \infty$ such that $r_{n_k}(x)$ and $\theta_{n_k}(x)$ converge everywhere in I_δ to $r_0(x)$ and $\theta_0(x)$.

Since $u'_k(x) = \sqrt{r_k(x)} \cos \theta_k(x)$ and $v'_k(x) = \sqrt{r_k(x)} \sin \theta_k(x)$, the sequences u'_{n_k} and v'_{n_k} converge in I_δ to u'_0 and v'_0 and the sequences u_{n_k} and v_{n_k} converge in I_δ to u_0 and v_0 .

Let $\delta \rightarrow 0$; using the diagonalization procedure, we obtain that the functions u_0 and v_0 belong to S_0 and

$$\int_0^1 ([u'_0(x)^2 + v'_0(x)^2]^{-1}) dx \leq 1, \quad \int_0^1 [u_0(x)^2 + v_0(x)^2 + \varepsilon(u'_0(x)^2 + v'_0(x)^2)] dx \leq m_\varepsilon.$$

Since m_ε is the minimal value and is positive, we indeed have the equalities. The proof is complete. $\square$

B. Let us analyze the solution obtained.

Let us show that $m_\varepsilon \geq \mu$, where $\mu > 0$ is independent of ε . Indeed, assume that for some sequence $\varepsilon_k \rightarrow 0$, we have $m_{\varepsilon_k} \rightarrow 0$. Each of the functions u'_ε and v'_ε has at most three zeros in I . Therefore, for any $\varepsilon > 0$, there exists an interval $I_\varepsilon = [a, b]$ of length $b - a = 1/12$, where both u_ε and v_ε are monotone. We set $a' = a + 1/48$ and $a'' = (a + b)/2 = a + 1/24$. The inequality

$$\int_0^1 (u_\varepsilon(x)^2 + v_\varepsilon(x)^2) dx \leq m_\varepsilon$$

implies that there exists a point $\theta \in [a, a']$ such that

$$u_\varepsilon(\theta)^2 + v_\varepsilon(\theta)^2 \leq 24\sqrt{m_\varepsilon}, \quad |u_\varepsilon(\theta)| + |v_\varepsilon(\theta)| \leq \sqrt{48m_\varepsilon}.$$

Since

$$\int_a^b (|u_\varepsilon(x)| + |v_\varepsilon(x)|) dx \leq \sqrt{2m_\varepsilon},$$

we have

$$\begin{aligned} \int_{\theta}^b (b-x)(|u'_{\varepsilon}(x)| + |v'_{\varepsilon}(x)|)dx &\leq (b-\theta)(|u_{\varepsilon}(\theta)| + |v_{\varepsilon}(\theta)|) \\ &+ \int_{\theta}^b (|u_{\varepsilon}(x)| + |v_{\varepsilon}(x)|)dx \leq \left(\frac{\sqrt{48}}{12} + \sqrt{2}\right) \sqrt{m_{\varepsilon}} \leq 3\sqrt{m_{\varepsilon}} \end{aligned}$$

and

$$\int_{a'}^{a''} (|u'_{\varepsilon}(x)| + |v'_{\varepsilon}(x)|)dx \leq 72\sqrt{m_{\varepsilon}} = w_{\varepsilon}.$$

The set $\{x; a' < x < a'', |u'_{\varepsilon}(x)| > \sqrt{w_{\varepsilon}}\}$ has measure $\leq \sqrt{w_{\varepsilon}}$. Therefore, the inequality $|u'_{\varepsilon}(x)| \leq \sqrt{w_{\varepsilon}}$ holds on a set of measure $\geq 1/48 - \sqrt{w_{\varepsilon}}$. The same is true for $v'_{\varepsilon}(x)$. Therefore, on a subset of measure $1/48 - 2\sqrt{w_{\varepsilon}}$, we have

$$u'_{\varepsilon}(x)^2 + v'_{\varepsilon}(x)^2 \leq 2w_{\varepsilon}.$$

But then

$$1 = \int_0^1 \frac{dx}{[u'_{\varepsilon}(x)^2 + v'_{\varepsilon}(x)^2]} \geq \frac{1/48 - 2\sqrt{w_{\varepsilon}}}{(2w_{\varepsilon})},$$

which is impossible since $w_{\varepsilon} \rightarrow 0$ if $\varepsilon \rightarrow 0$. Hence, $m_{\varepsilon} \geq \mu > 0$.

C. If in some subinterval $[a, b]$ of I , the functions r_0 and θ_0 are strictly monotone, then

$$\frac{d}{d\delta} F_{\varepsilon}[u_0 + \delta\varphi, v_0 + \delta\psi] = 0$$

for $\delta = 0$ and any $\varphi, \psi \in S_0 \times S_0$, with their supports in $[a, b]$, i.e., u_0 and v_0 satisfy the Euler-Lagrange equations

$$m_{\varepsilon} \frac{d}{dx} (Q(x)u'_0) + u_0 = C_1, \quad m_{\varepsilon} \frac{d}{dx} (Q(x)v'_0) + v_0 = C_2, \quad (9)$$

where C_1 and C_2 are constants and

$$Q(x) = -\frac{\varepsilon}{m_{\varepsilon}} + (u'_0(x)^2 + v'_0(x)^2)^{-2}.$$

Here $Q(x)$, u , and v depend on ε , but we drop the extra subindex. The functions r and θ cannot be constant simultaneously on any subinterval $[a, b]$ of I . In this case, the functions u_0 and v_0 are linear. Assume that this is not true and

$$u_0(x) = kx + c, \quad v_0(x) = k_1x + c_1, \quad a < x < b,$$

where $k > 0$. We set $\varphi(x) = u_1(x) - u_0(x)$, where u_1 is continuous,

$$u_1(x) = \begin{cases} u_0(x) & \text{if } x < a - \delta_1 \text{ or } x > b + \delta_2, \\ u_0(x) - \delta & \text{if } a < x < b, \end{cases}$$

where $\delta > 0$ is small. If $u(x) < kx + c$ for $a - \delta_0 < x < a$, then we set $u_1(x) = kx + c - \delta$ and define δ_0 by the condition $u_0(a - \delta_0) = k(a - \delta_0) + c - \delta$. Of course, it is possible only if δ is sufficiently small. If $u(x) > kx + c$ for $a - \delta_0 < x < a$, then we set $u_1(x) = u'_0(a - \delta_0)(x - a + \delta_0)$ on this interval and define δ_0 by the condition $u'_0(a - \delta_0)\delta_0 = \delta$. We apply the same construction on the interval $b < x < b + \delta_0$.

In order to make u_1 satisfy the condition

$$\int_0^1 u_1(x)dx = 0,$$

we must compensate the value of the integral on some other subinterval of $[0, 1]$, where u_0 satisfies the Lagrange equation, setting $u_1 = u_0 + \delta\varphi(x)$ with the support of φ on this subinterval. Then $u_1 \in S_0$ and

$$F_\varepsilon[u_1, v_0] - F_\varepsilon[u_0, v_0] \leq -2(b-a)\delta - 2m_\varepsilon \int_0^1 \frac{\varphi'(x)u_0'(x)dx}{(u_0'(x)^2 + v_0'(x)^2)^2} + o(\delta) = -2(b-a)\delta + o(\delta) < 0$$

for small $\delta > 0$, which contradicts to the optimality of u_0, v_0 . The case $k < 0$ is considered similarly.

If the function r_0 is monotone on $[a, b]$ but $\theta_0(x) = \text{const}$ on this interval, then $v_0'(x) = ku_0'(x)$ and $v_0(x) = ku_0(x) + c$. We can vary u_0 and v_0 in such a way that $\psi(x) = k\varphi(x)$ and obtain the equations

$$-\varepsilon u_0''(x) + m(u_0'(x)^{-3})' + u_0(x) = C_1, \quad -\varepsilon v_0''(x) + m(v_0'(x)^{-3})' + v_0(x) = C_2.$$

Then Eqs. (9) hold.

If the function θ_0 is monotone on $[a, b]$ but $r_0(x) = R = \text{const}$ on this interval, then $v_0'(x) = R \cos \theta_0(x)$, $u_0'(x) = R \sin \theta_0(x)$, and

$$\begin{aligned} & \int_a^b [u_0(t)^2 + v_0(t)^2 + \varepsilon(u_0'(t)^2 + v_0'(t)^2)] dt \int_a^b \frac{dt}{[(u_0'(t)^2 + v_0'(t)^2)]} \\ &= \varepsilon(b-a)^2 + (b-a)/R^2 \int_a^b [u_0(t)^2 + v_0(t)^2] dt. \end{aligned}$$

Therefore, the integral $\int_a^b (u^2 + v^2) dt$ takes its minimal value for $u = u_0$ and $v = v_0$. Consider a variation of θ such that $\theta_1(x) = \theta_0(x) + \delta\varphi(x)$ with small $\delta > 0$ and $\varphi \in C_0^\infty(a, b)$. We have

$$\int_a^b [u_0(t) \int_a^t \sin \theta_0(s) \varphi(s) ds - v_0(t) \int_a^t \cos \theta_0(s) \varphi(s) ds] dt = 0,$$

i.e.,

$$\int_a^b [\sin \theta_0(s) \int_s^b u_0(t) dt - \cos \theta_0(s) \int_s^b v_0(t) dt] \varphi(s) ds = 0$$

if

$$\int_a^b \sin \theta_0(s) \varphi(s) ds = 0, \quad \int_a^b \cos \theta_0(s) \varphi(s) ds = 0.$$

Therefore,

$$\sin \theta_0(s) \int_s^b u_0(t) dt - \cos \theta_0(s) \int_s^b v_0(t) dt = c_1 \sin \theta_0(s) + c_2 \cos \theta_0(s).$$

Since $\theta_0'(s) < 0$, we have

$$\int_s^b u_0(t) dt = c_1 + k \cos \theta_0(s), \quad \int_s^b v_0(t) dt = c_2 + k \sin \theta_0(s).$$

Therefore,

$$v_0(s) = -k\theta_0'(s) \cos \theta_0(s), \quad u_0(s) = k\theta_0'(s) \sin \theta_0(s), \quad u_0(s)^2 + v_0(s)^2 = k^2\theta_0'(s)^2.$$

Thus, the integral $\int_a^b \theta'(s)^2 ds$ has the minimal value under the condition $\int_a^b \theta'(s) ds = \text{const}$ if $\theta(x) = \theta_0(x)$. Then $\theta'(x) = -k = \text{const}$ and $\theta_0(x) = \theta_1 - kx$. Therefore,

$$u_0(x) = u_1 - k^2 \sin(\theta_1 - kx), \quad v_0(x) = v_1 + k^2 \cos(\theta_1 - kx)$$

and again Eqs. (9) hold.

Multiplying the first equation in (9) by u'_0 and the second equation by v'_0 , then adding and integrating, we obtain

$$-\varepsilon(u'_0(x)^2 + v'_0(x)^2) + \frac{3m_\varepsilon}{u'_0(x)^2 + v'_0(x)^2} + (u_0 - C_1)^2 + (v_0 - C_2)^2 = C_3 \quad (10)$$

with a constant C_3 . After integrating this equality over I , we see that

$$C_3 = C_1^2 + C_2^2 + 3m_\varepsilon + \int_0^1 [u_0(x)^2 + v_0(x)^2 - \varepsilon(u'_0(x)^2 + v'_0(x)^2)] dx (u'_0(x)^2 + v'_0(x)^2) dx.$$

Since

$$\varepsilon \int_0^1 [u'_0(x)^2 + v'_0(x)^2] dx = m_\varepsilon - \int_0^1 (u_0(x)^2 + v_0(x)^2) dx,$$

we have

$$C_3 = C_1^2 + C_2^2 + 2m_\varepsilon + 2 \int_0^1 (u_0(x)^2 + v_0(x)^2) dx.$$

If we set $\rho_0 = u'_0(0)^2 + v'_0(0)^2$, then (10) implies that $-\varepsilon\rho_0/m_\varepsilon + 3/\rho_0 < 4$. Note that, by the Cauchy inequality, $m_\varepsilon \geq \varepsilon$. Therefore, $3/\rho_0 < 4 + \rho_0$ or $\rho_0^2 + 4\rho_0 > 3$. Hence, $\rho_0 > 1/2$ and $Q(0) < 4$.

Since $-\varepsilon\rho_0/m_\varepsilon + 3/\rho_0 > 2$, we see that $\rho_0 < 3/2$ and $Q(0) > -5/9$.

Moreover, $\rho_0 = u'_0(1)^2 + v'_0(1)^2$ and $Q(1) = Q(0)$. Integrating Eqs. (9) over I , we see that

$$C_1 = m_\varepsilon [Q(1)u'_0(1) - Q(0)u'_0(0)], \quad C_2 = m_\varepsilon [Q(1)v'_0(1) - Q(0)v'_0(0)].$$

Therefore,

$$C_1^2 + C_2^2 \leq 2m_\varepsilon^2 Q(1)^2 [u'_0(1)^2 + v'_0(1)^2] + 2m_\varepsilon^2 Q(0)^2 [u'_0(0)^2 + v'_0(0)^2] = 4m_\varepsilon^2 Q(0)^2 \rho_0.$$

We have $Q(0)^2 \rho_0 \leq 24$. Hence, $C_1^2 + C_2^2 \leq 96m_\varepsilon^2$ and $C_1^2 + C_2^2$ is uniformly bounded in ε for small $\varepsilon > 0$.

If we multiply the first equation in (9) by $v_0 - C_2$ and the second equation by $u_0 - C_1$ and subtract, we obtain after integration

$$Q(x)[u'_0(x)(v_0(x) - C_2) - v'_0(x)(u_0(x) - C_1)] = C_4 \quad (11)$$

with some constant C_4 ,

$$C_4 = Q(0)[C_1 v'_0(0) - C_2 u'_0(0)] = m_\varepsilon Q(0)^2 [u'_0(1)v'_0(0) - v'_0(1)u'_0(0)].$$

Therefore,

$$|C_4| \leq 2m_\varepsilon Q(0)^2 \rho_0 \leq 24m_\varepsilon.$$

We set

$$r(x) = u'_0(x)^2 + v'_0(x)^2, \quad \rho(x) = (u_0(x) - C_1)^2 + (v_0(x) - C_2)^2.$$

Then

$$Q(x) = -\varepsilon/m_\varepsilon + r(x)^{-2} = \frac{m_\varepsilon - \varepsilon r^2}{m_\varepsilon r^2}$$

and by (10),

$$-\varepsilon r(x) + 3m_\varepsilon/r(x) + \rho(x) = C_3,$$

i.e.,

$$r(x) = \frac{6m_\varepsilon}{C_3 - \rho(x) + \sqrt{(C_3 - \rho(x))^2 + 12m_\varepsilon\varepsilon}}.$$

Solving the system

$$\begin{aligned} u'_0(x)^2 + v'_0(x)^2 &= r(x), \\ u'_0(x)(v_0(x) - C_2) - v'_0(x)(u_0(x) - C_1) &= \rho_1(x), \end{aligned}$$

where

$$\begin{aligned} \rho_1(x) &= \frac{C_4}{Q(x)} = \frac{C_4 m_\varepsilon r(x)^2}{m_\varepsilon - \varepsilon r(x)^2} = \frac{36C_4 m_\varepsilon^2}{(C_3 - \rho + R)^2 - 36m_\varepsilon\varepsilon} \\ R(x) &= \sqrt{(C_3 - \rho(x))^2 + 12m_\varepsilon\varepsilon}, \end{aligned}$$

we obtain

$$u'_0(x) = \frac{\rho_1(x)}{\rho(x)}(v_0(x) - C_2) \pm \frac{\rho_2(x)}{\rho(x)}(u_0(x) - C_1), \quad (12)$$

$$v'_0(x) = -\frac{\rho_1(x)}{\rho(x)}(u_0(x) - C_1) \pm \frac{\rho_2(x)}{\rho(x)}(v_0(x) - C_2), \quad (13)$$

where

$$\rho_2(x) = \sqrt{\rho(x)r(x) - \rho_1(x)^2}.$$

This implies that

$$(u_0(x) - C_1)u'_0(x) + (v_0(x) - C_2)v'_0(x) = \pm\sqrt{\rho(x)r(x) - \rho_1(x)^2}.$$

Therefore,

$$\rho'(x)^2 = 4\rho_2(x)^2 = G_\varepsilon(\rho(x)), \quad (14)$$

where

$$G_\varepsilon(\rho) = 4 \frac{6\rho[(C_3 - \rho + R)^2 - 36m_\varepsilon\varepsilon]^2 - (36m_\varepsilon^2 C_4)^2 (C_3 - \rho + R)}{(C_3 - \rho + R)[(C_3 - \rho + R)^2 - 36m_\varepsilon\varepsilon]^2}.$$

Moreover,

$$\rho(0) = \rho(1) = C_1^2 + C_2^2.$$

Since $\rho(0) < 3/2$, we see that $Q(0) > 4/9 - \varepsilon/m_\varepsilon > 1/3$ for small $\varepsilon > 0$.

D. Let us show that the function Q is symmetric, i.e., $Q(x) = Q(1 - x)$. As we showed above, $Q(0) = Q(1)$ and

$$u'_0(1) = u'_0(0) + C_1/Q(0)m_\varepsilon, \quad v'_0(1) = v'_0(0) + C_2/Q(0)m_\varepsilon.$$

Therefore,

$$\begin{aligned} \rho'(1) &= 2(u_0(1) - C_1)u'_0(1) + 2(v_0(1) - C_2)v'_0(1) = -2(C_1u'_0(1) + C_2v'_0(1)) \\ &= \rho'(0) - 2(C_1^2 + C_2^2)/Q(0)m_\varepsilon < \rho'(0). \end{aligned}$$

Since $\rho'(1) = \pm\rho'(0)$ by (14), this means that $\rho'(1) = -\rho'(0)$. Therefore, (14) implies that the function $r_\varepsilon(x)$ is even and the function Q_ε is also even.

Now we show that the minimal value of the functional F_ε in $S_1 \times S_2$ coincides with m_ε . It suffices to verify that one of the functions u_0 and v_0 is odd and the other is even.

Let us show that $u_0 \not\equiv 0$. Indeed, if $F_\varepsilon[0, v_0] = m_\varepsilon$, then

$$\frac{d}{d\delta} F_\varepsilon[\sqrt{\delta}\varphi, v_0] = \int_0^1 (\varphi(x)^2 + \varepsilon\varphi'(x)^2)dx - m_\varepsilon \int_0^1 \frac{\varphi'(x)^2}{v'_0(x)^4}dx$$

for $\delta = 0$. Let $[a, b]$ be a subinterval of I where $v'_0(x)^4 \leq c$. We set $b - a = 4l$,

$$\varphi(x) = \begin{cases} x - a, & x \in (a, a + l), \\ -x + a + 2l, & x \in (a + l, a + 3l), \\ \varphi(x) = x - b, & x \in (a + 3l, b). \end{cases}$$

Then

$$\int_0^1 \varphi(x) dx = 0, \quad \int_0^1 \frac{\varphi'(x)^2}{v'_0(x)^4} dx \geq \frac{4l}{c}.$$

We can take l and ε sufficiently small such that

$$\int_a^b (\varphi^2 + \varepsilon \varphi'(x)^2) dx = 4 \int_a^{a+l} (x - a)^2 dx + \varepsilon l = 4 \left(\frac{l^3}{3} + \varepsilon l \right) < \frac{4m_\varepsilon l}{c}.$$

Therefore, $\frac{d}{d\delta} F_\varepsilon[\sqrt{\delta}\varphi, v_0] < 0$ for $\varepsilon = 0$ and m_ε is not the minimal value of F_ε .

Similarly, we can verify that $v_0 \not\equiv 0$. The functions u_0 and v_0 are linearly independent since the relation $u = kv$ implies the equality $F_\varepsilon[u, v] = F_\varepsilon[0, v]$.

Let us show that the functions u_0 and v_0 cannot both be odd. If this would be so, then $C_1 = C_2 = 0$ and we can assume that u_0 and v_0 are positive in $]0, 1/2[$. Then by the Sturm theorem, they are linearly dependent, which is impossible, as we have shown.

For the same reason, the functions u_0 and v_0 cannot both be even. In this case, the functions $u_0 - C_1$ and $v_0 - C_2$ satisfy the same homogeneous equation $m(Qz')' + z = 0$ and their derivatives vanish at $x = 1/2$; therefore, they are linearly dependent, i.e., $k_1(u_0 - C_1) + k_2(v_0 - C_2) = 0$. Since $u_0(0) = v_0(0)$, this implies that $k_1 u_0(x) + k_2 v_0(x) = 0$.

We know that

$$u'_0(0)^2 + v'_0(0)^2 = u'_0(1)^2 + v'_0(1)^2, \quad C_1 v'_0(0) - C_2 u'_0(0) = C_1 v'_0(1) - C_2 u'_0(1).$$

If $C_1 \neq 0$ and $C_2 \neq 0$, then either $u'_0(0) = u'_0(1)$ and $v'_0(0) = v'_0(1)$ or $u'_0(0) = -u'_0(1)$ and $v'_0(0) = -v'_0(1)$. In the first case, both u_0 and v_0 are odd, and in the second case, they are both even. Since this is impossible, one of the constants C_j vanishes. Assume that $C_1 = 0$. Then $C_2 \neq 0$ and u_0 is odd. Since $v'_0(0)^2 = v'_0(1)^2$ and v_0 cannot be odd, it is even.

E. Now we consider the following problem.

Problem F. Find the functions u and v of class S_0 giving the minimal value of the functional

$$F[u, v] = \int_0^1 [u(x)^2 + v(x)^2] dx \int_0^1 (u'(x)^2 + v'(x)^2)^{-1} dx.$$

Lemma 2. Problem F has a solution (u, v) , which is the limit in $C^1(I)$ of some sequence $(u_{\varepsilon_k}, v_{\varepsilon_k})$ of solutions of Problems F_{ε_k} as $\varepsilon_k \rightarrow 0$. These functions are linearly independent, belong to $C^\infty(0, 1)$, are analytic on this interval, and satisfy Eqs. (7), (8). Moreover, we can assume that u is odd and v is even, so that

$$F[u, v] = \inf_{\tilde{u} \in S_0} F[\tilde{u}, \tilde{v}] = \inf_{\substack{\tilde{u} \in S_1 \\ \tilde{v} \in S_2}} F[\tilde{u}, \tilde{v}].$$

Proof. Let u_ε and v_ε be the functions from $S_1 \times S_2$ minimizing the functional F_ε . The function u_ε is positive on $]0, 1/2[$ and, by the Sturm theorem, $v_\varepsilon - C_\varepsilon$ has a unique zero in this interval.

Since $C_3 \leq 4m_\varepsilon + 8m_\varepsilon^2$ and $C_4 \leq 2m_\varepsilon$, we can assume that $m_\varepsilon \rightarrow m$, $C_\varepsilon = C_{2\varepsilon} \rightarrow C$, $C_{3\varepsilon} \rightarrow C_3$, and $C_{4\varepsilon} \rightarrow C_4$. From (10), we see that

$$u'_\varepsilon(0)^2 + v'_\varepsilon(0)^2 \rightarrow \frac{3}{4}, \quad Q(0) \rightarrow \frac{16}{9}.$$

We can assume that $v'_\varepsilon(0) \rightarrow k$ and $u'_\varepsilon(0) \rightarrow \sqrt{3/4 - k^2}$.

As $\varepsilon \rightarrow 0$, we see from (14) that

$$G_\varepsilon(\rho) \rightarrow G(\rho) = \frac{12\rho(C_3 - \rho) - 324m^2C_4^2}{(C_3 - \rho)^3}.$$

The function G takes negative values at $\rho = 0$ and at $\rho = C_3$. It has two real, positive roots, R_1 and R_2 . Since r_ε takes its values in the interval between two real roots of G_ε , we obtain that $2R_2 \geq r_\varepsilon(x) \geq R_1/2$ for all small ε and $0 \leq x \leq 1$.

Equalities (12) and (13) imply that the first derivatives of the functions u_ε and v_ε are uniformly bounded. Since

$$\begin{aligned} u''_\varepsilon(x) \left(-\frac{\varepsilon}{m_\varepsilon} + \frac{3u'_\varepsilon(x)^2 - v'_\varepsilon(x)^2}{(u'_\varepsilon(x)^2 + v'_\varepsilon(x)^2)^3} \right) + 4v''_\varepsilon(x) \frac{u'_\varepsilon(x)v'_\varepsilon(x)}{(u'_\varepsilon(x)^2 + v'_\varepsilon(x)^2)^3} &= \frac{C_1 - u_\varepsilon(x)}{m_\varepsilon}, \\ v''_\varepsilon(x) \left(-\frac{\varepsilon}{m_\varepsilon} + \frac{3v'_\varepsilon(x)^2 - u'_\varepsilon(x)^2}{(u'_\varepsilon(x)^2 + v'_\varepsilon(x)^2)^3} \right) + 4u''_\varepsilon(x) \frac{u'_\varepsilon(x)v'_\varepsilon(x)}{(u'_\varepsilon(x)^2 + v'_\varepsilon(x)^2)^3} &= \frac{C_2 - v_\varepsilon(x)}{m_\varepsilon}, \end{aligned}$$

we have

$$\begin{aligned} D_\varepsilon(x)u''_\varepsilon(x) &= (C_1 - u_\varepsilon(x)) \left(-\frac{\varepsilon}{m_\varepsilon} + \frac{3v'_\varepsilon(x)^2 - u'_\varepsilon(x)^2}{(u'_\varepsilon(x)^2 + v'_\varepsilon(x)^2)^3} \right) - (C_2 - v_\varepsilon(x)) \frac{4u'_\varepsilon(x)v'_\varepsilon(x)}{(u'_\varepsilon(x)^2 + v'_\varepsilon(x)^2)^3}, \\ D_\varepsilon(x)v''_\varepsilon(x) &= (C_2 - v_\varepsilon(x)) \left(-\frac{\varepsilon}{m_\varepsilon} + \frac{3u'_\varepsilon(x)^2 - v'_\varepsilon(x)^2}{(u'_\varepsilon(x)^2 + v'_\varepsilon(x)^2)^3} \right) - (C_1 - u_\varepsilon(x)) \frac{4u'_\varepsilon(x)v'_\varepsilon(x)}{(u'_\varepsilon(x)^2 + v'_\varepsilon(x)^2)^3}, \end{aligned}$$

where

$$D_\varepsilon = \frac{4\varepsilon^2}{3m_\varepsilon} - 3m_\varepsilon \left(\frac{\varepsilon}{3m_\varepsilon} + \frac{1}{(u'_\varepsilon(x)^2 + v'_\varepsilon(x)^2)^2} \right)^2$$

tends to $-3m/\rho(x)^4$ as $\varepsilon \rightarrow 0$. Therefore, the second derivatives of u_ε and v_ε are uniformly bounded:

$$\sup_x (|u''_\varepsilon(x)| + |v''_\varepsilon(x)|) \leq A.$$

Hence the set of functions u_ε and v_ε is compact in $C^1(I)$ and we can find subsequences converging in $C^1(I)$ as $\varepsilon \rightarrow 0$, $u_\varepsilon \rightarrow u$, and $v_\varepsilon \rightarrow v$.

In particular, the last system takes, after passing to the limit, the following form:

$$\begin{aligned} \frac{3m}{\rho} u'' &= (3v'(x)^2 - u'(x)^2)(u - C_1) - 4u(x)v'(x)(v - C_2), \\ \frac{3m}{\rho} v'' &= -4u(x)v'(x)(u - C_1) + (3u'(x)^2 - v'(x)^2)(v - C_2). \end{aligned}$$

Now let μ be the minimum in Problem F. Since always $F[u, v] \leq F_\varepsilon[u, v]$, we have $\mu \leq m$. On the other hand, if $\mu < m$, then there exist $u_1 \in S_1$ and $v_1 \in S_2$ such that $F[u_1, v_1] < m - \delta$ with $\delta > 0$. Then

$$F_\varepsilon[u_1, v_1] < m - \delta + \varepsilon \int_0^1 [u'_1(x)^2 + v'_1(x)^2] dx \int_0^1 [u'_1(x)^2 + v'_1(x)^2]^{-1} dx < m - \frac{\delta}{2} < m_\varepsilon - \frac{\delta}{4}$$

for small $\varepsilon > 0$, which is impossible. Therefore, $\mu = m$ and the functions u and v yield a solution of Problem F.

We set $H_\varepsilon(\rho) = G_\varepsilon(C_3 - \rho)$. If $\varepsilon \rightarrow 0$, then $H_\varepsilon(\rho) \rightarrow H(\rho) = c_1\rho^3 - \rho^4 - c_2$, where

$$c_1 = 4m + \left(\frac{32}{9}mk\right)^2, \quad c_2 = \left(\frac{2^{18}}{3^5}\right)m^5k^2\left(\frac{3}{4} - k^2\right),$$

$m = \lim m_\varepsilon$, $k = v'(0)$. Therefore, there exist positive limit values of the real roots R_1 and R_2 . They serve as the bounds for the limit function $\rho(t)$. Moreover, $R_2 < c_1 = \lim_{\varepsilon \rightarrow 0} C_3$ and, by (10), $\rho(x) \geq 3m_\varepsilon/b_1$.

In any case, we see that

$$Q(x) = Q_\varepsilon(x) \rightarrow \frac{(c_1 - \rho(x))^2}{9m^2}.$$

Obviously, the function u is odd and v is even in $[0, 1]$. Moreover, the function u is positive in $]0, 1/2[$ and v is positive in $]0, t_p[$ and negative in $]t_p, 1/2[$. Passing to the limit, we see that the functions u and v satisfy Eqs. (7) and (8).

Let us show that $u \not\equiv 0$. Indeed, if $F[0, v] = m$, then

$$\frac{d}{d\varepsilon} F[\sqrt{\varepsilon}\varphi, v] = \int_0^1 \varphi(x)^2 dx - m \int_0^1 \frac{\varphi'(x)^2}{v'(x)^4} dx$$

for $\varepsilon = 0$. Let $[a, b]$ be a subinterval of I where $v'(x)^4 \leq c$. Set $b - a = 4l$,

$$\varphi(x) = \begin{cases} x - a, & x \in (a, a + l), \\ -x + a + 2l, & x \in (a + l, a + 3l), \\ x - b, & x \in (a + 3l, b). \end{cases}$$

Then

$$\int_0^1 \varphi(x) dx = 0.$$

We can take l sufficiently small such that

$$m \int_0^1 \frac{\varphi'(x)^2}{v'(x)^4} dx \geq \frac{ml}{c} > 4 \int_a^{a+l} (x - a)^2 dx = \frac{4l^3}{3},$$

i.e., $l^2 < 3m/4c$. Therefore, $\frac{d}{d\varepsilon} F[\sqrt{\varepsilon}\varphi, v] < 0$ at $\varepsilon = 0$ and m is not the minimal value of F .

Similarly, we can verify that $v \not\equiv 0$. The functions u and v are linearly independent since the equality $u = kv$ implies that $F[u, v] = F[0, v]$.

System (7), (8) is equivalent to the following system of four first-order equations:

$$u' = \frac{z}{Q_0}, \quad z' = -Mu, \quad v' = \frac{w}{Q_0}, \quad w' = M(v - c),$$

where $M = 1/m$ and $Q_0(x) = (u'(x)^2 + v'(x)^2)^{-2}$. Since $Q_0 \in C(I)$ and $Q_0 > 0$, we see that $u, v \in C^1(I)$. Therefore, $Q_0 u' \in C^1(I)$, $Q_0 v' \in C^1(I)$, and

$$(Q_0 u')^2 + (Q_0 v')^2 = \frac{1}{(u'^2 + v'^2)^2} \in C^1(I).$$

Hence, $Q_0 \in C^1(I)$ and $u, v, z, w \in C^2(I)$. We can continue and obtain that $u, v \in C^\infty(I)$. It is easy to see that the functions u, v, z, w , and Q are analytic in I . $\square$

F. We say that $u \in S_0(a, b)$, where $0 \leq a < b \leq 1$, if $u(a + x(b - a)) \in S_0$.

Lemma 3. Let f be a positive function from $C(0, 1)$ monotone in (a_j, a_{j+1}) , where $j = 0, 1, 2, 3$ and $0 = a_0 \leq a_1 \leq a_2 \leq a_3 \leq a_4 = 1$. Then there exist two functions u and v from S_0 such that

$$f = u'^2 + v'^2.$$

Proof. Set

$$u(x) = \int_0^x \sqrt{f(t)} \cos \theta(t) dt, \quad v(x) = \int_0^x \sqrt{f(t)} \sin \theta(t) dt.$$

Let Φ be the set of monotonically decreasing functions θ such that $0 \leq \theta(0) \leq \pi/2$ and $-5\pi/2 \leq \theta(1) \leq -2\pi$. It is easy to verify that $(u, v) \in S_0$ if and only if $\theta \in \Phi$ and

$$\begin{aligned} \int_0^1 \sqrt{f(t)} \cos \theta(t) dt &= 0, & \int_0^1 \sqrt{f(t)} \sin \theta(t) dt &= 0, \\ \int_0^1 t \sqrt{f(t)} \cos \theta(t) dt &= 0, & \int_0^1 t \sqrt{f(t)} \sin \theta(t) dt &= 0. \end{aligned}$$

We set

$$\begin{aligned} \alpha &= \int_0^1 \sqrt{f(t)} \cos \theta(t) dt, & \beta &= \int_0^1 \sqrt{f(t)} \sin \theta(t) dt, \\ \gamma &= \int_0^1 t \sqrt{f(t)} \cos \theta(t) dt, & \delta &= \int_0^1 t \sqrt{f(t)} \sin \theta(t) dt, \end{aligned}$$

and $B[\theta] = \alpha^2 + \beta^2 + \gamma^2 + \delta^2$.

Let μ be the minimum of B in Φ when the function f is fixed. It is easy to see that the set Φ is compact by the Helly theorem. Therefore, if $B[\theta_k] \rightarrow 0$, we can assume that $\theta_k(x) \rightarrow \theta_0(x)$ for almost all x and $B[\theta_0] = \mu$.

Let us show that $\mu = 0$. Assume that θ_0 is strictly monotone in a subdomain $\omega \subset [0, 1]$ and $\varphi \in C_0(\omega)$. Then

$$\frac{d}{d\varepsilon} B[\theta_0 + \varepsilon\varphi] = 0$$

for $\varepsilon = 0$, i.e.,

$$\int_0^1 \sqrt{f(t)} \left[-\alpha \sin \theta_0(t) + \beta \cos \theta_0(t) - \gamma t \sin \theta_0(t) + \delta t \cos \theta_0(t) \right] \varphi(t) dt = 0.$$

Therefore,

$$(\alpha + x\gamma) \sin \theta_0(x) = (\beta + x\delta) \cos \theta_0(x),$$

i.e., either $\mu = 0$ or

$$\sin \theta_0(x) = \frac{\beta + x\delta}{r(x)}, \quad \cos \theta_0(x) = \frac{\alpha + x\gamma}{r(x)}, \quad r(x)^2 = (\beta + x\delta)^2 + (\alpha + x\gamma)^2.$$

However, then u' and v' can change their signs only once, which is impossible for $\theta_0 \in \Phi$. Therefore, $\mu = 0$ and $(u_0, v_0) \in S_0$. The proof is complete. $\square$

G.

Lemma 4. Let Q be a continuous, positive function defined in $[0, 1/2]$, $Q(0) > 1$. Consider the following Sturm–Liouville problem:

$$(Q(x)y')' + \lambda y = 0 \quad \text{on} \quad [0, 1/2], \quad 2Q(0)y'(0) = \lambda y(0), \quad y'(1/2) = 0.$$

Its spectrum is discrete, $\lambda_j \rightarrow \infty$ as $j \rightarrow \infty$, $\lambda_0 = 0 < \lambda_1 < \dots$, and the eigenvalues are simple. For positive j , the number of zeros of the eigenfunction φ_j corresponding to λ_j is exactly j and the number of zeros of φ'_j in $[0, 1/2[$ is also j .

Proof. If $\lambda < 0$, then

$$-\int_0^{1/2} Q(x)y'(x)^2 dx + \lambda/2 y^2(0) + \lambda \int_0^{1/2} y^2(x) dx$$

and, therefore, $y \equiv 0$. If $\lambda = 0$, then $y(x) = \text{const.}$

Positive eigenvalues λ_j can be found as

$$\lambda_j = \inf_{y \in M_j} \int_0^{1/2} Q(x)y'(x)^2 dx,$$

where M_j is the set of functions $y \in H^1(0, 1/2)$ such that

$$\int_0^{1/2} y(x)^2 dx - y(0)^2/2 = 1, \quad \int_0^{1/2} y(x)\varphi_i(x) dt = y(0)\varphi_i(0)/2, \quad 0 \leq i < j.$$

Note that for $i = 0$, the last condition means that

$$y(0) = 2 \int_0^{1/2} y(x) dx$$

and, therefore,

$$y(0)^2 < 2 \int_0^{1/2} y(x)^2 dx$$

if $y(x) \not\equiv \text{const.}$ The rest of the proof is standard (see, e.g., [4, p. 166]). □

Lemma 5. Let

$$M = \sup_{Q \in \mathcal{A}} \inf_{\substack{u \in S_0 \\ v \in S_0}} \frac{\int_0^1 Q(x)(u'(x)^2 + v'(x)^2) dx}{\int_0^1 (u(x)^2 + v(x)^2) dx}.$$

Then $M \leq 1/m$, where m was defined in Lemma 2.

Proof. Let Q be a function from $\mathcal{A} \cap C^2(I)$. If Q has less than five intervals of monotonicity, then we can apply Lemma 3 and find $(u, v) \in S_0$ such that

$$f(x) = \frac{1}{\sqrt{Q}} = u'(x)^2 + v'(x)^2$$

in I . Since

$$\int_0^1 Q(x)^{1/2} dx = 1,$$

we have

$$\frac{\int_0^1 Q(x)(u'(x)^2 + v'(x)^2)dx}{\int_0^1 (u(x)^2 + v(x)^2)dx} = \frac{\int_0^1 (u'(x)^2 + v'(x)^2)^{-1}dx}{\int_0^1 (u(x)^2 + v(x)^2)dx} = \frac{1}{F[u, v]} \leq \frac{1}{m}.$$

If Q has more than four intervals of monotonicity, then we can choose a subinterval $[a, b]$ where Q has exactly four intervals of monotonicity and set $u = v = 0$ outside this subinterval. We can construct the functions u and v as above. Then

$$\int_a^b Q(x)^{1/2} dx \leq 1,$$

and the above arguments hold. $\square$

H.

Lemma 6. *Let u and v be the functions found in Lemma 2. Let $k = v'(0)$, $c_1 = 4m + (32mk/9)^2$, and $c_2 = 512mk\sqrt{3/4 - k^2}$. We set $r(x) = c_1 - u(x)^2 - (v(x) - C)^2$. Then*

$$r'(x)^4 r(x)^2 = 12m((c_1 - r(x))r(x)^3 - c_2^2(3m)^3).$$

The function r cannot take a constant value on any nonempty subinterval of I .

Proof. Multiplying Eqs. (7) and (8) by u' and v' , respectively, and adding, we obtain

$$-3m \frac{u'(x)u''(x) + v'(x)v''(x)}{(u'(x)^2 + v'(x)^2)^2} + u(x)u'(x) + (v(x) - C)v'(x) = 0.$$

After integration, we obtain

$$\frac{3m}{u'(x)^2 + v'(x)^2} + [u(x)^2 + (v(x) - C)^2] = c_1. \quad (15)$$

Integrating this equality over $(0, 1)$, we have

$$c_1 = 4m + C^2.$$

Setting $x = 0$ and using (7) and (8), we can see that

$$u'(0)^2 + v'(0)^2 = \frac{3}{4}, \quad Q(0) = \frac{16}{9}, \quad C = -\frac{32}{9}mv'(0).$$

On the other hand, multiplying Eqs. (7) and (8) by $v - C$ and u and subtracting, we see that

$$m \left(\frac{u'(x)(v(x) - C) - v'(x)u(x)}{(u'(x)^2 + v'(x)^2)^2} \right)' = 0,$$

i.e.,

$$u'(x)(v(x) - C) - v'(x)u(x) = c_2(u'(x)^2 + v'(x)^2)^2. \quad (16)$$

Setting $x = 0$, we see that

$$c_2 = -\frac{16}{9}Cu'(0) = \frac{512}{81}mu'(0)v'(0).$$

Set

$$\rho(x) = u(x)^2 + (v(x) - C)^2.$$

By (15) and (16), we have

$$u'(x)^2 + v'(x)^2 = \frac{3m}{c_1 - \rho},$$

$$u'(x)(v(x) - C) - v'(x)u(x) = c_2 \left(\frac{3m}{c_1 - \rho} \right)^2.$$

Solving this system for $u'(x)$ and $v'(x)$, we obtain that

$$\rho(x)u'(x) = c_2(v(x) - C)f(x)^2 \pm u(x)\sqrt{f(x)\rho(x) - c_2^2 f(x)^4}; \quad (17)$$

$$\rho(x)v'(x) = -c_2u(x)f(x)^2 \pm (v(x) - C)\sqrt{f(x)\rho(x) - c_2^2 f(x)^4}, \quad (18)$$

where $f(x) = 3m/(c_1 - \rho(x))$. Multiplying (17) by u and (18) by $v(x) - C$ and summing, we find that

$$\rho'(x) = \pm 2\sqrt{f(x)\rho(x) - c_2^2 f(x)^4}, \quad \rho(0) = C^2.$$

Setting $r(x) = c_1 - \rho(x)$, we see that $r(0) = 4m$ and

$$r'(x)^2 = 4 \left(3m \frac{c_1 - r(x)}{r(x)} - c_2^2 \frac{(3m)^4}{r(x)^4} \right) = \frac{12m}{r(x)^4} (c_1 r(x)^3 - r(x)^4 - c_2^2 (3m)^3).$$

Since $v'(0) = k$, we have $u'(0) = \sqrt{3/4 - k^2}$,

$$c_1 = 4m + \left(\frac{32}{9}mk \right)^2, \quad c_2 = \frac{512}{81}mk\sqrt{3/4 - k^2}.$$

The polynomial

$$P(r) = c_1 r^3 - r^4 - c_2^2 (3m)^3$$

has two real roots r_1 and r_2 such that

$$0 < r_1 < 4m < r_2.$$

The solution $r(x)$ monotonically decreases from $4m$ to r_1 on the interval $[0, p]$ and increases from r_1 to r_2 on $[p, t_1]$. After that it can oscillate between r_1 and r_2 several times and, at the end, $r(1/2) = r_2$ and $r(1) = 4m$. The function $r(x)$ cannot take a constant value on any subinterval of I because of its analyticity. $\square$

I. The optimal shape of the strongest column is defined as

$$Q_0(x) = \frac{r(x)^2}{9m^2}.$$

The values k and m can be found from the conditions

$$\theta(1/2) - \theta(0) = -\pi, \quad r(1/2) = r_2,$$

where $\theta = \arctan(v - c)/u$, and

$$\frac{d\theta}{dx} = -\frac{9m^2 c_2}{\rho(c_1 - \rho)^2} = -\frac{9m^2 c_2}{r^2(c_1 - r)};$$

this means that

$$2 \int_{r_1}^{4m} \frac{r^2 dr}{\sqrt{P(r)}} + \int_{r_2}^{4m} \frac{r^2 dr}{\sqrt{P(r)}} = \sqrt{3m},$$

$$2 \int_{r_1}^{4m} \frac{dr}{(c_1 - r)\sqrt{P(r)}} + \int_{4m}^{r_2} \frac{dr}{(c_1 - r)\sqrt{P(r)}} = \frac{2\pi}{c_2(3m)^{3/2}}.$$

The last system of two equations for k and m can be solved numerically. The minimal value of m corresponds to $p_1 = 2$ and $p_2 = 1$ and is equal to 0.01908.

The integrals involved in this system are improper. In order to work with them, we use the factorization

$$P(r) = (r - r_1)(a_0 + a_1 r + a_2 r^2 - r^3), \quad r_1 < r < 4m,$$

$$P(r) = (r_2 - r)(b_0 + b_1 r + b_2 r^2 + r^3), \quad 4m < r < r_2,$$

with

$$a_2 = c_1 - r_1, \quad a_1 = r_1 a_2, \quad a_0 = r_1 a_1, \quad b_2 = r_2 - c_1, \quad b_1 = r_2 b_2, \quad b_0 = r_2 b_1$$

and the substitution

$$r = r_1 + s^2, \quad r = r_2 - t^2$$

regularizing the integrals.

3. Proof of Theorem 1

A. Set

$$L[Q, u, v] = \frac{\int_0^1 Q(x)(u'(x)^2 + v'(x)^2)dx}{\int_0^1 (u(x)^2 + v(x)^2)dx}.$$

Let $u_0 = u$ and $v_0 = v$ be the functions found in Lemma 2 such that

$$\int_0^1 (u'_0(x)^2 + v'_0(x)^2)^{-2} dx = 1.$$

We set

$$Q_0(x) = (u'_0(x)^2 + v'_0(x)^2)^{-2}.$$

We know that $Q_0 \in C^\infty[0, 1]$ and $Q_0(x) = Q_0(1 - x)$.

Let us show that

$$L[Q_0, u_0, v_0] \geq \sup_{Q \in \mathcal{A}} \inf_{u, v \in S \times S} L[Q, u, v].$$

We have

$$L[Q_0, u_0, v_0] = \frac{\int_0^1 Q_0(x)(u'_0(x)^2 + v'_0(x)^2)dx}{\int_0^1 (u_0(x)^2 + v_0(x)^2)dx} = \frac{\int_0^1 (u'_0(x)^2 + v'_0(x)^2)^{-1}dx}{\int_0^1 (u_0(x)^2 + v_0(x)^2)dx} = \frac{1}{m},$$

since

$$\int_0^1 (u'_0(x)^2 + v'_0(x)^2)^{-1} dx = 1.$$

On the other hand, if Q is a positive function from $C^1(I)$ and

$$\int_0^1 Q^{1/2}(x)dx = 1,$$

then, by Lemma 5, we have $L[Q, u, v] \leq 1/m$.

Therefore, for any $Q \in \mathcal{A}$, we have

$$\inf_{\substack{u \in S \\ v \in S}} L[Q, u, v] \leq \inf_{\substack{u \in S_0 \\ v \in S_0}} L[Q, u, v] \leq \frac{1}{m} \equiv M.$$

Thus,

$$\sup_Q \inf_{\substack{u \in S \\ v \in S}} L[Q, u, v] \leq M, \quad \sup_Q \inf_{y \in S} L_1[Q, y] \leq M.$$

B. Now we show that

$$\inf_{y \in S} L_1[Q_0, y] = \inf_{\substack{u \in S \\ v \in S}} L[Q_0, u, v] = \inf_{\substack{u \in S_1 \\ v \in S_2}} L[Q_0, u, v] = M.$$

By Lemma 2, $u_0 \not\equiv 0$ and $v_0 \not\equiv 0$. Moreover, let

$$\inf_{u \in S_{\text{odd}}} L[Q_0, u, 0] = \mu.$$

This minimal value is attained for a function u_1 which is positive in $]0, 1/2[$. It satisfies the Lagrange equation

$$(Q_0 u_1'(x))' + \mu u_1(x) = 0,$$

$u_1(0) = 0$, and $u_1(1/2) = 0$.

The function u_0 is odd and positive in $]0, 1/2[$, and $(Q_0(x)u_0'(x))' + Mu_0(x) = 0$. Therefore,

$$(M - \mu) \int_0^{1/2} u_1(x)u_0(x)dx = 0.$$

The functions u_0 and u_1 are positive in $]0, 1/2[$. Therefore, $M = \mu$ and $u_1 = ku_0$.

C. Now let

$$\inf_{v \in S_{\text{even}}} L[Q_0, 0, v] = \mu_1.$$

This minimal value is attained for a function v_1 which is positive in $]0, p[$ and negative in $]p, 1/2[$. It satisfies the Lagrange equation

$$(Q_0 v_1'(x))' + \mu_1 v_1(x) = C',$$

$v_1(0) = 0$, and $v_1'(1/2) = 0$, where $C' = -2Q_0(0)v_1'(0) < 0$.

The function v_0 is positive in $]0, p_0[$ and negative in $]p_0, 1/2[$, and $(Q_0(x)v_0'(x))' + Mv_0(x) = C$ with $C = -2Q_0(0)v_0'(0) < 0$. Therefore, the functions $w_0 = v_0 - C/M$ and $w_1 = v_1 - C'/\mu_1$ satisfy the equations

$$(Q_0 w_0')' + Mw_0 = 0, \quad (Q_0 w_1')' + \mu_1 w_1 = 0.$$

Moreover,

$$\begin{aligned} w_0(0) &= -\frac{C}{M}, & M \int_0^{1/2} w_0(x)dx &= Q_0(0)w_0'(0), & \int_0^{1/2} w_0(x)dx &= -\frac{C}{2M}, \\ w_1(0) &= -\frac{C'}{\mu_1}, & \mu_1 \int_0^{1/2} w_1(x)dx &= Q_0(0)w_1'(0), & \int_0^{1/2} w_1(x)dx &= -\frac{C'}{2\mu_1}. \end{aligned}$$

Therefore, they are eigenfunctions of the Sturm–Liouville operator corresponding to the boundary problem

$$(Q_0 z')' + \lambda z = 0, \quad \lambda z(0) = 2Q_0(0)z'(0), \quad z'(1/2) = 0,$$

considered in Lemma 4. Since both w_0 and w_1 have exactly one zero in $[0, 1/2]$, they are linearly dependent, i.e., $w_1(x) = k'w_0(x)$. Therefore, $v_1(x) = k'v_0(x) + C'''$. But $v_0(0) = v_1(0) = 0$; therefore, $C''' = 0$, $v_1 = k'v_0$, and $\mu_1 = M$.

D. If $y \in S$, then $y = u + v$, where $u \in S_{\text{odd}}$ and $v \in S_{\text{even}}$. Therefore,

$$L_1[Q_0, y] = \frac{\int_0^{1/2} Q_0(x)[u'(x)^2 + v'(x)^2]dx}{\int_0^{1/2} [u(x)^2 + v(x)^2]dx} \geq M$$

since

$$\int_0^{1/2} Q_0(x)u'(x)^2 dx \geq M \int_0^{1/2} u(x)^2 dx, \quad \int_0^{1/2} Q_0(x)v'(x)^2 dx \geq M \int_0^{1/2} v(x)^2 dx,$$

as we have shown. We obtain

$$\inf_{y \in S} L_1[Q_0, y] = \inf_{\substack{u \in S_1 \\ v \in S_2}} L[Q_0, u, v] = L_1[Q_0, y_0] = M$$

and $y_0 = u_0 + v_0$.

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