Weak and Strong Convergence Theorems of Quasi-Nonexpansive Mappings in a Hilbert Spaces

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Abstract In this paper, we first prove the weak convergence for the Moudafi's iterative scheme of two quasi-nonexpansive mappings. Then we prove the weak convergence for the Moudafi's iterative scheme of quasi-nonexpansive mapping and nonexpansive mapping. Finally, we prove the strong convergence for the Moudafi's iterative scheme of two quasi-nonexpansive mappings. Our results generalize the recent results due to Iemoto and Takahashi.

Keywords Weak and strong convergence · Fixed point · Moudafi iteration process · Quasi-nonexpansive mapping

1 Introduction

Approximating fixed points of nonexpansive mappings by an iterative sequence has been investigated by several authors; see, e.g., [1–5] and others. Recently, Iemoto and Takahashi [6] obtained some fundamental properties for nonspreading mappings in a Hilbert space. Further, they studied the approximation of common fixed points of nonexpansive mappings and nonspreading mappings in a Hilbert space by using Moudafi's iterative scheme.

In this paper, we first prove the weak convergence of two quasi-nonexpansive mappings in a Hilbert space by using Moudafi's iterative scheme. Then we prove the weak convergence of quasi-nonexpansive mapping and nonexpansive mapping in a Hilbert space by using Moudafi's iterative scheme. Finally, we prove the strong convergence of two quasi-nonexpansive mappings satisfying condition **A** in a Hilbert

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space by using Moudafi's iterative scheme. Our results generalize the recent results due to Iemoto and Takahashi [6].

2 Preliminaries

Throughout this paper, we denote by H a real Hilbert space. Let C be a nonempty, closed, and convex subset of H . Then a mapping T of C into itself is called *nonexpansive* iff $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We denote by $F(T)$ the set of all fixed points of T , i.e., $F(T) := \{x \in C : Tx = x\}$. A mapping T from C into C is also called *quasi-nonexpansive* iff the set $F(T)$ of fixed points of T is nonempty and $\|Tx - y\| \leq \|x - y\|$ for all $x \in C$ and $y \in F(T)$. For two mappings S, T of C into itself, we consider the following iteration scheme (Moudafi [7]): $x_1 \in C$,

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n[\beta_n Sx_n + (1 - \beta_n)Tx_n] \quad (1)$$

for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1]$. If $\beta_n = 0$, then such an iteration scheme shrinks to that introduced by Mann [1]. In a Hilbert space, it is known that

$$\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2 \quad (2)$$

for all $x, y \in H$ and $\alpha \in \mathbb{R}$; see [8]. A Banach space E is called *uniformly convex* iff for each $\epsilon > 0$ there is a $\delta > 0$ such that for $x, y \in E$ with $\|x\|, \|y\| \leq 1$ and $\|x - y\| \geq \epsilon$, $\|x + y\| \leq 2(1 - \delta)$ holds. Let E be a smooth, strictly convex and reflexive Banach space, let J be the duality mapping of E and let C be a nonempty, closed, and convex subset of E . Then a mapping $T : C \rightarrow C$ is said to be *nonspreading* [6] iff

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x)$$

for all $x, y \in C$, where $\phi(x, y) := \|x\|^2 - 2\langle x, J(y) \rangle + \|y\|^2$ for all $x, y \in E$. In the case when E is a Hilbert space, we know that $\phi(x, y) = \|x - y\|^2$ for all $x, y \in E$. So, a nonspreading mapping $T : C \rightarrow C$ in a Hilbert space H is defined as follows:

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|x - Ty\|^2$$

for all $x, y \in C$. Thus, Iemoto and Takahashi [6] proved that it is equivalent to

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty \rangle$$

for all $x, y \in C$. We know that in a Hilbert space, if the set of fixed points of a nonspreading mapping is nonempty, then the nonspreading mapping is quasi-nonexpansive; see [6]. When $\{x_n\}$ is a sequence in E , then $x_n \rightarrow x$ ($x_n \rightharpoonup x$) will denote strong (weak) convergence of the sequence $\{x_n\}$ to x . A mapping $T : C \rightarrow H$ is said to be *demiclosed* at $y \in H$ [9] iff for any sequence $\{x_n\}$ in C , it follows from $x_n \rightarrow x$ and $Tx_n \rightarrow y$ that $x \in C$ and $T(x) = y$. If $I - T$ is demiclosed at zero, i.e., for any sequence $\{x_n\}$ in C , the conditions $x_n \rightarrow x$ and $x_n - Tx_n \rightarrow 0$ imply $x - Tx = 0$.

We know that H satisfies *Opial's condition* [10], that is, for any sequence $\{x_n\}$ in H , $x_n \rightarrow x$ implies that

$$\limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\|$$

for all $y \in H$ with $y \neq x$. All Hilbert spaces and l^p ($1 < p < \infty$) satisfy Opial's condition, while L^p with $1 < p \neq 2 < \infty$ do not. Two mappings $S, T : C \rightarrow C$, where C is a subset of E , are said to satisfy condition **A** [11] (cf. [12]) iff there exists a nondecreasing function $f : [0, \infty] \rightarrow [0, \infty]$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in [0, \infty]$ such that

$$\frac{1}{2}(\|x - Tx\| + \|x - Sx\|) \geq f(d(x, \mathbf{F}))$$

for all $x \in C$, where $\mathbf{F} = F(S) \cap F(T) \neq \emptyset$ and $d(x, \mathbf{F}) = \inf_{z \in \mathbf{F}} \|x - z\|$.

3 Weak and Strong Convergence Theorems

We first begin with the following lemma.

Lemma 3.1 [4] *Let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative real numbers such that $\sum_{n=1}^{\infty} b_n < \infty$ and*

$$a_{n+1} \leq a_n + b_n$$

for all $n \geq 1$. Then $\lim_{n \rightarrow \infty} a_n$ exists.

Lemma 3.2 [13] *Let E be a uniformly convex Banach space. Let $x, y \in E$. If $\|x\| \leq 1$, $\|y\| \leq 1$, and $\|x - y\| \geq \epsilon > 0$, then $\|\lambda x + (1 - \lambda)y\| \leq 1 - 2\lambda(1 - \lambda)\delta(\epsilon)$ for λ with $0 \leq \lambda \leq 1$.*

Our Theorem 3.1 carries over Theorem 4.1(i), (iii) of Iemoto and Takahashi [6] to quasi-nonexpansive mappings.

Theorem 3.1 *Let H be a Hilbert space and C be a nonempty, closed and convex subset of H , and let S, T be two quasi-nonexpansive mappings of C into itself such that $I - T, I - S$ are demiclosed at zero with $F(S) \cap F(T) \neq \emptyset$. Suppose that, for any x_1 in C , $\{x_n\}$ is defined by (1). Then the following hold:*

- (i) *If $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$ and $\sum_{n=1}^{\infty} (1 - \beta_n) < \infty$, then $\{x_n\}$ converges weakly to $p \in F(S)$;*
- (ii) *if $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$ and $\liminf_{n \rightarrow \infty} \beta_n(1 - \beta_n) > 0$, then $\{x_n\}$ converges weakly to $p \in F(S) \cap F(T)$.*

Proof For any $z \in F(S) \cap F(T)$, since

$$\|x_{n+1} - z\| = \|(1 - \alpha_n)x_n + \alpha_n[\beta_n Sx_n + (1 - \beta_n)Tx_n] - z\|$$

$$\begin{aligned}
&\leq (1 - \alpha_n)\|x_n - z\| + \alpha_n\|\beta_n Sx_n + (1 - \beta_n)Tx_n - z\| \\
&\leq (1 - \alpha_n)\|x_n - z\| + \alpha_n\{\beta_n\|Sx_n - z\| + (1 - \beta_n)\|Tx_n - z\|\} \\
&\leq (1 - \alpha_n)\|x_n - z\| + \alpha_n\{\beta_n\|x_n - z\| + (1 - \beta_n)\|x_n - z\|\} \\
&= \|x_n - z\|
\end{aligned}$$

for all $n \geq 1$. We readily see that

$$\lim_{n \rightarrow \infty} \|x_n - z\| (\equiv c) \quad (3)$$

exists, and thus $\{x_n\}$ is bounded. (i) put $z_{n+1} = (1 - \alpha_n)x_n + \alpha_n Sx_n$. Then we obtain

$$\begin{aligned}
\|x_{n+1} - z_{n+1}\| &= \|(1 - \alpha_n)x_n + \alpha_n[\beta_n Sx_n + (1 - \beta_n)Tx_n] - (1 - \alpha_n)x_n - \alpha_n Sx_n\| \\
&= \alpha_n\|\beta_n Sx_n + (1 - \beta_n)Tx_n - Sx_n\| \\
&= \alpha_n(1 - \beta_n)\|Tx_n - Sx_n\| \\
&\leq (1 - \beta_n)\|Tx_n - Sx_n\|,
\end{aligned}$$

where $\sup_{n \geq 1} \|Tx_n - Sx_n\| < \infty$. Since $\sum_{n=1}^{\infty} (1 - \beta_n) < \infty$, we obtain $\|x_n - z_n\| \rightarrow 0$ as $n \rightarrow \infty$ and thus

$$\lim_{n \rightarrow \infty} \|z_n - z\| = \lim_{n \rightarrow \infty} \|x_n - z\| = c. \quad (4)$$

From (2), we obtain

$$\begin{aligned}
\|z_{n+1} - z\|^2 &= \|(1 - \alpha_n)x_n + \alpha_n Sx_n - z\|^2 \\
&= \|(1 - \alpha_n)(x_n - z) + \alpha_n(Sx_n - z)\|^2 \\
&= (1 - \alpha_n)\|x_n - z\|^2 + \alpha_n\|Sx_n - z\|^2 - \alpha_n(1 - \alpha_n)\|x_n - Sx_n\|^2 \\
&\leq (1 - \alpha_n)\|x_n - z\|^2 + \alpha_n\|x_n - z\|^2 - \alpha_n(1 - \alpha_n)\|x_n - Sx_n\|^2 \\
&= \|x_n - z\|^2 - \alpha_n(1 - \alpha_n)\|x_n - Sx_n\|^2.
\end{aligned}$$

Thus,

$$\alpha_n(1 - \alpha_n)\|x_n - Sx_n\|^2 \leq \|x_n - z\|^2 - \|z_{n+1} - z\|^2.$$

Since $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$ and by (4), we obtain

$$\lim_{n \rightarrow \infty} \|Sx_n - x_n\| = 0. \quad (5)$$

Since $\{x_n\}$ is a bounded sequence, there exist the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_i}\} \rightharpoonup p$ and $\{x_{n_j}\} \rightharpoonup q$, respectively. Since $I - S$ is demiclosed at zero and by (5), we have $p, q \in F(S)$. We first show that for any $y \in F(S)$, $\lim_{n \rightarrow \infty} \|x_n - y\|$ exists. In fact, for any $y \in F(S)$,

$$\|z_{n+1} - y\| = \|(1 - \alpha_n)x_n + \alpha_n Sx_n - y\|$$

$$\begin{aligned}
&\leq (1 - \alpha_n)\|x_n - y\| + \alpha_n\|Sx_n - y\| \\
&\leq \|x_n - y\| \\
&\leq \|z_n - y\| + \|x_n - z_n\|.
\end{aligned}$$

By Lemma 3.1, $\lim_{n \rightarrow \infty} \|z_n - y\|$ exists. So, we see that $\lim_{n \rightarrow \infty} \|x_n - y\|$ exists by $x_n - z_n \rightarrow 0$ as $n \rightarrow \infty$. Next, we show $p = q$. If not, by Opial's condition,

$$\begin{aligned}
\lim_{n \rightarrow \infty} \|x_n - p\| &= \lim_{i \rightarrow \infty} \|x_{n_i} - p\| \\
&< \lim_{i \rightarrow \infty} \|x_{n_i} - q\| \\
&= \lim_{n \rightarrow \infty} \|x_n - q\| \\
&= \lim_{j \rightarrow \infty} \|x_{n_j} - q\| \\
&< \lim_{j \rightarrow \infty} \|x_{n_j} - p\| \\
&= \lim_{n \rightarrow \infty} \|x_n - p\|.
\end{aligned}$$

This is a contradiction. Hence, we have $p = q$. Hence, $\{x_n\}$ converges weakly to $p \in F(S)$.

(ii) By using (1), we obtain

$$x_{n+1} = \beta_n[(1 - \alpha_n)x_n + \alpha_n Sx_n] + (1 - \beta_n)[(1 - \alpha_n)x_n + \alpha_n Tx_n]$$

for all $n \geq 1$. Putting $V_n = \beta_n[(1 - \alpha_n)I + \alpha_n S] + (1 - \beta_n)[(1 - \alpha_n)I + \alpha_n T]$. Then $x_{n+1} = V_n x_n$. For any $z \in F(S) \cap F(T)$,

$$\begin{aligned}
\|V_n x_n - z\| &= \|\beta_n[(1 - \alpha_n)x_n + \alpha_n Sx_n] + (1 - \beta_n)[(1 - \alpha_n)x_n + \alpha_n Tx_n] - z\| \\
&\leq \beta_n \|(1 - \alpha_n)x_n + \alpha_n Sx_n - z\| + (1 - \beta_n) \|(1 - \alpha_n)x_n + \alpha_n Tx_n - z\| \\
&\leq \beta_n \|(1 - \alpha_n)x_n + \alpha_n Sx_n - z\| + (1 - \beta_n) \|x_n - z\| \\
&\leq \|x_n - z\|
\end{aligned}$$

and thus

$$\begin{aligned}
0 &\leq \|x_n - z\| - \beta_n \|(1 - \alpha_n)x_n + \alpha_n Sx_n - z\| - (1 - \beta_n) \|x_n - z\| \\
&= \beta_n (\|x_n - z\| - \|(1 - \alpha_n)x_n + \alpha_n Sx_n - z\|) \\
&\leq \|x_n - z\| - \|V_n x_n - z\|.
\end{aligned}$$

By using (3),

$$0 \leq \|x_n - z\| - \|V_n x_n - z\| = \|x_n - z\| - \|x_{n+1} - z\| \rightarrow c - c = 0 \quad (6)$$

as $n \rightarrow \infty$. So, we obtain

$$\begin{aligned}
0 &\leq (1 - \beta_n) \beta_n (\|x_n - z\| - \|(1 - \alpha_n)x_n + \alpha_n Sx_n - z\|) \\
&\leq (1 - \beta_n) (\|x_n - z\| - \|V_n x_n - z\|).
\end{aligned}$$

Since $\liminf_{n \rightarrow \infty} \beta_n(1 - \beta_n) > 0$, it follows from (6) that

$$\lim_{n \rightarrow \infty} (\|x_n - z\| - \|(1 - \alpha_n)x_n + \alpha_n Sx_n - z\|) = 0. \quad (7)$$

Since

$$\|Sx_n - z\| \leq \|x_n - z\|$$

and by Lemma 3.2 and Takahashi [14], we have

$$\begin{aligned} & \|(1 - \alpha_n)x_n + \alpha_n Sx_n - z\| \\ &= \|(1 - \alpha_n)(x_n - z) + \alpha_n(Sx_n - z)\| \\ &\leq \|x_n - z\| \left[1 - 2\alpha_n(1 - \alpha_n)\delta_E \left(\frac{\|Sx_n - x_n\|}{\|x_n - z\|} \right) \right]. \end{aligned}$$

Hence, we obtain

$$2\alpha_n(1 - \alpha_n)\|x_n - z\|\delta_E \left(\frac{\|Sx_n - x_n\|}{\|x_n - z\|} \right) \leq \|x_n - z\| - \|(1 - \alpha_n)x_n + \alpha_n Sx_n - z\|.$$

Since δ_E is strictly increasing, continuous, and by $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$ and (7), we obtain

$$\lim_{n \rightarrow \infty} \|Sx_n - x_n\| = 0.$$

Since $I - S$ is demiclosed at zero and by using the same method in the proof of (i), we obtain that if $x_{n_i} \rightharpoonup p$, then $p \in F(S)$. We also prove that $p \in F(T)$. In fact, we obtain that for any $z \in F(S) \cap F(T)$,

$$\begin{aligned} \|V_n x_n - z\| &= \|\beta_n[(1 - \alpha_n)x_n + \alpha_n Sx_n] + (1 - \beta_n)[(1 - \alpha_n)x_n + \alpha_n T x_n] - z\| \\ &\leq \beta_n \|(1 - \alpha_n)x_n + \alpha_n Sx_n - z\| + (1 - \beta_n) \|(1 - \alpha_n)x_n + \alpha_n T x_n - z\| \\ &\leq \beta_n \|x_n - z\| + (1 - \beta_n) \|(1 - \alpha_n)x_n + \alpha_n T x_n - z\| \\ &\leq \|x_n - z\| \end{aligned}$$

and hence

$$\begin{aligned} 0 &\leq (1 - \beta_n)(\|x_n - z\| - \|(1 - \alpha_n)x_n + \alpha_n T x_n - z\|) \\ &\leq \|x_n - z\| - \|V_n x_n - z\|. \end{aligned}$$

So, we have

$$\begin{aligned} 0 &\leq \beta_n(1 - \beta_n)(\|x_n - z\| - \|(1 - \alpha_n)x_n + \alpha_n T x_n - z\|) \\ &\leq \beta_n(\|x_n - z\| - \|V_n x_n - z\|). \end{aligned}$$

Since $\liminf_{n \rightarrow \infty} \beta_n(1 - \beta_n) > 0$, it follows from (6) that

$$\lim_{n \rightarrow \infty} (\|x_n - z\| - \|(1 - \alpha_n)x_n + \alpha_n T x_n - z\|) = 0. \quad (8)$$

Since

$$\|Tx_n - z\| \leq \|x_n - z\|$$

and by Lemma 3.2 and Takahashi [14], we have

$$\begin{aligned} & \|(1 - \alpha_n)x_n + \alpha_n Tx_n - z\| \\ &= \|(1 - \alpha_n)(x_n - z) + \alpha_n(Tx_n - z)\| \\ &\leq \|x_n - z\| \left[1 - 2\alpha_n(1 - \alpha_n)\delta_E \left(\frac{\|Tx_n - x_n\|}{\|x_n - z\|} \right) \right]. \end{aligned}$$

Hence, we obtain

$$2\alpha_n(1 - \alpha_n)\|x_n - z\|\delta_E \left(\frac{\|Tx_n - x_n\|}{\|x_n - z\|} \right) \leq \|x_n - z\| - \|(1 - \alpha_n)x_n + \alpha_n Tx_n - z\|.$$

Since δ_E is strictly increasing, continuous, and by $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$ and (8), we obtain

$$\lim_{n \rightarrow \infty} \|Tx_n - x_n\| = 0.$$

Since $\{x_{n_i}\} \rightharpoonup p$ and $I - T$ is demiclosed at zero, then $p \in F(T)$. Let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $\{x_{n_j}\} \rightharpoonup q$. Then we obtain $p = q$. If not, by Opial's condition and (3),

$$\begin{aligned} \lim_{n \rightarrow \infty} \|x_n - p\| &= \lim_{i \rightarrow \infty} \|x_{n_i} - p\| \\ &< \lim_{i \rightarrow \infty} \|x_{n_i} - q\| \\ &= \lim_{n \rightarrow \infty} \|x_n - q\| \end{aligned}$$

and by using similar method, we have

$$\lim_{n \rightarrow \infty} \|x_n - q\| < \lim_{n \rightarrow \infty} \|x_n - p\|.$$

This is a contradiction. Hence, we have $p = q$. Therefore, $\{x_n\}$ converges weakly to $p \in F(S) \cap F(T)$. $\square$

Our Theorem 3.2 carries over Theorem 4.1(ii) of Iemoto and Takahashi [6] to a quasi-nonexpansive mapping.

Theorem 3.2 *Let H be a Hilbert space and C be a nonempty, closed, and convex subset of H , and let S be a quasi-nonexpansive mapping of C into itself and T be a nonexpansive mapping of C into itself such that $F(S) \cap F(T) \neq \emptyset$. Suppose that, for any x_1 in C , $\{x_n\}$ is defined by (1), where $\{\alpha_n\}$ and $\{\beta_n\}$ are chosen so that $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$ and $\sum_{n=1}^{\infty} \beta_n < \infty$. Then $\{x_n\}$ converges weakly to $p \in F(T)$.*

Proof For any $z \in F(S) \cap F(T)$. Put $z_{n+1} = (1 - \alpha_n)x_n + \alpha_nTx_n$. Then

$$\begin{aligned}\|x_{n+1} - z_{n+1}\| &= \|(1 - \alpha_n)x_n + \alpha_n[\beta_nSx_n + (1 - \beta_n)Tx_n] \\ &\quad - (1 - \alpha_n)x_n - \alpha_nTx_n\| \\ &= \alpha_n\|\beta_nSx_n + (1 - \beta_n)Tx_n - Tx_n\| \\ &= \alpha_n\beta_n\|Tx_n - Sx_n\| \\ &\leq \beta_nM,\end{aligned}\tag{9}$$

where $M = \sup_{n \geq 1} \|Tx_n - Sx_n\| < \infty$. Since $\sum_{n=1}^{\infty} \beta_n < \infty$, we obtain $\|x_n - z_n\| \rightarrow 0$ as $n \rightarrow \infty$. Since

$$\begin{aligned}\|z_{n+1} - z\| &= \|(1 - \alpha_n)x_n + \alpha_nTx_n - z\| \\ &\leq (1 - \alpha_n)\|x_n - z\| + \alpha_n\|Tx_n - z\| \\ &\leq \|x_n - z\| \\ &\leq \|z_n - z\| + \|x_n - z_n\|,\end{aligned}$$

and by Lemma 3.1, $\lim_{n \rightarrow \infty} \|z_n - z\|$ exists. So, we see that $\lim_{n \rightarrow \infty} \|x_n - z\|$ exists by $x_n - z_n \rightarrow 0$ as $n \rightarrow \infty$, and thus $\{x_n\}$ is bounded. By using (9), we obtain

$$\begin{aligned}\|x_{n+1} - z\| &\leq \|x_{n+1} - z_{n+1}\| + \|z_{n+1} - z\| \\ &\leq \beta_nM + \|z_{n+1} - z\|\end{aligned}$$

and thus

$$\|x_{n+1} - z\| - \beta_nM \leq \|z_{n+1} - z\|.\tag{10}$$

Since

$$\|Tx_n - z\| \leq \|x_n - z\|$$

and by Lemma 3.2 and Takahashi [14], we have

$$\begin{aligned}\|z_{n+1} - z\| &= \|(1 - \alpha_n)x_n + \alpha_nTx_n - z\| \\ &= \|(1 - \alpha_n)(x_n - z) + \alpha_n(Tx_n - z)\| \\ &\leq \|x_n - z\| \left[1 - 2\alpha_n(1 - \alpha_n)\delta_E \left(\frac{\|Tx_n - x_n\|}{\|x_n - z\|} \right) \right].\end{aligned}$$

Hence, by (10), we obtain

$$\begin{aligned}2\alpha_n(1 - \alpha_n)\|x_n - z\|\delta_E \left(\frac{\|Tx_n - x_n\|}{\|x_n - z\|} \right) &\leq \|x_n - z\| - \|z_{n+1} - z\| \\ &\leq \|x_n - z\| - \|x_{n+1} - z\| + \beta_nM.\end{aligned}$$

Since δ_E is strictly increasing, continuous, $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$ and $\sum_{n=1}^{\infty} \beta_n < \infty$, we obtain

$$\liminf_{n \rightarrow \infty} \|Tx_n - x_n\| = 0. \quad (11)$$

We note that $Tx_{n+1} - x_{n+1} = Tx_{n+1} - U_n x_n + (1 - \alpha_n)(U_n x_n - x_n)$, where $U_n = \beta_n S + (1 - \beta_n)T$. Then we have that

$$\begin{aligned} & \|x_{n+1} - Tx_{n+1}\| \\ & \leq \|Tx_{n+1} - Tx_n\| + \|Tx_n - U_n x_n\| + (1 - \alpha_n)\|U_n x_n - x_n\| \\ & \leq \|x_{n+1} - x_n\| + \|Tx_n - U_n x_n\| + (1 - \alpha_n)\|U_n x_n - x_n\| \\ & = \|(1 - \alpha_n)x_n + \alpha_n U_n x_n - x_n\| + \|Tx_n - U_n x_n\| + (1 - \alpha_n)\|U_n x_n - x_n\| \\ & = \alpha_n\|U_n x_n - x_n\| + \|Tx_n - U_n x_n\| + (1 - \alpha_n)\|U_n x_n - x_n\| \\ & = \|U_n x_n - x_n\| + \|Tx_n - (\beta_n S x_n + (1 - \beta_n)Tx_n)\| \\ & = \|\beta_n(Sx_n - x_n) + (1 - \beta_n)(Tx_n - x_n)\| + \beta_n\|Tx_n - Sx_n\| \\ & \leq (1 - \beta_n)\|x_n - Tx_n\| + \beta_n(\|x_n - Sx_n\| + \|Tx_n - Sx_n\|) \\ & \leq (1 - \beta_n)\|x_n - Tx_n\| + \beta_n(\|x_n - Tx_n\| + \|Tx_n - Sx_n\| + \|Tx_n - Sx_n\|) \\ & \leq \|x_n - Tx_n\| + 2\beta_n\|Tx_n - Sx_n\| \\ & \leq \|x_n - Tx_n\| + 2\beta_n M. \end{aligned}$$

Since $\sum_{n=1}^{\infty} \beta_n < \infty$ and by Lemma 3.1, $\lim_{n \rightarrow \infty} \|x_n - Tx_n\|$ exists. Hence, by (11), we obtain

$$\lim_{n \rightarrow \infty} \|Tx_n - x_n\| = 0. \quad (12)$$

Since $\{x_n\}$ is bounded, there exist the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_i}\} \rightharpoonup p$ and $\{x_{n_j}\} \rightharpoonup q$, respectively. We first show that $p, q \in F(T)$. In fact, if $Tp \neq p$, then by Opial's condition and (12), we obtain

$$\begin{aligned} \limsup_{i \rightarrow \infty} \|x_{n_i} - p\| & < \limsup_{i \rightarrow \infty} \|x_{n_i} - Tp\| \\ & \leq \limsup_{i \rightarrow \infty} \{\|x_{n_i} - Tx_{n_i}\| + \|Tx_{n_i} - Tp\|\} \\ & \leq \limsup_{i \rightarrow \infty} \|x_{n_i} - p\|. \end{aligned}$$

This is a contradiction. Hence we obtain $Tp = p$. Similarly, we obtain $q \in F(T)$. Next, as in the proof of (i) in Theorem 3.1, $\{x_n\}$ converges weakly to $p \in F(T)$. $\square$

Theorem 3.3 *Let H be a Hilbert space and C be a nonempty, closed, and convex subset of H , and let S, T be two quasi-nonexpansive mappings of C into itself satisfying condition **A** with $\mathbf{F} = F(S) \cap F(T) \neq \emptyset$. Suppose that, for any x_1 in C , $\{x_n\}$ is defined by (1), where $\{\alpha_n\}$ and $\{\beta_n\}$ are chosen so that $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$*

and $\liminf_{n \rightarrow \infty} \beta_n(1 - \beta_n) > 0$. Then $\{x_n\}$ converges strongly to a common fixed point of S and T .

Proof As in the proof of (ii) in Theorem 3.1, we obtain

$$\lim_{n \rightarrow \infty} \|Tx_n - x_n\| = \lim_{n \rightarrow \infty} \|Sx_n - x_n\| = 0. \quad (13)$$

By using Condition A, we obtain

$$f(d(x_n, \mathbf{F})) \leq \frac{1}{2} \{ \|Tx_n - x_n\| + \|Sx_n - x_n\| \}$$

for all $n \geq 1$. As in the proof of Theorem 3.1,

$$\|x_{n+1} - z\| \leq \|x_n - z\|, \quad (14)$$

for any $z \in \mathbf{F}$. Taking the infimum over all $z \in \mathbf{F}$ on both sides, we see that $\lim_{n \rightarrow \infty} d(x_n, \mathbf{F}) (\equiv k)$ exists. We first claim that $\lim_{n \rightarrow \infty} d(x_n, \mathbf{F}) = 0$. In fact, assume that $k = \lim_{n \rightarrow \infty} d(x_n, \mathbf{F}) > 0$. Then we can choose $n_0 \in N$ such that $0 < \frac{k}{2} < d(x_n, \mathbf{F})$ for all $n \geq n_0$. By using Condition A and (13), we obtain

$$0 < f\left(\frac{k}{2}\right) \leq f(d(x_n, \mathbf{F})) \leq \frac{1}{2} \{ \|Tx_n - x_n\| + \|Sx_n - x_n\| \} \rightarrow 0$$

as $n \rightarrow \infty$. This is a contradiction. So, we obtain $k = 0$. Next, we claim that $\{x_n\}$ is a Cauchy sequence. Let $\epsilon > 0$ be given. Since $\lim_{n \rightarrow \infty} d(x_n, \mathbf{F}) = 0$, there exists $n_0 \in N$ such that for all $n \geq n_0$, we obtain

$$d(x_n, \mathbf{F}) < \frac{\epsilon}{2}. \quad (15)$$

Let $n, m \geq n_0$ and $p \in \mathbf{F}$. Then, by (14), we obtain

$$\begin{aligned} \|x_n - x_m\| &\leq \|x_n - p\| + \|x_m - p\| \\ &\leq 2[\|x_{n_0} - p\|]. \end{aligned}$$

Taking the infimum over all $p \in \mathbf{F}$ on both sides and by (15), we obtain

$$\|x_n - x_m\| \leq 2[d(x_{n_0}, \mathbf{F})] < \epsilon$$

for all $n, m \geq n_0$. This implies that $\{x_n\}$ is a Cauchy sequence. Let $\lim_{n \rightarrow \infty} x_n = q$. Since $\mathbf{F}$ is closed, we obtain $q \in \mathbf{F}$. Hence, $\{x_n\}$ converges strongly to a common fixed point of S and T . $\square$

The following is an example of a quasi-nonexpansive mapping which is not non-expansive mapping.

Example 3.1 [15] Let $\mathbb{R}$ denote the reals with the usual norm and $C = [-\pi, \pi]$. Let $T : C \rightarrow C$ be defined by

$$Tx = x \cos x$$

for all $x \in C$.

The following is an example of a quasi-nonexpansive mapping which are not a nonspreading mapping and a nonexpansive mapping.

Example 3.2 Let $\mathbb{R}$ denote the reals with the usual norm and $C = [0, 2]$. Let $T : C \rightarrow C$ be defined by

$$Tx = \frac{1}{3}(x^2 + 2)$$

for all $x \in C$, where $x + 2 \leq 3$, $x \in [0, 1]$ and $x + 2 = 3$, $x > 1$. Clearly $F(T) = \{1, 2\}$. T is a quasi-nonexpansive mapping since if $x \in [0, 2]$ and $z = 1$, then

$$|Tx - z| = |Tx - 1| = \left| \frac{1}{3}(x^2 + 2) - 1 \right| = \frac{1}{3}|x^2 - 1| \leq |x - z|.$$

If $x \in [0, 2]$ and $z = 2$, then

$$|Tx - z| = |Tx - 2| = \left| \frac{1}{3}(x^2 + 2) - 2 \right| = \frac{1}{3}|x^2 - 4| \leq |x - z|.$$

But it is not a nonspreading mapping. In fact, if we take $x = 2$ and $y = \sqrt{2}$, then we obtain $|Tx - Ty|^2 = 0.444$, $|x - y|^2 = 0.343$, $2\langle x - Tx, y - Ty \rangle = 0$, and thus

$$|Tx - Ty|^2 = 0.444 > 0.343 = |x - y|^2 + 2\langle x - Tx, y - Ty \rangle.$$

Also, it is not a nonexpansive mapping. In fact, if we take $x = 2$ and $y = \sqrt{2}$, then we obtain

$$|Tx - Ty| = 0.666 > 0.586 = |x - y|.$$

4 Concluding Remarks

Our results, Theorems 3.1 and 3.2, carry over Theorem 4.1 of Iemoto and Takahashi to quasi-nonexpansive mappings. We study the strong convergence of the Moudafi's iterative scheme under the assumption that two quasi-nonexpansive self-mappings satisfying condition A. We give an example of a quasi-nonexpansive mapping which are not a nonspreading mapping and a nonexpansive mapping.

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