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Phenomena in Inverse Stackelberg Games, Part 1: Static Problems

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Abstract Games are considered in which the role of the players is a hierarchical one. Some players behave as leaders, others as followers. Such games are named after Stackelberg. In the current paper, a special type of these games is considered, known in the literature as inverse Stackelberg games. In such games, the leader (or: leaders) announces his strategy as a mapping from the follower (or: followers) decision space into his own decision space. Arguments for studying such problems are given. The routine way of analysis, leading to a study of composed functions, is not very fruitful. Other approaches are given, mainly by studying specific examples. Phenomena in problems with more than one leader and/or follower are studied within the context of the inverse Stackelberg concept. As a side issue, expressions like "two captains on a ship" and "divide and conquer" are given a mathematical foundation.

Keywords Games · Transaction costs · Stackelberg games · Inverse Stackelberg games · Composed functions

1 Introduction: Stackelberg Equilibria and Terminology

This paper deals with various types of Stackelberg problems, with a clear emphasis on *inverse Stackelberg* problems, to be defined later on. Such problems are usually treated within the context of game theory. In the simplest form, there are two players, called leader (L) and follower (F) respectively, each having its own cost function,

$$J_{\mathrm{L}}(u_{\mathrm{L}}, u_{\mathrm{F}}),\ \ J_{\mathrm{F}}(u_{\mathrm{L}}, u_{\mathrm{F}}),$$

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where $u_L, u_F \in R$. The leader chooses u_L, the follower u_F. Each player wants to choose its own decision variable in such a way as to minimize its own cost function. Without giving an equilibrium concept, the problem as stated so far is not well defined. Such an equilibrium concept could for instance be one named after Nash or after Pareto [1]. In the Stackelberg equilibrium concept, one player, the leader, announces his decision u_L, which is subsequently made known to the other player, the follower. With this knowledge, the follower chooses his u_F. Hence, u_F becomes a function of u_L, written as

$$u_F = l_F(u_L),$$

which is determined through the relation

$$\min_{u_F} J_F(u_L, u_F) = J_F(u_L, l_F(u_L)).$$

For the sake of simple exposition, it is assumed that this minimum exists and that it is unique for each possible choice u_L of the leader. The function $l_F(\cdot)$ is sometimes called a reaction function (i.e., it indicates how the follower will react upon the leader decision). Before the leader announces his decision u_L, he will realize how the follower will react and hence the leader will choose and subsequently announce u_L so as to minimize

$$J_L(u_L, l_F(u_L)).$$

Example 1.1 Suppose that

$$J_L(u_L, u_F) = (u_F - 5)^2 + u_L^2, \qquad J_F(u_L, u_F) = u_L^2 + u_F^2 - u_L u_F.$$

The reaction curve l_F is given by $u_F = \frac{1}{2}u_L$ and hence u_L will be chosen so as to minimize

$$\left(\frac{1}{2}u_L - 5\right)^2 + u_L^2,$$

which results immediately in $u_L = 2$. With this decision of the leader, the follower will choose $u_F = 1$. The costs for the players are given by respectively 20 and 3.

Another equilibrium concept, to be dealt with now, is the *inverse Stackelberg equilibrium* as it was introduced in [2]. The leader does not announce the scalar u_L, as above, but a function $\gamma_L(\cdot)$ which maps u_F into u_L. Examples of games with such information structure are:

- The leader is the bank and the follower the investor. The investor can buy stocks, with the bank as intermediator, with the money he has in his savings account. Suppose he buys stocks worth u_F Euros. Then the bank will charge him $\gamma_L(u_F)$ as transaction costs. The function $\gamma_L(\cdot)$ has been made known by the bank before the actual transaction takes place [3, 4].
- The leader is a producer of electricity in a liberalizing market and the follower is the market (a group of clients) itself. The price of electricity is set to $\gamma_L(u_F)$, where u_F is the amount of electricity traded.

Given the function $\gamma_{\mathrm{L}}(\cdot)$, the follower will make his choice u_{F} according to

$$u_{\mathrm{F}}^* = \arg\min_{u_{\mathrm{F}}} J_{\mathrm{F}}(\gamma_L(u_{\mathrm{F}}), u_{\mathrm{F}}). \tag{1}$$

Optimizing quantities are provided with an asterix. The leader, before announcing his $\gamma_{\mathrm{L}}(\cdot)$, will of course realize how the follower will play and he should exploit this knowledge in order to choose the best possible γ-function, such that ultimately his own cost function J_{L} becomes as small as possible. Symbolically, we could write

$$\gamma_{\mathrm{L}}^*(\cdot) = \arg\min_{\gamma_{\mathrm{L}}(\cdot)} J_{\mathrm{L}}(\gamma_L(u_{\mathrm{F}}(\gamma_{\mathrm{L}}(\cdot))), u_{\mathrm{F}}(\gamma_{\mathrm{L}}(\cdot))). \tag{2}$$

In this way, one enters the field of composed functions [5], which is known to be a notoriously complex area.

Example 1.2 In terms of Example 1.1, (1) becomes

$$u_{\mathrm{F}}^* = \arg\min_{u_{\mathrm{F}}} = \arg\min(\gamma_{\mathrm{L}}^2(u_{\mathrm{F}}) + u_{\mathrm{F}}^2 - \gamma_{\mathrm{L}}(u_{\mathrm{F}})u_{\mathrm{F}}).$$

Thus, u_{F}^* has become a function of the yet unknown γ_{L} and this function is substituted into (2) leading to

$$\gamma_{\mathrm{L}}^* = \arg\min((u_{\mathrm{F}}^* - 5)^2 + \gamma_{\mathrm{L}}^2(u_{\mathrm{F}}^*)).$$

From here onward, it turns out to be difficult to proceed in an analytic way. However, there is a trick that often works as is exemplified by the following example.

Example 1.3 Suppose that the cost functions are those of Example 1.1. If both the follower and the leader would be so kind as to minimize $J_{\mathrm{L}}(u_{\mathrm{L}}, u_{\mathrm{F}})$, the follower totally disregarding his own cost function, the leader would obtain his team minimum, i.e. $J_{\mathrm{L}}(0,5) = 0$. Now, the leader should choose the curve $u_{\mathrm{L}} = \gamma_{\mathrm{L}}(u_{\mathrm{F}})$ in such a way that the minimum $u_{\mathrm{L}} = 0,\ u_{\mathrm{F}} = 5$ lies on this curve and moreover that this curve does not have other points in common with the set

$$J_{\mathrm{F}}(u_{\mathrm{L}}, u_{\mathrm{F}}) = u_{\mathrm{L}}^2 + u_{\mathrm{F}}^2 - u_{\mathrm{L}}u_{\mathrm{F}} \le J_{\mathrm{F}}(0,5) = 25.$$

An example of such a curve is $u_{\mathrm{L}} = 2u_{\mathrm{F}} - 10$. With this choice of the leader, the best for the follower to do is to minimize

$$J_{\mathrm{F}}(2u_{\mathrm{F}} - 10, u_{\mathrm{F}}),$$

which leads to $u_{\mathrm{F}} = 5$. Hence $u_{\mathrm{L}} = 0$ and, to our surprise, the leader obtains his team minimum in spite of the fact that the follower minimizes his own cost function (however, subject to the constraint $u_{\mathrm{L}} = \gamma_{\mathrm{L}}(u_{\mathrm{F}}) = 2u_{\mathrm{F}} - 10$).

Other examples exist in which the leader cannot obtain his team minimum and such problems are harder to deal with as exemplified by the following example.

Example 1.4 This is a continuation of Example 1.3. We add the constraints $-4 \leq u_{\mathrm{L}} \leq +3$ and $-5 \leq u_{\mathrm{F}} \leq 7$, which the two players must obey. Although the absolute minimum of the leader cost function is a feasible point in this rectangle, the leader will not be able to enforce it here. The worst that can happen to the follower is characterized by $\min u_{\mathrm{F}} \max u_{\mathrm{L}} J_{\mathrm{F}}$ which is realized for $u_{\mathrm{F}} = -2$, $u_{\mathrm{L}} = -4$ resulting in $J_{\mathrm{F}}(-4, -2) = 12$. Subsequently, consider the team problem

$$\begin{aligned} &\min_{u_{\mathrm{L}}, u_{\mathrm{F}}} \; J_{\mathrm{L}}(u_{\mathrm{L}}, u_{\mathrm{F}}), \\ &\text{s.t.} \quad J_{\mathrm{F}}(u_{\mathrm{L}}, u_{\mathrm{F}}) \leq J_{\mathrm{F}}(-4, -2) = 12. \end{aligned}$$

The calculation to find this solution, though straightforward, does not lead to a "nice" answer. The solution will simply be indicated by $u_{\mathrm{L}}^{\dagger}(\approx 0.51)$, $u_{\mathrm{F}}^{\dagger}(\approx 3.69)$. An optimal choice for the leader is

$$u_{\mathrm{L}} = \gamma_{\mathrm{L}} = \begin{cases} -4, & \text{for } -5 \leq u_{\mathrm{F}} < u_{\mathrm{F}}^{\dagger} - \epsilon, \\ u_{\mathrm{L}}^{\dagger}, & \text{for } u_{\mathrm{F}}^{\dagger} - \epsilon \leq u_{\mathrm{F}} \leq 7, \end{cases}$$

where ϵ is an arbitrarily small positive number. The quantity ϵ being greater than zero makes the solution unique; $\epsilon = 0$ would lead to a nonunique response of the follower (i.e., either $u_{\mathrm{F}} = -2$ or $u_{\mathrm{F}} = u_{\mathrm{F}}^{\dagger}$, where the latter is clearly preferred by the leader).

The philosophy in most cases (as in the current paper) is first to get an impression of what the leader can achieve and subsequently try to find a strategy to really achieve this goal. If one does not have any clue as to what the leader can achieve (in terms of minimal costs), hardly anything is known.

The origin of the Stackelberg equilibrium concept goes back to [7]. A steady stream of publications has appeared since then, see e.g. [2, 6, 8, 9] and the references therein. Originally applications were found in economics, but nowadays Stackelberg problems are recognized to be present in many other areas; see e.g. [10–13].

The organization of this paper is as follows. In Sect. 2, a particular inverse Stackelberg problem will be introduced; it is a game with the bank as leader and the investor as follower. In Sect. 3, problems with more than one leader and/or more than one follower are considered. In these problems the leaders, as well as the followers, will play according to the Nash equilibrium concept among themselves. Hardly any theory is available in this area and the contribution is to show, by means of various examples, what kind of phenomena exist and what kind of problems occur. The word 'static' in the title refers to problems in which there is no time evolution; there is no underlying model described by a differential or difference equation. The only dynamics that shows up is that the follower acts after the leader. For both players, it concerns a one-shot game. In the subsequent paper [14], *dynamic* inverse Stackelberg problems will be considered; time evolution plays an essential role in the sense that both players act more than once.

2 Specific Static Problem

We will consider the following static problem:

$$\min_u (f(u) + \gamma(u)), \qquad \max_{\gamma(\cdot)} \gamma(u),$$

subject to $\gamma(\cdot) \geq 0$ and $\gamma(0) = 0$. A possible interpretation is an investor who wants to maximize its wealth $-f(u)$, equivalently, wants to minimize its loss $f(u)$, where u represents the amount of stocks to buy or to sell. The buying or selling is done through a bank which wants to maximize its transaction costs $\gamma(u)$. If there is no trade ($u = 0$), the transactions costs are zero ($\gamma(0) = 0$). The investor is secured of a maximum cost $f(0)$ (equivalently: a minimum income $-f(0)$) by playing $u = 0$. Therefore he will only take u-values into consideration for which $f(u) \leq f(0)$. This set of u-values is called U.

Example 2.1 As a specific f take $f(u) = (u-1)^2 + 1$, then $U = [0, 2]$. An upper bound for the investor's criterion is $f(0) = 2$. Suppose that the bank chooses

$$\gamma(u) = \begin{cases} (f(0) - f(u))(1-\epsilon), & \text{if } 0 \leq u \leq 2, \\ \geq 0, & \text{elsewhere}, \end{cases} \tag{3}$$

where ϵ is a small positive number. Obviously $u^* = 1$, the investor criterion is $2 - \epsilon$ and the bank profit is $1 - \epsilon$. The conclusion is that the bank reaps essentially all the investor profits (the latter would have been $\min_u f(u) = 1$ if the transaction costs would have been identically zero). Note that the optimal γ-function of the bank is nonunique; another choice, equally advantageous to the bank, would be

$$\gamma(u) = \begin{cases} 1-\epsilon, & \text{if } u \neq 0, \\ 0, & \text{if } u = 0, \end{cases}$$

where ϵ is a small positive number. If one wants to adhere to the nondecreasing property of γ, then (3) could be replaced by

$$\gamma(u) = \begin{cases} (f(0) - f(u))(1-\epsilon), & \text{if } 0 \leq u \leq 1, \\ 1 - \epsilon + (1-u)^2, & \text{if } u \geq 1, \end{cases}$$

and for negative u: $\gamma(u) = \gamma(-u)$, without altering the results.

Theorem 2.1 *An upper bound for the profit of the bank is $f(u=0) - f(u=u^*)$, where u^* is the optimal decision of the investor in absence of transaction costs*, *i.e.* $u^* = \arg\min f(u)$.

The proof is straightforward:

$$f(u^*) + \gamma(u) \leq f(u) + \gamma(u) \leq f(0) + \gamma(0) = f(0),$$

which leads immediately to $\gamma(u) \leq f(u=0) - f(u=u^*)$. That the leader essentially reaps all profits from the follower is a known phenomenon. In [13], one can for instance read: "It is a remarkable feature of these problems (i.e., contract design games) that the leader always takes all, pushing the follower to zero utility".

3 More Leaders and Followers

This section has been inspired by modelling the price of electricity on a free market [15]. Only static problems will be considered. The idea is that there are $\overline{P}$ producers of electricity which offer their products to $\overline{M}$ markets. These products are identical apart from possibly the price. For the sake of simplicity, we will assume $\overline{P}=2$, $\overline{M}=2$. The producers, called P_3 and P_4, will be the leaders and the markets, P_1 and P_2, are the followers. The two leaders among themselves will play Nash, as will the two followers among themselves. Assume that each of the two followers (i.e. the markets) chooses an amount of electricity $u_i \in R$, $i=1,2$, respectively, with no restrictions concerning bounds for the time being. Their cost functions, to be minimized are J_i, $i=1,2$. In addition, the two leaders (i.e. the producers) choose $u_i \in R$, $i=3,4$, again with no restrictions for the time being. These decision variables are not chosen as scalars directly, but as functions of $u_i \in R$, $i=1,2$ (think of $u_i \in R$, $i=3,4$, as being prices that depend on the quantities $u_i \in R$, $i=1,2$, chosen by the followers). As before, we will write $u_i = \gamma_i(u_1,u_2)$, $i=3,4$. The cost functions of the leaders, to be minimized, are J_i, $i=3,4$. The direct dependence of the cost functions on the various decision variables will be as follows:

$$J_1(u_3,u_4,u_1),\ J_2(u_3,u_4,u_2),\ J_3(u_3,u_1,u_2),\ J_4(u_4,u_1,u_2). \tag{4}$$

The interpretation of $J_1(u_3,u_4,u_1)$ for instance is that the value of the criterion is determined by the total amount of electricity requested and this will depend on the price setting of the two producers, $u_i=\gamma_i(u_1,u_2)$, $i=3,4$. In practice, the market will buy an amount u_{1_i} from producer i, with $u_{1_3}+u_{1_4}=u_1$, so as to get the electricity at the lowest price. The cost function J_1 does not directly depend on the amount of electricity required by the second market. The cost function of the first producer, i.e. P_3, depends on the total amount of electricity sold, i.e. on u_1 and u_2, its own price setting, but not directly on the price setting of the other producer.

The dependence of the cost functions on the decision variables as expressed by (4) is of course not the most general one. However, the restriction to this subclass of possible cost functions will lead already to novel phenomena and the analysis is simpler than in the general case.

In the remainder of this section, we will consider two subsections, one with one leader and two followers, and the other one with one follower and two leaders. No realistic cost functions are envisaged.

3.1 Special Case: One Leader, Two Followers

Example 3.1 Consider a game with two followers and one leader, specifically,

$$J_1 = u_1^2 - u_1u_3, \qquad J_2 = u_2^2 - 2u_2u_3,$$

$$J_3 = u_3^2 + 2u_1u_3 + 5u_2u_3 + u_1^2 + u_2^2 + 4u_3^2.$$

The absolute minimum of J_3 is achieved for $u_1 = -8/25$, $u_2 = -20/25$ and $u_3 = 8/25$. Call this point A. This point would be achieved if all players would help minimize J_3. The leader will try to obtain his team minimum by choosing the coefficients α_i in

$$u_3 = \gamma_3(u_1, u_2) = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3$$

properly. If he is successful with linear functions, there is no necessity to consider the larger class of nonlinear functions. We derive three (linear) equations for the coefficients α_i. The first one is obtained by the fact that the absolute minimum must lie on the curve $u_3 = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3$. The second and third ones are obtained by $\frac{\partial J_i(u_i, \gamma_3(u_1,u_2))}{\partial u_i} = 0$, $i = 1, 2$. The equations are

$$\begin{aligned} -8\alpha_1 - 20\alpha_2 + 25\alpha_3 &= 8, \\ 16\alpha_1 - 20\alpha_2 - 25\alpha_3 &= 16, \\ -8\alpha_1 + 80\alpha_2 - 50\alpha_3 &= 40, \end{aligned}$$

which results in

$$\alpha_1 = 3, \qquad \alpha_2 = \frac{7}{5}, \qquad \alpha_3 = \frac{12}{5}.$$

Checking the second-order conditions leads to wrong signs of the second order derivatives in the supposed solution (in spite of the fact that each cost function J_i has a minimum with respect to its "own" decision variable u_i). Hence, the solution obtained cannot be the correct one. If the leader would instead announce the nonlinear strategy

$$u_3 = \gamma_3(u_1, u_2) = 3u_1 + \frac{7}{5}u_2 + \frac{12}{5} + \mu_1\left(u_1 + \frac{8}{25}\right)^2 + \mu_2\left(u_2 + \frac{20}{25}\right)^2,$$

with constants $\mu_1 > \frac{25}{4}$ and $\mu_2 > \frac{9}{8}$, then also the second order conditions are fulfilled. Thus, a nonlinear strategy for the leader does work in order for him to obtain his absolute minimum.

We continue with a slightly different problem in order to show that a linear strategy for the leader sometimes can work. The cost functions are now changed to

$$\begin{aligned} J_1 &= u_1^2 - u_1u_3 + 2u_3^2, \\ J_2 &= u_2^2 - 2u_2u_3 + 5u_3^2, \\ J_3 &= u_3^2 + 2u_1u_3 + 5u_2u_3 + u_1^2 + u_2^2 + 4u_3^2. \end{aligned}$$

Note that the only difference is that we have added a u_3 term to J_1 and J_2; J_3 has not been changed. We could perform the same analysis as above to reach the correct solution this time. We will, however, proceed in a slightly different way to achieve the (same) solution. The absolute minimum of J_3 is still achieved for $u_1 = -8/25$, $u_2 =$

$-20/25$ and $u_3 = 8/25$ (i.e. point A). Consider the constant level curve $J_1(u_1, u_3)$ through this point, i.e. $J_1(u_1, u_3) = 928/625$. This curve determines u_3 as a function of u_1. By taking the total derivative of $J_1(u_1, u_3) = 928/625$ with respect to u_1, one obtains $\frac{\partial u_3}{\partial u_1} = \frac{3}{5}$ in point A. By considering the constant level curve $J_2(u_2, u_3)$ through the same point, one obtains similarly $\frac{\partial u_3}{\partial u_2} = \frac{7}{15}$. Hence, if a linear γ_3 function exists, it must be of the form

$$u_3 = \gamma_3(u_1, u_2) = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3$$

with $\alpha_1 = \frac{3}{5}$, $\alpha_2 = \frac{7}{15}$. Now, α_3 is obtained by the fact that the curve $u_3 = \gamma_3$ must pass through A. This yields $\alpha_3 = \frac{332}{375}$.

The γ-function thus obtained satisfies all requirements and the leader obtains his team minimum with this choice. In other words, he cannot do better.

In the example above, we studied the problem with two followers and one leader. The analysis introduced can be extended directly to N followers and one leader. The adagium "divide and conquer" applies in the example above (for the leader). It is conjectured that this adagium holds generically, at least as long as there are no restrictions on the decision variables (compare Examples 1.3 and 1.4). What one has to find is a function $u_3 = \gamma_3(u_1, u_2)$ such that

$$u_3^* = \gamma_3(u_1^*, u_2^*),\ u_1^* = \arg\min J_1(u_1, \gamma_3(u_1, u_2^*)),\ u_2^* = \arg\min J_2(u_2, \gamma_3(u_1, u_2^*)),$$

where (u_1^*, u_2^*, u_3^*) is the absolute minimum of J_3. It is even conjectured that this statement holds for cost functions with a more general dependence on the decision variables as in (4).

In the following example, we will study the problem with two leaders and one follower. In this case, there is not an obvious point (as the point A above) on which *both* leaders will agree at the outset. Hence, we will not start with such a point.

3.2 Special Case: One Follower, Two Leaders

Example 3.2 The follower has the cost function

$$J_1 = u_1^2 + u_3^2 + u_4^2,$$

and the leaders have the cost functions

$$J_3 = (u_1 - 1)^2 + (u_3 - 1)^2, \qquad J_4 = (u_1 - 2)^2 + (u_4 - 1)^2,$$

respectively. Suppose that the two leaders will choose their functions γ_i as

$$u_3 = \gamma_3(u_1) = \alpha_1 u_1 + \alpha_2, \qquad u_4 = \gamma_4(u_1) = \beta_1 u_1 + \beta_2.$$

In the three-dimensional (u_1, u_2, u_3) space, these two planes have a line of intersection and the follower is forced to choose the best point (i.e., with the minimum value of J_1) on this line of intersection. This leads to

$$u_1 = -\frac{\alpha_1\alpha_2 + \beta_1\beta_2}{1 + \alpha_1^2 + \beta_1^2}.$$

Realising this choice of the follower, the two leaders will substitute this choice into their own γ functions and subsequently into their own cost functions. Thus, these cost functions become functions of the parameters α_i and β_i, $i = 1, 2$, only. By setting

$$\frac{\partial J_3(\alpha_1, \alpha_2, \beta_1, \beta_2)}{\partial \alpha_i} = 0, \qquad \frac{\partial J_4(\alpha_1, \alpha_2, \beta_1, \beta_2)}{\partial \beta_i} = 0,$$

i.e., the necessary conditions for a Nash equilibrium, one obtains four equations with four unknowns. The solutions have been obtained by means of Maple.[1] They are

$$\alpha_1 = -5, \qquad \alpha_2 = 10, \qquad \beta_1 = -2, \qquad \beta_2 = 5,$$

with corresponding $u_1 = 2,\ u_3 = 0,\ u_4 = 1$ and

$$\alpha_1 = -1, \qquad \alpha_2 = 2, \qquad \beta_1 = 2, \qquad \beta_2 = -2,$$

with corresponding $u_1 = 1,\ u_3 = 1,\ u_4 = 0$. Besides, some other solutions were indicated which result from the roots of a fourth order polynomial.

Let us study the first solution given in more detail. It turns out that the second-order conditions are fulfilled. Hence, a correct solution has been obtained. It is striking that the resulting u-values coincide with the absolute minimum of the second leader (moreover, the second solution obtained corresponds to the absolute minimum of the first leader).

It is claimed now that the solution obtained is only locally optimal. If the second leader sticks to

$$u_4 = \gamma_4(u_1) = -2u_1 + 5,$$

it is claimed now that the first leader can do better than

$$u_3 = \gamma_3(u_1) = -5u_1 + 10,$$

namely by playing

$$u_3 = \gamma_3(u_1) = +5u_1 - 4.$$

With this choice of γ_3, while γ_4 remains the same, i.e. $\gamma_4(u_1) = -2u_1 + 5$, the first leader obtains his team minimum ($u_1 = u_3 = 1$). How has this latter γ_3 function been obtained? Answer: by substituting the function $u_4 = \gamma_4(u_1) = -2u_1 + 5$ into the cost functions J_1 and J_3. This now is a game for one leader and one follower. The leader can obtain his team minimum in this case.

If the first leader chooses $u_3 = \gamma_3(u_1) = +5u_1 - 4$, then the second leader might be willing to change his γ_4 function. Indeed, that is the case. With $u_3 = \gamma_3(u_1) = +5u_1 - 4$ (fixed) and $u_4 = \gamma_4(u_1) = -32u_1 + 65$, the resulting (u_1, u_3, u_4) coincides with the absolute minimum of the second leader! If the leaders continue with

[1] O. Pourtallier (INRIA) is acknowledged for having carried out these calculations.

alternately adapting their optimal functions, we obtain

$$
\begin{aligned}
u_4 &= \gamma_4(u_1) = -2u_1 + 5,\\
u_3 &= \gamma_3(u_1) = +5u_1 - 4,\\
u_4 &= \gamma_4(u_1) = -32u_1 + 65,\\
u_3 &= \gamma_3(u_1) = +1055u_1 - 1054,\\
u_4 &= \gamma_4(u_1) = -1114082u_1 + 2228165,
\end{aligned}
$$

et cetera. Obviously this algorithm does not converge. In this sequence, if P_3 is the last actor, then his choice of $\gamma_3(u_1)$ leads to the absolute minimum of J_3. On the other hand, if P_4 is the last actor, then his choice of $\gamma_4(u_1)$ leads to the absolute minimum of J_4. By drawing a three-dimensional picture, it will be clear that linear γ-functions cannot lead to a Nash solution.

In the following theorem, (u_{1,J_3}, u_{3,J_3}) refers to the pair (u_1, u_3) that minimizes J_3 (in the example $u_1 = 1,\ u_3 = 1$). Similarly, (u_{1,J_4}, u_{4,J_4}) refers to the pair (u_1, u_4) that minimizes J_4 (in the example $u_1 = 2,\ u_4 = 1$).

Theorem 3.1 *If $u_{1,J_3} \neq u_{1,J_4}$, a Nash solution between the leaders does not exist.*

This theorem holds irrespective of the class of $\gamma_i(u_1)$ functions, $i = 3, 4$, one admits. If these functions are allowed to be discontinuous (even with an infinite number of discontinuity points), the theorem remains true.

Proof Given $\gamma_4(\cdot)$, the other player will choose $\gamma_3(\cdot)$ with $u_{3,J_3} = \gamma_3(u_{1,J_i})$ in such a way that J_1 reaches its minimal value at the point $u_{1,J_3}, u_{3,J_3}, \gamma_4(u_{1,J_3})$ on the intersection of the two curves $u_i = \gamma_i(u_1),\ i = 3, 4$. This choice of $\gamma_3(\cdot)$ leads to a solution of the game which coincides with the team minimum of J_3. On his turn, the other leader (who chooses $\gamma_4(\cdot)$) can choose his $\gamma_4(\cdot)$ function in such a way that the solution of the game coincides with the team minimum of J_4. Due to the fact that $u_{1,J_3} \neq u_{1,J_4}$ these two solutions obtained (one with γ_4 given and finding the optimal γ_3, and the other one with γ_3 given and finding the optimal γ_4) will never be consistent. □

Remark 3.1 As long as $u_{1,J_3} \neq u_{1,J_4}$, the well-known consequences of "two captains on a ship" seem to apply. If $u_{1,J_3} = u_{1,J_4}$, then the Nash solution exists; it leads to the team minima of both leaders. Essential in the proof above is that there are no restrictions on the decision variables, e.g. of the kind $L_i \leq u_i \leq U_i$. With such restrictions, the problem may become solvable, even if $u_{1,J_3} \neq u_{1,J_4}$, as shown in the following example.

Example 3.3 Let us consider the cost functions of Example 3.2 once more, but now with the constraints $-1 \leq u_i \leq +3$, $i = 1, 2, 3$. The roles of the players also remain the same. We will let the two leaders alternately minimize their cost functions and see whether this algorithm converges.

We start by assuming γ_3 to be given with $u_3=\gamma_3(u_1)\equiv 0$. A two-player Stackelberg game results with P_4 as leader and P_1 as follower. Their cost functions are respectively

$$J_1=u_1^2+u_4^2,\qquad J_4=(u_1-2)^2+(u_4-1)^2.$$

An optimal choice for P_4 is

$$u_4=\gamma_4(u_1)=\begin{cases}3, & \text{if } u_1\neq 2,\\ 1, & \text{if } u_1=2.\end{cases}\tag{5}$$

As a result of this choice P_1 will choose $u_1=2$. Subsequently $u_4=1$ and P_4 has realized his team minimum. Note that many other choices for γ_3 are possible with the same result, e.g.

$$u_4=\gamma_4(u_1)=\begin{cases}-2u_1+5, & \text{if } 1\le u_1\le 3,\\ 3, & \text{if } -1\le u_1\le +1.\end{cases}$$

We will continue with the first choice for γ_4 given, i.e. (5). Keeping this function fixed, the other leader P_3 will now choose his optimal $\gamma_3(u_1)$ function. Equation (5) is substituted into J_1 leading to

$$J_1=u_1^2+u_3^2+\begin{cases}9, & \text{if } u_1\neq 2,\\ 1, & \text{if } u_1=2.\end{cases}$$

It is easily verified now that an optimal solution for P_3 is

$$u_3=\gamma_3(u_1)=\begin{cases}3, & \text{if } u_1\neq 1,\\ 1, & \text{if } u_1=1.\end{cases}\tag{6}$$

This leads to the team minimum of P_3. Also in this case, the optimal γ_3 is not unique.

We now fix γ_3 as given in (6) and study the best answer by P_4. P_4 cannot obtain his team minimum anymore, since J_1 prefers playing $u_1=1$ to $u_1=2$, whatever the choice of $\gamma_4(\cdot)$. The worst that can happen to player P_1 is the outcome 11 which is realized for $u_1=1,\ u_4=\gamma_4(u_1=1)=3,\ u_3=\gamma_3(u_1=1)=1$. Hence, P_4 should consider $\min_{u_1,u_4} J_4(u_1,u_4)$ subject to $J_1(u_1,u_3=\gamma_3(u_1),u_4)\le J_1(u_1=1,u_3=1,u_4=3)=11$. This leads to

$$u_4=\gamma_4(u_1)=\begin{cases}3 & \text{if } u_1\neq 2\left(\frac{\sqrt{2}}{\sqrt{5}}-\epsilon_1\right),\\[2ex] \frac{\sqrt{2}}{\sqrt{5}}-\epsilon_1 & \text{if } u_1=2\left(\frac{\sqrt{2}}{\sqrt{5}}-\epsilon_1\right)\end{cases}\tag{7}$$

as a possible choice for P_4. The value $\epsilon_1>0$ has been added so as to make the choice for P_1 unique after (7) has been announced. For $\epsilon_1=0$, player P_1 has two choices, but one of them is preferred by P_4.

If we continue in this way, ultimately we find that

$$\gamma_3(u_1) = \begin{cases} 3, & \text{if } u_1 \neq 1, \\ 1, & \text{if } u_1 = 1, \end{cases} \qquad \gamma_4(u_1) = \begin{cases} 3, & \text{if } u_1 \neq \sqrt{\frac{8}{5}}, \\ \sqrt{\frac{2}{5}}, & \text{if } u_1 = \sqrt{\frac{8}{5}}, \end{cases}$$

apart from ϵ-terms, is a stable Nash solution, which, however, leads to a nonunique minimum of the follower. Follower chooses either $u_1 = 1$ or $u_1 = \sqrt{\frac{8}{5}}$. The first choice leads to $J_1^* = 11$, $J_3^* = 0$, $J_4^* = 5$ and the second one to $J_1^* = 11$, $J_3^* = 4.07$, $J_4^* = 0.69$. Follower apparently is rather beneficial to one leader of his choice.

4 Conclusions

The theory of inverse Stackelberg problems is a difficult one, especially if there is more than one follower and/or more than one leader involved. The surface has been scratched only. The mathematical difficulty lies in the fact that composed functions form part of the analysis and the study of such functions is known to be notoriously complex. This paper showed some phenomena which can show up in the solutions of simple static inverse Stackelberg problems.

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