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Spectral Properties of the Linearized Boltzmann Operator in L^p for $1 \leq p \leq \infty$

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Abstract The spectrum of both classical and relativistic Boltzmann operator for hard interactions in a whole space is shown to be independent of p in L^p for $1 \leq p < \infty$. It consists of a half-plane $\operatorname{Re} \lambda \leq -\nu_0$ and countably many branches in the strip $-\nu_0 < \operatorname{Re} \lambda \leq 0$. Moreover the resolvent set is independent of p for all $1 \leq p \leq \infty$ but in L^∞ this operator possesses in addition to the continuous also an uncountable set of point spectrum.

Keywords Boltzmann operator · Spectral theory · Hard interaction · Relativistic Boltzmann operator

1 Introduction

Our aim in this paper is to present a complete spectral theory of the linearized Boltzmann operator for hard, cut-off interaction both for classical [1, 2] and relativistic [3–5] cases.

The history of serious investigation of the spectral properties of the linearized Boltzmann operator began with pioneering work of Grad [1] who showed that the linearized collision operator L can be decomposed, with suitable assumption on the form of a cross-section, as $L = -\nu(\xi) + K$, where $\nu(\xi)$, called collision frequency, is a continuous function of velocity ξ, and the operator K compact in $L^2(d_3\xi)$. With these results and Schechter theorem [6] he was able to locate an essential spectrum of the classical collision operator in $L^2(d_3\xi)$. Most of the following works were devoted to investigation of properties of the spectrum in $L^2(d_3\xi)$ improving results of Grad and establishing these properties of the semi-group $\exp[tL]$ which were necessary for constructing various existence proofs for nonlinear Boltzmann equation for system close to the global equilibrium. We shall mention results of Ellis

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and Pinsky [7], Nishida and Imai [8] and important paper of Nikolaenko [9]. One of the most interesting in this field was the work of Klaus [10] who considered the spectrum of Boltzmann operator in L^p with $p \neq 2$. His approach required additional smoothing properties of the operator K which one can quite easily prove for classical hard spheres model but which are not always available for general form of interaction especially for relativistic interactions. Palczewski [11, 12], using techniques developed in the neutron transport theory, described the spectrum of the linearized Boltzmann operator with periodic boundary conditions. These results were extended to the system with external potential in the work of Tabota and Eshima [13] and in [14] for time dependent forces. An extensive review of various developments in this field was given by Ukai and Yang [15] and Villani [16]. The recent works on the parametric approach to the spectral problem of the Boltzmann operator was presented by Mouhot and Strain [17]. The properties of the semi-group $\exp[Bt]$ solving the Cauchy problem for the Boltzmann equation were presented in [15, 16, 18] for classical equation and in [5, 19] for the relativistic case.

For hard interactions for classical and relativistic gases the main result is that the spectrum of the Boltzmann operator $L = -\nu + K$ in $L^p(R^3)$ is the same for all p, $1 \leq p \leq \infty$. This clearly shows that estimates obtained for the spectral properties of the Boltzmann operator in $L^2(R^3)$ hold in all L^p, as for example the spectral gap estimated by Baranger and Mahout [20]. In particular we have shown that the eigenfunctions of the Boltzmann operators decay at infinity faster than any power of ξ^n, similar to the behaviour of the eigenfunction for Maxwell molecules. For the family of Fourier transformed operators B_k defined as $L + i\bar{k}\bar{\xi}$ we show that the spectrum is the same in all $L^p(R^3)$, $1 \leq p \leq \infty$, and that the operators B_k are sectorial for any k, $|k| < \infty$ but with angle defining the sector unfortunately $|k|$ dependent and as $|k| \to \infty$ the angle tends to π. For a full integro-differential operator $B^p = \nabla\xi + L_0$ in $L^p(R^3 \times R^3)$ we were able to prove the following Main Theorem:

Theorem 1.1 *For all hard interactions operator B^p for $1 \leq p < \infty$ in $L^p_{x,\xi}$ has spectrum independent on p. This spectrum consists of half-plane $\operatorname{Re}\lambda \leq -\nu_0$ and in the strip $-\nu_0 < \operatorname{Re}\lambda \leq 0$ of finite if $[\nu(\xi) - \nu(0)]^{-1} \in L^r_{loc}(r > 1)$ or otherwise infinite number of branches $\lambda = \lambda_i(k)$, $k \in [0, k^i_{max}]$. These are continuous functions of k and $\operatorname{Re}\lambda_i(k^i_{max} = -\nu_0)$. For $p = \infty$ the continuous part of the spectrum also covers the half-plane $\operatorname{Re}\lambda < -\nu_0$ but $\lambda_i(k)$ for $|k| \in [0, k^i_{max}]$ belong to uncountable point spectrum of the operator B^∞. For all $1 \leq p \leq \infty$ a half-plane $\operatorname{Re}\lambda > 0$ lies entirely in the resolvent set of the operators B^p.*

This paper is organized as follows: in Sect. 2 we introduce linearized Boltzmann operator and prove an important theorem concerning compactness of the K operator in L^p, in Sect. 3 a detailed analysis of the spectrum of the collision operator in $L^p(d_3\xi)$ is given. In Sect. 4 we present similar analysis of the spectrum of the Fourier transformed Boltzmann operator in $L^p(d_3\xi)$ and Sect. 5 contains the main theorem of this paper describing a spectrum of the Boltzmann operator in $L^p(R^3 \times R^3)$ spaces for $1 \leq p \leq \infty$.

2 The linearized Boltzmann operator

I denote as $L^p_{x,\xi}$ the space of measurable functions of two independent variables $x \in R^3$ and $\xi \in R^3$, which are integrable with p-th power $1 \leq p < \infty$. The norm in these spaces is defined as:

$$\|\phi\|_{L^p_{x,\xi}} = \left[\int d_3x d_3\xi \left|\phi(x,\xi)\right|^p\right]^{1/p}, \tag{2.1}$$

and for $p = \infty$ we denote as $L^\infty_{x,\xi}$ the space of essentially bounded measurable functions with a norm:

$$\|\phi\|_{L^\infty_{x,\xi}} = \operatorname{ess} \sup_{x\in R^3, \xi\in R^3} \left|\phi(x,\xi)\right|. \tag{2.2}$$

A pairing between different L^p spaces is provided by scalar product defined for $\phi \in L^p_{x,\xi}$ and $\psi \in L^q_{x,\xi}$ with $p^{-1} + q^{-1} = 1$ as:

$$(\phi, \psi) = \int d_3x d_3\xi \bar{\phi}(x,\xi)\psi(x,\xi). \tag{2.3}$$

I denote as L^p_x (respectively L^p_ξ) the space of functions of $x \in R^3$ ($\xi \in R^3$) integrable with p-th power with a norm:

$$\|f\|_{L^p_x} = \left[\int d_3x \left|f(x)\right|^p\right]^{1/p}, \tag{2.4}$$

for $1 \le p < \infty$ and for $p = \infty$ with a sup norm

$$\|f\|_{L^\infty_x} = \operatorname{ess} \sup_{x\in R^3} \left|f(x)\right|. \tag{2.5}$$

In the following we will make use of a partial Fourier transform in x of the functions from $L^p_{x,\xi}$; $1 \le p < \infty$ defined for all $k \in R^3$ as

$$\phi(k,\xi) = \int d_3x e^{-ikx}\phi(x,\xi). \tag{2.6}$$

My aim is to consider spectral properties of the operator B defined for $\phi \in L^p_{x,\xi}$ as:

$$B\phi(x,\xi) = -\eta(\xi)\nabla\phi(x,\xi) + L[\phi](x,\xi), \tag{2.7}$$

where:

$$\eta \cdot \nabla\phi(x,\xi) = \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\phi(x,\xi)\eta_i, \tag{2.8}$$

$$\eta(\xi) = \begin{cases} \xi & \text{for classical Boltzmann operator} \\ \xi/(1+\xi^2)^{1/2} & \text{for relativistic operator} \end{cases} \tag{2.9}$$

and the integral operator L is a Boltzmann collision operator linearized around global equilibrium state (global Maxwellian, denoted as M in classical or Jütner function [4], denoted as J in the relativistic case) and is given by following integrals:

(i) for non-relativistic case,

$$\begin{aligned} L[\phi](x,\xi) = M^{1/2} \int d_3\xi_1 d\Omega |\xi - \xi_1| \sigma\big(|\xi - \xi_1|, \theta\big) M\big[\phi\big(x,\xi_1'\big)M_1'^{-1/2} \\ + \phi\big(x,\xi'\big)M'^{-1/2} - \phi(x,\xi_1)M_1^{-1/2} + \phi(x,\xi)M^{-1/2}\big] \end{aligned} \tag{2.10a}$$

(ii) for relativistic case,

$$L[\phi](x,\xi) = \frac{J^{1/2}}{2\xi_0} \int d_3\xi_1 d\Omega \frac{gs^{1/2}}{\xi_{10}} \sigma(g,\theta)$$

$$\times J\big[\phi\big(x,\xi_1'\big)J_1'^{-1/2}+\phi\big(x,\xi'^{-1/2}\big)J'^{-1/2}-\phi(x,\xi_1)J_1^{-1/2}-\phi(x,\xi)J^{-1/2}\big] \tag{2.10b}$$

where $\xi_0=(1+\xi^2)^{1/2}$, ξ,ξ_1 are momenta before and ξ',ξ_1' after the collision and $\sigma(g,\theta)$ is the scattering cross-section. We have denoted as $\xi^\mu=(\xi_0,\xi)$ a vector in a flat Minkowski space with signature $(+,-,-,-)$. Variables g,s can be expressed with the help of these four-momenta as:

$$s=\big[\big(\xi^\mu+\xi_1^\mu\big)(\xi_\mu+\xi_{1\mu})\big]^{1/2},$$
$$2g=\big[\big(\xi^\mu-\xi_1^\mu\big)(\xi_\mu-\xi_{1\mu})\big]^{1/2}.$$

I have assumed that both velocity of light and the rest mass of particles are equal to 1.

Properties of the operator L depend on the form of the cross-section $\sigma(g,\theta)$ which is the only place in the Boltzmann operator where the detailed form of interactions of the system enters. In the following we will concentrate on the spectral properties of the Boltzmann operator for a class of cross-sections corresponding to the hard, repulsive cut-off interactions i.e. for cross-sections fulfilling following conditions:

(i) for classical case [1, 2],

$$\sigma\big(|\xi-\xi_1|,\theta\big)\le B|\xi-\xi_1|^\beta(\sin\theta)^\alpha, \tag{2.11a}$$
$$\sigma\big(|\xi-\xi_1|,\theta\big)>B'\frac{|\xi-\xi_1|^\epsilon}{1+|\xi+\xi_1|}, \tag{2.11b}$$

with $\beta>-2,\alpha>-1$ and some constants B, B' and $\epsilon>0$;

(ii) for relativistic operator [4, 21],

$$\sigma(g,\theta)\le\big(Bg^\beta+B'g^{-\alpha}\big)(\sin\theta)^\gamma, \tag{2.12a}$$

with: $\gamma>-2,\alpha>4,0\le\beta\le\gamma+2$,

$$\sigma(g,\theta)\ge\hat{B}\frac{g^{\beta'+1}}{c_0+g}(\sin\theta)^{\gamma'}, \tag{2.12b}$$

with: $\gamma'>-2,a>4,0\le\beta'\le\gamma'+2$.

For cross-sections fulfilling conditions (2.11a) or (2.12a) it was shown in [1, 2] for classical and in [4] for relativistic Boltzmann operator that L can be written as:

$$\begin{aligned}L&=-\nu(\xi)+K,\\ \nu(\xi)&=\nu\big(|\xi|\big)\end{aligned} \tag{2.13}$$

and if in addition the cross-sections fulfill conditions (2.11b) or (2.12b), then there exist constants C, C', D, D', δ and δ' such that

$$C\big(1+|\xi|\big)^\delta\le\nu(\xi)\le C'\big(1+|\xi|\big)^\delta, \tag{2.14a}$$

for classical case and corresponding bounds for relativistic collision frequency have the following form:

$$D(\xi_0)^{\delta'}\le\nu(\xi)\le D'(\xi_0)^{\delta'}. \tag{2.14b}$$

For all cross sections fulfilling conditions (2.11a), (2.11b) in classical case and in relativistic case for $\sigma(g,\theta)$ such that:

$$\sigma(g,\theta) \geq \frac{\hat{B} g^{\hat{\beta}+1}}{c_0 + g} (\sin\theta)^{\hat{\gamma}}, \tag{2.15}$$

where $\hat{\gamma} > -1, 0 \leq \hat{\beta} \leq \hat{\gamma} + 1$. the operator K is α-smooth i.e. for $\phi \in L^2_\xi$ $K\phi \in L^{2-\alpha}_\xi$ with some $\alpha > 0$. This property for classical case was shown in Grad's paper [1] and improved in [22], whereas for the relativistic case it is a consequence of estimates given in [4, 5].

For the relativistic case the operator K has the following form:

$$K = K_2 - K_1 \tag{2.16}$$

and

$$K_i\big[f(\mathbf{r},\mathbf{p},t)\big] = \int d_3 p_1 k_i(\mathbf{p},\mathbf{p_1}) f(\mathbf{r},\mathbf{p},t), i = 1,2 \tag{2.17}$$

$$k_1(\mathbf{p},\mathbf{p_1}) = \frac{g s^{\frac{1}{2}}}{p_0 p_{10}} \exp\left[-\frac{\tau+\tau_1}{2}\right] \int_0^\pi d\Theta \sin\Theta \sigma(g,\Theta) \tag{2.18a}$$

$$\begin{aligned} k_2(\mathbf{p},\mathbf{p_1}) = \frac{1}{4} \frac{s^{\frac{3}{2}}}{p p_0 p_{10}} \int_0^\infty \exp\left[-\frac{(1+x^2)^{\frac{1}{2}}(\tau+\tau_1)}{2}\right] \\ \times \sigma\left[\frac{g}{\sin(\psi/2)}, \psi\right] \frac{1+(1x^2)^{1/2}}{(1+x^2)^{1/2}} I_0\left[\frac{|\mathbf{p}\wedge\mathbf{p_1}|}{2g} x\right] \end{aligned} \tag{2.18b}$$

$$v(p) = \int d_3 p_1 k_1(\mathbf{p},\mathbf{p_1}) \exp\left[\frac{\tau-\tau_1}{2}\right] \tag{2.19}$$

and

$$\tau = u^\mu p_\mu \tag{2.20a}$$

$$\tau_1 = u^\mu p_{1\mu} \tag{2.20b}$$

$$\sin(\psi/2) = 2^{1/2} g\big[g^2 - M^2 + \big(g^2 + M^2\big)\big(1+x^2\big)^{1/2}\big]^{-1/2} \tag{2.21}$$

$\mathbf{pp_1}$ is a vector product of $\mathbf{p}$ and $\mathbf{p_1}$, calculated in the rest frame of the gas [in this frame $u_\mu = (1,0,0,0)$]; the explicit expression for $|\mathbf{pp_1}|$ has the form

$$|\mathbf{pp_1}| = \big[4g\big(\tau\tau_1 - g^2 - M^2\big) - M^2(\tau-\tau_1)^2\big]^{1/2} \tag{2.22}$$

while for the classical hard sphere model:

$$K = K_2 - K_1 \tag{2.23}$$

and

$$K_1\big(\mathbf{v},\mathbf{v}'\big) = \frac{1}{8\pi} |\mathbf{v}-\mathbf{v}'| \exp\left[-\frac{\mathbf{v}^2 + \mathbf{v}'^{\mathbf{2}}}{4}\right] \tag{2.24a}$$

$$K_2(\mathbf{v},\mathbf{v}') = \frac{1}{2\pi}\frac{1}{|\mathbf{v}-\mathbf{v}'|}\exp\left[-\frac{1}{8}\left(|\mathbf{v}-\mathbf{v}'|^2 + \frac{(v^2-v'^2)^2}{|\mathbf{v}-\mathbf{v}'|^2}\right)\right] \tag{2.24b}$$

$$\nu(v) = \nu_0 + b|v| \tag{2.25}$$

For detailed analysis of ν and K operators see [1, 5] for the classical and [3–5, 21] for the relativistic case. Note that for all cut off inverse power low potentials with $n \geq 4$ leading to hard interactions, our assumptions are fulfilled.

See [1, 5] for details for classical and [4] for relativistic K operator.

An important property of the operator K, considered as an operator from L^p_ξ into L^p_ξ, $1 \leq p \leq \infty$, that we will frequently use in this paper is provided by

Lemma 2.1 *For cross-sections fulfilling conditions* (2.11a) *or* (2.12a) *the operator* K *is a compact operator in* $L^p(d_3\xi)$ *for* $1 < p < \infty$ *and it is weakly compact operator in* L^1_ξ *and* L^∞_ξ.

Proof For non-relativistic operator the compactness of K was proved in [1, 2] for $p = 2$, and in [22] for other p. This last proof includes also weak compactness of K in L^1_ξ and L^∞_ξ. For relativistic case the proof of compactness in L^2_ξ was given in [4]. The proof given in [22] can be easily applied to the relativistic operator in L^p_ξ spaces with $1 < p < \infty$ implying compactness of the relativistic operator K in these spaces. For $p = 1$ it is relatively easy to check using results of [4] and the method from [22] that operators K_n defined as $X_n K X_n$ with X_n being the characteristic function of a set $\Omega_n = A_n \cap B_n$ where:

$$A_n = \left\{(\bar{\xi},\bar{\xi}_1) : |\xi - \xi_1| \geq \frac{1}{n}\right\}, \tag{2.26a}$$

$$B_n = \left\{(\bar{\xi},\bar{\xi}_1) : |\xi| \leq n\right\}, \tag{2.26b}$$

are weakly compact operators in $L^\infty(\Omega_n)$ as K_n generate for every n an equi-bounded and equi-continuous mapping. We have shown in [4] for relativistic operator that $\lim_{n\to\infty} \|K - X_n K X_n\| = 0$ in operator norm. This means that K is weakly compact in L^∞_ξ and from the Schauder theorem the operator K in L^1_ξ is weakly compact as a dual operator of a weakly compact one. □

Operator K is bounded as an operator from L^p_ξ to L^∞_ξ for $3/2 < p \leq \infty$ (see [22] for discussion of this property). In general $K\phi$ behaves better than ϕ for $\phi \in L^p_\xi$ and we have $K\phi \in L^\infty_\xi$ for $\phi \in L^2_\xi$, $(1+|\xi|)^{r+1} K\phi \in L^\infty_\xi$ if $(1+|\xi|)^r\phi \in L^\infty_\xi$ [1, 4, 5, 22]. Further properties of operator K are given in Appendix.

3 Spectrum of the operator B_0 in L^p_ξ for $1 \leq p \leq \infty$

I denote as B^p_0 the operator $L = -\nu(\xi) + K$ considered as an operator from L^p_ξ into L^p_ξ with $1 \leq p \leq \infty$. These operators are densely defined and closed in L^p_ξ for $1 \leq p \leq \infty$.

I consider first the problem of locating the essential spectrum of these operators. For $1 < p < \infty$ it is quite simple as the operator K is a compact operator in L^p_ξ for such p and the Schechter theory of compact perturbation is applicable.

If $p = 1$ or $p = \infty$ operator K is only weakly compact, but in these cases application of Voigt Theorem is possible. We use this theorem in the following formulation.

Lemma 3.1 (Voigt theorem [30]) *Let T be a closed operator in E and let Ω be a component of $\rho_{ess}(T)$. Let the operator B be a T-power compact on $\Omega \cap \rho(T)$, and let $I - (B(\lambda - T)^{-1})^N$ be invertible in $B(E)$ for some $\lambda \in \Omega \cap \rho(T)$ (N from the definition of T-power compactness of B). Then $\Omega \subset \rho_{ess}(T + B)$ and B is $(T + B)$-power compact on $\Omega \cap \rho(T + B)$.*

See also Ribaric and Vidav [23] for more general presentation of this theorem.

The above Lemma and index theory [24] lead to the following:

Theorem 3.1 *For $1 \le p \le \infty$ $\sigma_{ess}(B_0^p)$ is independent of p and is equal to the set $\Gamma = \{\lambda \in R; \lambda = -\nu(\xi)$ for some $\xi \in R^3\}$.*

Proof As $B_0^p = -\nu(\xi) + K$ with K being a compact operator in L_ξ^p for $1 < p < \infty$ it follows from the Schechter theory of compact perturbation [6] that $\sigma_{ess}(B_0^p) = \sigma_{ess}(-\nu(\xi))$ and this last is equal to Γ for all $1 < p < \infty$.

The problem of location of the essential spectrum of the operator B_0^1 or B_0^∞ is more complicated. For $p = 1$ Lemma 3.1 provides a partial answer locating the essential resolvent set $\rho_{ess}(B_0^1)$.

I set in Lemma 3.1. $K = B$, $Tf = -\nu f$ and as $\nu(\xi)$ is a continuous function for cross-sections fulfilling assumptions (2.11a) and (2.12a) the spectrum of T is continuous and equal to $\Gamma = \text{Range}(-\nu)$, $\xi \in R^3$. If Ω denotes the unbounded connected component of C/Γ, we see that for $\lambda \in C$ such that $\text{Re}\,\lambda \to \infty$ or $|\text{Im}\,\lambda| \to \infty$ $\lim \|K(\lambda - T)^{-1}\| = 0$. This shows that for such $\lambda \in \Omega$ operator $I - K(\lambda - T)^{-1}$ is invertible. Moreover operator $K(\lambda - T)^{-1}$ is compact in L^p, $1 < p < \infty$, and power compact for $p = 1$ and from Lemma 3.1 follows that $\Gamma \subset \rho_{ess}[-\nu + K]$.

Thus it remains to show that $\Gamma \subset \sigma(B_0^1)$. To this end, we consider operator B_0^∞. As $B_0^\infty = (B_0^1)'$ and B_0^1 is closed in $L^1(d_3\xi)$, it follows that for $\lambda \notin \sigma(B_0^1)$ ranges of both operators $B_0^1 - \lambda$ and $B_0^\infty - \lambda$ are closed in L_ξ^1 and L_ξ^∞ respectively. Moreover, the following relations are fulfilled:

$$\text{nul}\big(B_0^\infty - \lambda\big) = \text{def}\big(B_0^1 - \lambda\big), \tag{3.1a}$$

$$\text{nul}'\big(B_0^\infty - \lambda\big) = \text{def}'\big(B_0^1 - \lambda\big), \tag{3.1b}$$

$$\text{def}\big(B_0^\infty - \lambda\big) = \text{nul}\big(B_0^1 - \lambda\big), \tag{3.2a}$$

$$\text{def}'\big(B_0^\infty - \lambda\big) = \text{nul}'\big(B_0^1 - \lambda\big), \tag{3.2b}$$

where $\text{nul}(B)$ and $\text{def}(B)$ denote the nullity and deficiency of the operator B and $\text{nul}'(B)$ and $\text{def}'(B)$ the approximate nullity and deficiency respectively. From Eqs. (3.1a), (3.1b) and (3.2a), (3.2b) follows that for such λ $\text{ind}(B_0^1 - \lambda) = \text{ind}(B_0^\infty - \lambda) = 0$. We see then that the resolvent set of B_0^∞ is identical to the resolvent set of the operator B_0^1. Assume now that $\Gamma \not\subset \sigma(B_0^1)$. This means that there exists $\lambda \in \Gamma$ such that $\lambda \notin \rho(B_0^1)$ and $\lambda \notin \rho(B_0^\infty)$ and this means in fact that both operators $(B_0^1 - \lambda)^{-1}$ and $(B_0^\infty - \lambda)^{-1}$ are bounded operators in L_ξ^1 and L_ξ^∞ respectively. Application of the interpolation theorem of Riesz-Thorin [25] leads now to the conclusion that also operators $(B_0^p - \lambda)^{-1}$ are bounded in L_ξ^p for all p such that $1 < p < \infty$ but this is in contradiction with our result that $\Gamma \subset \sigma_{ess}(B_0^p)$. Thus we see that $\Gamma \subset \sigma(B_0^1)$ and in fact combining this with the previous result $\Gamma \supset \sigma(B_0^1)$ we obtain finally that $\sigma(B_0^1) = \sigma(B_0^\infty) = \Gamma$. ☐

I consider now the more difficult problem of the point spectrum of the operators B_0^p. From the general theory of closed operators [26] it follows that for $p^{-1}+q^{-1}=1$ we have relations:

$$\sigma_{pp}\big(B_0^p\big)=\sigma_{pp}\big(B_0^q\big). \tag{3.3}$$

Since for $1\le p\le r\le 2$ from the fact that $f\in L^p\cap L^z$ follows that $f\in L^r$ and we have also following inclusions:

$$\sigma_{pp}\big(B_0^1\big)\subset\sigma_{pp}\big(B_0^p\big)\subset\sigma_{pp}\big(B_0^r\big)\subset\sigma_{pp}\big(B_0^2\big). \tag{3.4}$$

The main problem is to show opposite inclusions. To this end we need additional properties of the operators B_0^p. In general we can appeal to the smoothing properties of the operator $\nu^{-1}K$.

We note here that in the following we need the smoothing properties of the operator $\nu^{-1}K$ rather than K alone, which is easier to prove. For example for hard interactions resulting in essentially unbounded $\nu(\xi)$ at infinity for which $\nu(\xi)$ behaves like $C|\xi|^\alpha$, $\alpha>0$ for $|\xi|\to\infty$ we see that $\nu^{-1}K$ is α-smooth for any bounded operator K. Taking this into account we see that for all interactions fulfilling (2.11a), (2.11b), (2.12a) and (2.15) $\nu^{-1}K$ is α-smooth with $\alpha>0$. If $\alpha>1$, it means that $\nu^{-1}K\phi$ decay sufficiently fast at infinity to be in L^1_ξ and in such case we put $\alpha=1$.

Theorem 3.2 *For any σ fulfilling conditions* (2.11a), (2.11b), (2.12a) *and* (2.15) *for* $1\le p\le r\le 2$, *the point spectrum* $\sigma_{pp}(B_0^2)\subset\sigma_{pp}(B_0^p)\subset\sigma_{pp}(B_0^r)\subset\sigma_{pp}(B_0^1)$

Proof Let λ and $\phi\in L^2_\xi$ be an eigenvalue and corresponding eigenfunction of the operator B_0^2 i.e. solution of the following equation:

$$[-\nu+K]\phi=\lambda\phi;\quad \phi\in D\big(B_0^2\big). \tag{3.5}$$

I assume first that $\lambda>-\nu_0$. In this case Eq. (3.5) is equivalent to the following:

$$\phi=(\nu+\lambda)^{-1}K\phi;\quad \phi\in L^2_\xi. \tag{3.6}$$

It follows from the α-smoothing of $(\nu+\lambda)^{-1}K$ that $(\nu+\lambda)^{-1}K\phi\in L^{2-\alpha}_\xi$ thus $\phi\in L^{2-\alpha}_\xi$. Iterating this we obtain $\phi\in L^p_\xi$ with $p\in[1,2]$.

Now we consider the more complicated case of eigenvalues embedded in the continuum $\lambda<-\nu_0$. In this case the operator $(\nu+\lambda)^{-1}$ is unbounded and we introduce a projector operator $P(\xi,\delta)$ defined for $\phi\in L^2_\xi$ as:

$$\big[P(\xi_1,\delta)\phi\big](\xi)=\begin{cases}\phi(\xi) & \text{if } \xi\in B(\xi,\delta)\\ 0 & \text{otherwise}\end{cases} \tag{3.7}$$

where $B(\xi_1,\delta)$ is a ball of radius δ with a centre at ξ_1 in R^3. The operator $Q(\xi_1,\delta)=I-P(\xi_1,\delta)$.

If $\lambda\in\mathrm{Range}(-\nu)$ then the $(\nu+\lambda)^{-1}$ can become unbounded for some ξ_1 such that $\nu(\xi)+\lambda\to 0$ as $\xi\to\xi_1$. In this case the operator $Q(\nu+\lambda)Q$ is invertible, the inverse in bounded and $P(\nu+\lambda)P$ has compact support and can be treated separately. In case there exist more ξ_i such that $\nu(\xi)+\lambda\to 0$ as $\xi\to\xi_i$, $i=1,\dots,N$, we introduce operator $P=\sum_{i=1}^N P(\xi_i,\delta_i)$ and $Q=1-P$ and the rest of the proof is the same.

With these operators Eq. (3.5) can be rewritten as:

$$-P(\xi_1,\delta)\big[\nu(\xi)+\lambda_1\big]P(\xi_1,\delta)\phi_i(\xi)+P(\xi_1,\delta)K\phi_i(\xi)=0, \tag{3.8a}$$

$$-Q(\xi_1,\delta)\big[\nu(\xi)+\lambda_1\big]Q(\xi_1,\delta)\phi_i(\xi)+Q(\xi_1,\delta)K\phi_i(\xi)=0. \tag{3.8b}$$

For any ξ_1 and $\delta>0$ it is easy to see that $P(\xi_1,\delta)\phi\in L^2_{\xi,loc}$ thus it is also in $L^p_{\xi,loc}$ for $1\le p\le 2$.

It remains to examine the $Q(\xi_1,\delta)\phi$. It is a solution of the following equation:

$$Q(\xi_1,\delta)\phi=\big[Q(\xi_1,\delta)(\nu+\lambda)\big]^{-1}Q(\xi_1,\delta)K\phi. \tag{3.9}$$

The operator $[Q(\xi_1,\delta)(\nu+\lambda)]^{-1}Q(\xi_1,\delta)K$ has the same smoothing properties as the operator $\nu^{-1}K$ and $[Q(\xi_1,\delta)(\nu+\lambda)]^{-1}$ is bounded for a chosen λ. The same arguments as before lead to the following results:

$$Q(\xi_1,\delta)\phi\in L^p_\xi;\quad 1\le p\le 2. \tag{3.10}$$

As $\phi=P(\xi_1,\delta)\phi+Q(\xi_1,\delta)\phi$, we see that $\phi\in L^p_\xi$ for $1\le p\le 2$ and we have in fact, for $1\le r\le p\le 2$, the following:

$$\sigma_{pp}\big(B_0^2\big)\subset\sigma_{pp}\big(B_0^p\big)\subset\sigma_{pp}\big(B_0^r\big)\subset\sigma_{pp}\big(B_0^p\big) \tag{3.11}$$

□

From the Theorem 3.1 using duality and recalling the fact that the residual spectrum of the operator B_0^2 is empty [5, 9], we obtain:

Corollary 3.1 *The spectrum of the operator B_0 in L^p_ξ is independent on p for $1\le p\le\infty$.*

It is well known from the theory of compact perturbation that an accumulation point of the spectrum may occur only on the boundary of the continuous part of the spectrum. In a case of the operator B_0 this means that the accumulation point can be located at $-\nu_0$ only. That such situation can occur for the Boltzmann operator, is shown by solution for the classical Maxwell potential. Nikolaenko [9] claimed similar property of the spectrum of the Boltzmann operator for hard spheres gas. His proof was criticized by Klaus [9] who shows that with respect to perturbation $ik\xi$ the point $-\nu_0$ is by no means exceptional rising doubts if there may be an accumulation point of the spectrum for such models. In fact when the spectrum is not too much concentrated at $\nu(\xi=0)$, meaning that $\exists r>1$ such that $[\nu(\xi)-\nu(0)]^{-1}\in L^r_{loc}$, then the accumulation point does not occur. A following theorem settles this problem for a wide class of hard interactions including also the classical hard sphere gas model with agreement with Klaus observation:

Theorem 3.3 *For such $\nu(\xi)$ that $|\nu(\xi)-\nu(0)|^{-1}\in L^r_{\xi,loc}$, with some $r>1$, there is no accumulation point of the spectrum nor eigenvalue with infinite multiplicity in the spectrum of the operator B_0^p for $1\le p\le\infty$.*

Proof As the spectrum of the operator B_0 is independent of p, we consider operator B_0^2 in L^2_ξ. We assume that there is an accumulation point of the spectrum at $\nu_0=\nu(|\xi|=0)$. This means that we can find $\{\lambda_i\}_{i=1}^\infty\in C$ and $\{\phi_i\}_{i=1}^\infty\in L^2$ such that

$$\|\phi_i\|_{L^2}=1$$

and

$$[-\nu+\kappa]\phi_i=\lambda_i\phi_i, \tag{3.12}$$

and

$$\lim_{i\to\infty}\lambda_i=-\nu_0. \tag{3.13}$$

As $\nu_0=\nu(0)$, and introducing operators $P(0,\delta)$ and $Q(0,\delta)$ with $\delta>0$, we can rewrite Eq. (3.19) as:

$$P(0,\delta)(\nu+\lambda_i)P(0,\delta)\phi_i=P(0,\delta)K\phi_i, \tag{3.14a}$$

$$Q(0,\delta)(\nu+\lambda_i)Q(0,\delta)\phi_i=Q(0,\delta)K\phi_i. \tag{3.14b}$$

As the K operator is a compact operator in L^p, $1<p<\infty$, the following strong limit exists:

$$s-\lim_{i\to\infty}P(0,\delta)(\nu+\lambda_i)P(0,\delta)\phi_i=g\in L^2. \tag{3.15}$$

As $|\nu-\nu_0|^{-1}\in L^r_{loc}$, then for $|\xi|<\delta$ the following estimate holds for Hölder inequality:

$$\begin{aligned}\|P(0,\delta)\phi_i\|_{L^1}&=\int_{B(0,\delta)}|\phi_i(\xi)|d\xi=\int_{B(0,\delta)}\big(\nu(\xi)+\lambda_i\big)\big(\nu(\xi)+\lambda_i\big)^{-1}|\phi_i(\xi)|d\xi\\&\le\left(\int_{B(0,\delta)}\Big[\big(\nu(\xi)+\lambda_i\big)^{-1}\Big]^r d\xi\right)^{1/r}\left(\int_{B(0,\delta)}\Big[\big(\nu(\xi)+\lambda_i\big)|\phi_i(\xi)|\Big]^q d\xi\right)^{1/q}\\&=\left(\int_{B(0,\delta)}\Big[\big(\nu(\xi)+\lambda_i\big)^{-1}\Big]^r d\xi\right)^{1/r}\|P(0,\delta)(\nu+\lambda_i)P(0,\delta)\phi_i\|_{L^q}\end{aligned} \tag{3.16}$$

and, at the limit as $i\to\infty$, we get:

$$\|P(0,\delta)\phi_i\|_{L^1}\le\big\|P(0,\delta)|\nu-\nu_0|^{-1}\big\|_{L^r}\|g\|_{L^q} \tag{3.17}$$

with $1/r+1/q=1$.

We see that the set of functions of common compact support $\{P(0,\delta)\phi_i\}_{i=1}^{\infty}$ is a uniformly bounded set in L^1_ξ. For any set $A\subset R^3$ with $\mu(A)<\delta$ we have:

$$\int_A|P(0,\delta)\phi_i|d\xi\le\mu(A)\left[\int_A|P(0,\delta)\phi_i|^q d\xi\right]^{1/q}\le\mu(A)\cdot\|g\|_{L^q}\le\delta\cdot\|g\|_{L^q}. \tag{3.18}$$

Thus according to the Dunford-Pettis Lemma the sequence $\{P(0,\delta)\phi_i\}_{i=1}^{\infty}$ are elements of a weakly compact set in L^1_ξ and we can choose subsequence converging weakly in L^1_ξ. We denote this limit as ψ i.e.

$$\psi=w-\lim_{i\to\infty}P(0,\delta)\phi_i. \tag{3.19}$$

Taking sufficiently large δ and choosing a subsequence if necessary, we can replace Eqs. (3.14a), (3.14b) by the following:

$$Q(0,\delta)\phi_i=\big[Q(0,\delta)(\nu+\lambda_i)-Q(0,\delta)kQ(0,\delta)\big]^{-1}KP(0,\delta)\phi_i. \tag{3.20}$$

$\{P(0,\delta)\phi_i\}_{i=1}^{\infty}$ are elements of a weakly compact set in $L^1(d_3\xi)$ and K is a weakly compact operator in this space thus we see that the set $\{P(0,\delta)\phi_i\}_{i=1}^{\infty}$ contains a strongly converging in L^1 subsequence. The operator $-Q(0,\delta)(\nu+\lambda_i)+Q(0,\delta)KQ(0,\delta)$ has properties similar to the operator $-\nu+K$ i.e. QKQ is a compact in L^2 operator and $-Q\nu$ has continuous spectrum contained in a set $]-\infty,-\nu(\delta)]$. The accumulation point in the spectrum of this operator can occur at $\lambda=-\nu(\delta)<-\nu_0$. Thus dropping to the subsequence if necessary, we can select a sequence $\{KP(0,\delta)\phi_{i_n}\}$ such that operators $[Q(\nu+\lambda_{i_n})-QKQ]^{-1}$ are uniformly bounded on this sequence. As K is a weakly compact operator in L^1, it follows that the corresponding sequence $Q(0,\delta)\phi_{i_n}$ converges in $L^1(d_3\xi)$ strongly. We call this limit $Q(0,\delta)\psi$.

From the fact that $[P(0,\delta)(\nu+\lambda_i)]^{-1}\in L^1(d_3\xi)$ it follows that:

$$P(0,\delta)\phi_i=\big(P(0,\delta)\nu+\lambda_i\big)^{-1}P(0,\delta)K\phi_i=w_i g_i.$$

Moreover, we see that the following strong limits in L^1 exist:

$$s-\lim_{i\to\infty} g_i=g,$$

$$s-\lim_{i\to\infty} w_i=w,$$

$$s-\lim_{i\to\infty} g_i w_i=gw.$$

From the fact that $P(0,\delta)(\nu-\nu_0)^{-1}\in L^p$, it follows that in L^1 with a help of estimate:

$$\int\big|P(0,\delta)(\nu+\lambda_i)\big|^{-1}\big|P(0,\delta)k\phi_i\big|\le\big\|P(0,\delta)|\nu+\nu_0|^{-1}\big\|_p\|PK\phi_i\|_q \tag{3.21}$$

we see that $P(0,\delta)\phi_i\to P(0,\delta)\psi$ strongly in L^1_ξ, and we finally obtain:

$$s-\lim_{i\to\infty}\phi_i=\psi\in L^1_\xi. \tag{3.22}$$

Taking now the limit in Eq. (3.13) we obtain:

$$s-\lim_{i\to\infty}[-\nu+\lambda_i]\phi_i=[-\nu+\nu_0]\psi \tag{3.23}$$

This shows that $-\nu_0\in\sigma_{pp}(B_0^p)$ for $1\le p\le\infty$ with corresponding eigenfunction ψ which we can normalize in L^2_ξ. On the other hand, eigenfunctions belonging to different eigenvalues are orthogonal in $L^2(d_3\xi)$, thus for all i $(\phi_i,\psi)_{L^2}=0$ but we have shown $\psi=w-\lim\phi_i$ in L^1 and as $\phi_i\in L^1\cap L^\infty$ we shall have $(\psi,\psi)_{L^2}=\lim_{i\to\infty}(\psi,\phi_i)=0$. This contradicts the fact that $\|\psi\|_{L^2}=1$, thus $-\nu_0$ cannot be an accumulation point. □

Remark The bound on $|\nu(\xi)-\nu(0)|^{-1}$ used in Theorem 3.3 is probably not an optimal one, but as it is well known from the exact solutions for classical Maxwell model or Lorentz gas [27] if $\nu(\xi)=\text{const}$ there is an accumulation point of the spectrum or eigenvalues with infinite multiplicity. These examples show that a bound of this kind is necessary for Theorem 3.3 to be true. We want to mention that for cross-sections behaving like g^β with $\beta>0$ in relativistic case and like $|\xi-\xi_1|^\gamma$; $\gamma>0$ in the classical case Theorem 3.3 shows absence of both accumulation points or eigenvalues with infinite multiplicity in the spectrum of the Boltzmann operator.

4 Spectrum of the operators B_k in L^p_ξ for $1 \le p \le \infty, k \in R^3$

The analysis of Fourier transformed operator B_k in L^2_ξ was first done in [7] with an excellent extension in [11] for the classical operator and in [5] for the relativistic one. For the reader's convenience we cite the main results of these papers in the beginning of this section. Operators $B^p_k = -\nu(\xi) + K - ik\eta(\xi)$, considered as operators from L^p_ξ into L^p_ξ, are closed operators with a domain $D(B^p_k) = \{f \in L^p_\xi : \|(|\nu| + |k\eta(\xi)|)f\|_{L^p_\xi} < \infty\}$. For the relativistic operator $\eta(\xi) = \xi/(1+\xi^2)$ and the domain is the same as for the operators B^p_0. In the classical case $\eta(\xi) = \xi$ and for all interaction except hard spheres gas model $D(B^p_k) \subset D(B^p_0)$.

I describe first the general properties of the spectrum of the operators B^p_k:

(i) $\mathrm{Re}(B^2_k\phi, \phi) = (B^2_0\phi, \phi)$. Therefore $\sigma(^2_k)$ lies in the left half-plane.
(ii) $\sigma(B^2_k)$ is symmetric with respect to the real axis, if $\psi(\bar{\xi})$ is an eigenfunction belonging to the eigenvalue λ of the operator B^2_k then $\phi^*(-\bar{\xi})$ is an eigenfunction belonging to the eigenvalue λ^* of the operator $B'^2_k = B^2_{-k}$.
(iii) $\sigma(B^2_k)$ consists of the continuum $\varGamma_k = -(i\bar{k}\bar{\eta}(\bar{\xi}) + \nu(\bar{\xi}); \bar{\xi} \in R^3)$ and a discrete part consisting of isolated eigenvalues with finite multiplicity. These can accumulate only on the boundary of $\varGamma_k$.
(iv) The residual spectrum of the B^2_k is empty.

These properties are also true for the operators B^p_k for $p \ne 2$ and as $|\nu(\xi) + ik\eta(\xi)|^{-1} \le |\nu(\xi)|^{-1}$ we see that the presence of $ik\eta(\xi)$ can improve both the smoothing properties of the operator $[\nu(\xi) + ik\eta(\xi)]^{-1}K$ and change the spectral concentration of the continuous part of the spectrum making it more uniform. This is more important for classical models where $\lim_{|\xi|\to\infty} |\nu(\xi) + ik\xi|^{-1} = 0$ a.e. and $\lim_{|k|\to\infty} |\nu(\xi) + ik\xi|^{-1} = 0$ a.e. we see that both for large k or ξ the operator becomes more regular than ν^{-1}.

I formulate the theorems below with the slightly more general requirements for regularity of operators $Q(\nu(\xi) + ik\eta(\xi))^{-1}QK$ and $(\nu(\xi) + ik\eta(\xi) - \lambda)^{-1}$ with the understanding that this assumptions are fulfilled by all scattering cross-sections defined as hard interactions including hard sphere gas.

Theorem 4.1 *For all cross-sections such that operator* $Q(\bar{\xi}^*, \delta)[\nu(\bar{\xi}) + i\bar{k}\bar{\eta}(\bar{\xi})]^{-1}Q(\bar{\xi}^*, \delta)K$ *with some* ξ^* *and* $\delta > 0$ *has smoothing properties with* $\alpha > 0$, *the spectrum of the operators* B^p_0 *in* $L^p(d_3\xi)$ *is independent of p for* $1 \le p \le \infty$.

Proof The proof follows exactly proofs of Theorems 3.1 and 3.3 with only changes that we shall use the fact that $(B^p_k)' = B^q_{-k}$ with $p^{-1} + q^{-1} = 1$.

We note that in particular Theorem 4.1 shows that eigenfunctions of the operators B_k $\phi^i_k \in L^1_\xi \cap L^\infty_\xi$ for all $k \in R^3$ and also that the residual spectrum of the operators B^p_k is empty for all p and k. □

Theorem 4.2 *Let us assume that there exists* $r > 1$ *such that* $|\nu(\xi) + ik\eta(\xi) - \nu(\xi_1) - ik\eta(\xi_1)|^{-1} \in L^r_{\xi,loc}$, *then there is no accumulation point nor eigenvalue with infinite multiplicity at the spectral point* $\lambda = -\nu(\xi_1) + ik\eta(\xi_1)$.

Proof The proof follows the proof of Theorem 3.3 with the only change that now instead of orthogonality of eigenfunctions belonging to different eigenvalues we shall apply the biorthogonality relations

$$\left(\psi^i_{-k}, \phi^j_k\right) = \delta_{ij} \quad \text{for } \lambda_i(k) \ne \lambda_j(k) \tag{4.1}$$

valid for left and right eigenfunctions of the operators B_k^p. □

Theorems 4.1 and 4.2 describe the properties of the spectrum of operators B_k for a given $k \in R^3$. Now we consider the behaviour of the spectrum as a function of this parameter. For $k = 0$, according to Theorem 3.3, operator B_0 has point spectrum consisting of eigenvalues of finite multiplicity. There may be finite or infinite number of eigenvalues and some of them may lie in the continuous part of the spectrum. In general, if the perturbation $ik\eta(\xi)$ is turned on, some of these eigenvalues may disappear even for arbitrary small value of $|k|$. Such situation cannot occur for the relativistic Boltzmann operator with discrete part of the spectrum, as in this case the operator $ik\eta(\xi)$ is bounded in $L^p(d_3\xi)$ for any k and Kato theory of analytical perturbation [26] is applicable. Similar situation occurs for classical hard spheres model where the perturbation $ik\xi$ is relatively bounded with respect to $-\nu + K$ and the theory of relatively bounded perturbation [26] assures existence of eigenvalues of the perturbed operator for sufficiently small values of $|k|$. However, if $\lambda \in \Gamma$ it is not possible to exclude such possibility and as we will show in the next section, it can occur for the Boltzmann operator.

I first prove that $\lambda_i(k)$ are continuous functions of k:

Theorem 4.3 *Let $\lambda_i(k)$ exist for $|k| \in]k_1, k_2[$ and let $\lambda_i(k) \notin \Gamma_k$ and $\lim_{k\to k_l} \lambda_i(k) \notin \Gamma_k$, $l = 1, 2$. Then it exists for $|k| \in [k_1, k_2]$ and $\lim_{k\to k_l} \lambda_i(k) = \lambda_i(k_l)$, $l = 1, 2$. If in addition the assumptions of Theorem 4.2 are fulfilled then the same is true also for $\lambda_i(k) \in \Gamma_k$.*

Proof We consider first $\mathrm{Re}\,\lambda_i(k) \notin \Gamma_k$ for $|k| \in]k_1, k_2[$ and let $\lim_{k\to k_{1,2}} \lambda_i(k) \notin \Gamma_k$. For all $k \in]k_1, k_2]$ exist functions $\lambda(k)$ and $\phi_k(\xi)$ such that $\|\phi_k\|_{L^2(d_3\xi)} = 1$ and

$$\big[\nu(\xi) + K - ik\eta(\xi)\big]\phi_k(\xi) = \lambda(k)\phi_k(\xi), \tag{4.2}$$

For every sequence $k_n \to k_2$ we can write

$$\phi_k(\xi) = \big[\nu(\xi) + ik\eta(\xi) + \lambda(k)\big]^{-1} K\phi_k(\xi). \tag{4.3}$$

I consider now a following sequence

$$\psi_{k_n}(\xi) = \big[\nu(\xi) + ik_2\eta(\xi) + \lambda(k_2)\big]^{-1} K\phi_{k_n}(\xi). \tag{4.4}$$

The operator $[\nu(\xi) + ik_2\eta(\xi) + \lambda(k_2)]^{-1}K$ is compact in $L^2(d_3\xi)$ and the sequence ϕ_{k_n} is bounded in $L^2(d_3\xi)$ thus the sequence ψ_{k_n} contains strongly converging subsequence. We denote this limit as $\psi_{k_2} = s - \lim_{n\to\infty} \phi_{k_n}$. My aim is to show that $\psi_{k_2}(\xi)$ is an eigenfunction with corresponding eigenvalue $\lambda(k_2)$. To this end we consider the following estimates:

$$\begin{aligned}\|\phi_{k_n} - \psi_{k_2}\|_{L^2(d_3\xi)} &= \|\phi_{k_n} - \psi_{k_n} + \psi_{k_n} - \psi_{k_2}\|_{L^2(d_3\xi)} \\ &\le \|\phi_{k_n} - \psi_{k_n}\|_{L^2(d_3\xi)} + \|\psi_{k_n} - \psi_{k_2}\|_{L^2(d_3\xi)},\end{aligned} \tag{4.5}$$

and the first term on the r.h.s. can be estimated as follows:

$$\|\phi_{k_n} - \psi_{k_n}\|_{L^2} \le \Big\|\Big\{\big[\nu(\xi) + ik_n\eta(\xi) + \lambda(k_n)\big]^{-1} - \big[\nu(\xi) + ik_2\eta(\xi) + \lambda(k_2)\big]^{-1}\Big\}K\phi_{k_n}\Big\|_{L^2}, \tag{4.6}$$

and as $w-\lim_{n\to\infty}\{[\nu(\xi)+ik_n\eta(\xi)+\lambda(k_n)]^{-1}-[\nu(\xi)+ik_2\eta(\xi)+\lambda(k_2)]^{-1}\}\phi_{k_n}=0$ then the expression on the r.h.s. of Eq. (4.6) converges to zero strongly in $L^2(d_3\xi)$ similarly as the second term in Eqs. (4.5), (4.6) and for sufficiently large n we obtain for arbitrary $\epsilon>0$

$$\|\phi_{k_n}-\psi_{k_2}\|_{L^2(d_3\xi)}<\epsilon. \tag{4.7}$$

This shows that $\psi_{k_2}=s-\lim_{n\to\infty}\phi_{k_n}$, and in fact means that ψ_{k_2} is a solution of the following equation:

$$\bigl[-\nu(\xi)+K-ik_n\eta(\xi)\bigr]\psi_{k_2}=\lambda(k_2)\psi_{k_2}. \tag{4.8}$$

Similar arguments hold for $k\to k_1$.

For $\operatorname{Re}\lambda(k)\le-\nu_0$ this problem is more complicated and we use operators $P(\xi^*,\delta)$ and $Q(\xi^*,\delta)$. Again, it is enough to consider the case $k\to k_2$ only. We assume that $-\nu(\xi^*)=\operatorname{Re}\lambda(k_2)$. Similar as in the previous case we consider the eigenproblem for the operator B_k^2 for $|k|\in]k_1,k_2[$:

$$\bigl[-\nu(\xi)+K-ik\eta(\xi)\bigr]\phi_k(\xi)=\lambda(k)\phi_k(\xi). \tag{4.9}$$

This equation can be rewritten as:

$$P\bigl(\xi^*,\delta\bigr)\bigl[\nu(\xi)+ik\eta(\xi)+\lambda(k)\bigr]P\bigl(\xi^*,\delta\bigr)\phi_k(\xi)=P\bigl(\xi^*,\delta\bigr)K\phi_k(\xi), \tag{4.10a}$$

$$Q\bigl(\xi^*,\delta\bigr)\bigl[\nu(\xi)+ik\eta(\xi)+\lambda(k)\bigr]Q\bigl(\xi^*,\delta\bigr)\phi_k(\xi)=Q\bigl(\xi^*,\delta\bigr)K\phi_k(\xi). \tag{4.10b}$$

As the operator K is compact in L^2 and ϕ_k are elements of a bounded set in L^2 we see that

$$s-\lim_{k\to k_2}P\bigl(\xi^*,\delta\bigr)\bigl[\nu(\xi)+ik\eta(\xi)+\lambda(k)\bigr]P\bigl(\xi^*,\delta\bigr)\phi_k(\xi)=P\bigl(\xi^*,\delta\bigr)g_{k_2}(\xi),$$

with $\|g_{k_2}\|_{L^2(d_3\xi)}=1$. With the help of the second assumption which states that $[\nu(\xi)+ik\eta(\xi)+\lambda(k)]^{-1}\in L^r_{\xi,loc}$ we can repeat part of the proof of Theorem 3.3 and show that $P(\xi^*,\delta)\phi_k$ are elements of a weakly compact set in $L^1(d_3\xi)$, and we can subtract a weakly converging subsequence $P(\xi^*,\delta)\phi_k$. We denote this weak limit as ψ_{k_2}:

$$\psi_{k_2}=w-\lim_{n\to\infty}\phi_{k_2}\text{ in }L^1(d_3\xi). \tag{4.11}$$

With a sufficiently large δ Eq. (4.10b) can be written in the following form:

$$Q\bigl(\xi^*,\delta\bigr)\phi_k(\xi)=\bigl\{Q\bigl(\xi^*,\delta\bigr)\bigl[\nu(\xi)+ik\eta(\xi)+\lambda(k)\bigr]\bigr\}^{-1}Q\bigl(\xi^*,\delta\bigr)K\phi_k(\xi). \tag{4.12}$$

For functions $Q(\xi^*,\delta)\phi_k(\xi)$ the reasoning from the first part of the proof can be applied leading to the conclusions that $Q(\xi^*,\delta)\phi_k(\xi)$ contains subsequence strongly converging in $L^1(d_3\xi)$, namely we have

$$Q\bigl(\xi^*,\delta\bigr)\phi_{k_2}(\xi)=s-\lim_{n\to\infty}Q\bigl(\xi^*,\delta\bigr)\phi_{k_n}(\xi). \tag{4.13}$$

Taking now into account simple facts that

$$w-\lim_{n\to\infty}P\bigl(\xi^*,\delta\bigr)\bigl[\nu(\xi)+ik_n\eta(\xi)+\lambda(k_n)\bigr]=P\bigl(\xi^*,\delta\bigr)\bigl[\nu(\xi)+ik_2\eta(\xi)+\lambda(k_2)\bigr],$$

and

$$w-\lim_{n\to\infty}Q\bigl(\xi^*,\delta\bigr)\bigl[\nu(\xi)+ik_n\eta(\xi)+\lambda(k_n)\bigr]=Q\bigl(\xi^*,\delta\bigr)\bigl[\nu(\xi)+ik_2\eta(\xi)+\lambda(k_2)\bigr],$$

and that a weakly compact operator sends weakly compact sequences into a norm converging ones, we obtain that $\psi_{k_2} \in L^1(d_3\xi)$ solves in $L^1(d_3\xi)$ the following equation

$$\left[-\nu(\xi) + K - ik\eta(\xi)\right]\psi_{k_2}(\xi) = \lambda(k_2)\psi_{k_2}(\xi). \tag{4.14}$$

This means that ψ_{k_2} is an eigenfunction of the operator B_k^1 with eigenvalue $\lambda(k_2)$ and as the spectrum is independent of p it is also an eigenfunction of the operator B_k^2 with the same eigenvalue. Having this it is easy to show that $\psi_{k_2} = s - \lim_{k\to k_2} \psi_k$, in $L^2(d_3\xi)$ and this ends the proof. As the arguments for any point $k_l \in [k_1, k_2]$ are similar it is easy to show also that if $\lambda(k)$ exists for $|k| \in [k_1, k_2]$ then $\lambda(k)$ is continuous functions of k in this interval. □

In the preceding theorem we tacitly assumed that $|\lambda(k) < \infty|$. A following lemma shows that if $|\mathrm{Re}\,\lambda(k)| < \infty$ then $|\mathrm{Im}\,\lambda(k)| < \infty$.

Lemma 4.1 *Let* $A_\lambda = (\nu + ik\eta + \lambda)$ *then*

$$\lim_{\substack{|\mathrm{Im}\,\lambda|\to\infty \\ |\mathrm{Re}\,\lambda| = \alpha < \infty}} \left\| K A_\lambda^{-1} \right\| = 0$$

Proof For sufficiently large $|\mathrm{Im}\,\lambda|$ with $|\mathrm{Re}\,\lambda| = \alpha$ the operator $[\nu + ik\eta + \lambda]^{-1}$ is bounded in L_ξ^p for $1 \le p \le \infty$. Moreover, for such λ the operator $K A_\lambda^{-1}$ is a compact operator as a composition of a compact and bounded operators. Let us assume on the contrary that there exist sequences λ_n with $|\mathrm{Re}\,\lambda_n| = \alpha$, $|\mathrm{Im}\,\lambda_n| \to \infty$ as $n \to \infty$, and $\phi^{\lambda n} \in L_\xi^2$; $\|\phi^{\lambda n}\|_2 = 1$ such that:

$$\left\| K A_{\lambda n}^{-1} \phi^{\lambda n} \right\| \nrightarrow 0. \tag{4.15}$$

It follows from the compactness of k that $A_{\lambda n}^{-1}\phi^{\lambda n} \nrightarrow 0$ weakly, but for any $\psi \in L^2$ we have:

$$\left(\psi, A_{\lambda n}^{-1}\phi^{\lambda n}\right) \le \left(\int |\psi|^1 \left|A_{\lambda n}^{-1}\right|\right)^{1/2} \left\|\phi_1^{\lambda n}\right\|_2 \tag{4.16}$$

and as $\lim_{n\to\infty} A_{\lambda n}^{-2} = 0$ a.e. we see that $\lim_{n\to\infty} \int |\psi|^2 |A_{\lambda n}^{-2}| = 0$. This shows that $A_{\lambda n}^{-1}\phi^{\lambda n} \to 0$ weakly and this contradiction shows:

$$\lim_{\substack{|\mathrm{Im}\,\lambda|\to\infty \\ |\mathrm{Re}\,\lambda| = \alpha}} \left\| K A_\lambda^{-1} \right\| = 0 \tag{4.17a}$$

By multiplying $K\phi = A_\lambda\phi$ by A_λ^{-1} we obtain $A_\lambda^{-1}K\phi = \phi$ and further by introducing $\phi = A_\lambda^{-1}\psi$ we obtain $K A_\lambda^{-1}\psi = \psi$ and see that:

$$\lim_{\substack{|\mathrm{Im}\,\lambda|\to\infty \\ |\mathrm{Re}\,\lambda| = \alpha}} \left\| A_\lambda^{-1} K \right\| = 0 \tag{4.17b}$$

Equations (4.17a) and (4.17b) prove the Lemma. □

From this Lemma it is easy to see the following:

Corollary 4.1 *If* $\lim_{k\to k_0} |\mathrm{Re}\,\lambda(k)| < \infty$ *then* $\lim_{k\to k_0} |\mathrm{Im}\,\lambda(k)| < \infty$.

The behaviour of eigenvalues $\lambda_i(k)$ for such k_ϵ that $\lambda_i(k)$ exist and $\operatorname{Re}\lambda_i(k) > -\nu_0$ is described by the following:

Proposition 4.1 *For every $0 < \epsilon < \nu_0$, exists such k_ϵ, that for $|k| > k_\epsilon$ the half-plane $\operatorname{Re}\lambda > -\epsilon$ is free from eigenvalues of the operator B_k.*

Proof For the spectrum of the operator B_k^2 this proposition was proved in [10] in the case of classical operator for hard spheres model. In [5] more general proof for relativistic hard interaction was presented for B_k^2 which applies also to classical operator with hard interaction. As the spectrum is independent of p this result is valid for the spectrum of all operators B_k^p with $1 \le p \le \infty$. □

Theorem 4.1, Proposition 4.1 and the theory of the relatively bounded perturbation solve almost completely the problem of behaviour of the eigenvalues $\lambda_i(k)$ of the operators B_k as functions of k for such k that $\operatorname{Re}\lambda_i(k) > -\nu_0$. These eigenvalues are either constant for $|k| \in [k_1^i, k_2^i]$ where $[k_1^i, k_2^i]$ are intervals of their existence or they change continuously with k for $|k| \in [0, k_{max}^i]$ and $\operatorname{Re}\lambda_i(k_{max}^i) = -\nu_0$.

To see this, it is enough to observe that if an eigenvalue $\lambda(k)$ exists for $|k| \in [k_1, k_2]$, then, as the operator $i(k-k_1)\eta(\xi)$ is relatively bounded with respect to the operator B_{k_1}, there exists $\delta_1 > 0$ such that for $|k-k_1| < \delta_1$ the $\lambda(k)$ exists for $|k| \in [k_1, -\delta_1, k]_1$ and has asymptotic expansion of the form $\lambda(k) = \lambda(k_1) + |k-k_1|\Delta\lambda + O(k^2)$. We can continue this as long as $|k_n - \delta_n| > 0$ and we see that either $\lambda(k)$ exists for all $|k| \in [0, k^{max}]$ with $\lambda(0) = \lambda_i$, this last being an eigenvalue of the collision operator L and $\lambda(k^{max}) \in \Gamma_k max$, or $\lambda(k)$ is constant, or $\lambda(k)$ exists for $|k| \in [k_1, k_2]$ and $\lambda(k_1) \in \Gamma_{k_1}, \lambda(k_2) \in \Gamma_{k_2}$. In particular, this result implies that eigenvalues of the collision operator L located in the region $-\nu_0 < \lambda_i \le 0$ can disappear when the perturbation $ik\eta(\xi)$ is added, but a new eigenvalue $\lambda(k) \ne \text{const}$ can emerge only from the continuum.

This shows that on any $|k| < \infty$ the spectrum of B_k is contained in the sector of the left complex half-plane $\operatorname{Re}\lambda < 0$, $\arg\lambda < \alpha < \pi$. This shows that the B_k^p are sectorial operators—see [28] for an excellent review of such operators. We have:

Corollary 4.2 *For any $|k| < \infty$ B_k^p are sectorial in L^p $1 \le p \le \infty$ with the angle $\alpha(k) < \pi$.*

As for $|k| \to \infty$ continuous part of the spectrum covers the whole half-space $\operatorname{Re}\lambda < -\nu_0$, the property cannot be extended to the original operator B in $R^3 \times R^3$ acting in the physical space.

The smoothing properties of the operator $[\nu(\xi) + ik\eta(\xi)]^{-1}K$ expressed as:

$$\left[\nu(\xi) + ik\eta(\xi)\right]^{-1} K\phi \in L_\xi^\infty \quad \text{for } p \in L_\xi^2, \tag{4.18}$$

$$\left(1+\xi^2\right)^{r+1/2}\left[\nu(\xi) + ik\eta(\xi)\right]^{-1} K\phi \in L_\xi^\infty \quad \text{if } \left(1+\xi^2\right)^r \phi \in L_\xi^\infty, \tag{4.19}$$

lead to the high degree of regularity of the eigenfunctions $\phi_k(\xi)$ as the functions of ξ.

We define spaces $L_\xi^{p,l}$ as:

$$\phi \in L_\xi^{p,l} \quad \text{if } \phi \in L_\xi^p \text{ and } (1+\xi)^l \phi \in L_\xi^p. \tag{4.20}$$

Clearly $L_\xi^{p,l} \subset L^p$ for $l > 0$. Iterating N times the equation for eigenfunction B_k^p

$$\phi = [\nu + ik\eta + \lambda]^{-1} K\phi \tag{4.21}$$

and letting $N \to \infty$, we see that:

Corollary 4.3 *The spectrum of the operator B_k^p is the same in all $L^{p,l}$ for $1 \le p \le \infty$ and $l \in R^+$.*

The proof for $\operatorname{Re}\lambda > -\nu_0$ is obvious now and for $\operatorname{Re}\lambda < \nu_0$ with the help of the Q and P operators repeating arguments in the Theorem above we obtain required result.

It follows then the $\phi_k(\xi)$ are not only in $L_\xi^\infty \cap L_\xi^1$ but decay faster than any power of ξ^n for $|\xi| \to \infty$. The well known example of such functions are the eigenfunctions for classical Maxwell potential.

5 The spectrum of the operator B in $L^p_{x,\xi}$ for $1 \le p \le \infty$

In this section we prove the main theorem of this paper describing the spectrum of the Boltzmann operator in $L^p_{x,\xi}$ spaces. Denoting as B^p the operator B considered as a operator from $L^p_{x,\xi}$ into $L^p_{x,\xi}$ we have the following

Theorem 5.1 *For all hard interactions operator B^p for $1 \le p < \infty$ in $L^p_{x,\xi}$ has spectrum independent of p. This spectrum consists of half-plane $\operatorname{Re}\lambda \le -\nu_0$ and in the strip $-\nu_0 < \operatorname{Re}\lambda \le 0$ of finite if $[\nu(\xi) - \nu(0)]^{-1} \in L^r_{loc}(r > 1)$ or otherwise infinite number of branches $\lambda = \lambda_i(k)$, $k \in [0, k^i_{max}]$. These are continuous functions of k and $\operatorname{Re}\lambda_i(k^i_{max}) = -\nu_0$. For $p = \infty$ the continuous part of the spectrum also covers the half-plane $\operatorname{Re}\lambda < -\nu_0$ but $\lambda_i(k)$ for $|k| \in [0, k^i_{max}]$ belong to uncountable point spectrum of the operator B^∞. For all $1 \le p \le \infty$ a half-plane $\operatorname{Re}\lambda > 0$ lies entirely in the resolvent set of the operators B^p.*

Proof We consider first the operator B^2 in $L^p_{x,\xi}$. As the Fourier transform is an isomorphism of L_x^2 and L_k^2, it follows that the spectrum of the operator B^2 is a sum of the spectral sets of the operators B_k for all $k \in R^3$. We see that it consists of the continuous part covering the whole half-plane $\operatorname{Re}\lambda < -\nu_0$ and of countable number of branches $\lambda_i(k)$, $i = l, 2, \ldots$. where $\lambda_i(k)$ are the eigenvalues of the operator B_k^2 in L_ξ^p. We note that as this direct sum assumes all values of the parameter k, it is clear that the spectrum is symmetric with respect to the real axis. □

Now we consider the operators B^1 and B^∞ . We note first that if $\lambda \in \sigma(B^1)$, then $\lambda \in \sigma(B^\infty)$. To see this, we observe that if there exists such a sequence of $\phi_n \in L^1_{x,\xi}$ that $\lim_{n\to\infty} \|(B^1 - \lambda)\phi_n\|_{L^1} = 0$ with $\|\phi_n\|_{L^1} = 1$, then for the sequence $\bar{\phi}_n$ we have $\lim_{n\to\infty} \|((B^1)' - \bar{\lambda})\bar{\phi}_n\|_{L^1} = 0$. This shows that $\bar{\lambda}$ belongs to the spectrum of the operator $(B^1)'$ in $L^1_{x,\xi}$ and as $(B^1)' = B^\infty$, it follows that $\lambda \in \sigma(B^\infty)$. Applying now the interpolation theorem to the resolvent operators $(B^1 - z)^{-1}$ and $(B^\infty - z)^{-1}$, we see that $\sigma(B^2) \subseteq \sigma(B^1)$ and $\sigma(B^2) \subseteq \sigma(B^\infty)$. Thus the essential difficulty lies in showing that $\sigma(B^\infty) \subseteq \sigma(B^2)$.

Assume first that $\operatorname{Re}\lambda > -\nu_0$ and that there exists a sequence $\phi_n \in L^\infty_{x,\xi}$ with $\|\phi_n\|_{L^\infty} = 1$ such that

$$\big(B^\infty - \lambda\big)\phi_n = w_n, \tag{5.1}$$

and $\lim_{n\to\infty} \|w_n\|_{L^\infty} = 0$.

I now use the fact that if functions $\phi_n(x, \xi)$ and $w_n(x, \xi)$ fulfill relation (5.1), then functions $\psi_n(x, \xi) = \phi_n(x - t_n, \xi)$, $r_n(x, \xi) = w_n(x - t_n, \xi)$ with arbitrary choice of t_n also fulfill this relation. We denote as A_n a sequence of sets contained in R^6 such that ess inf $|\phi_n| > 1-\epsilon$ for $(x,\xi) \in A_n$. Such a set exists for each n as ess sup $|\phi_n| = 1$. Equation (5.1) can be rewritten to the following form

$$\phi_n = S(\lambda)K\phi_n + S(\lambda)w_n \tag{5.2}$$

where the explicit form of the operator $S(\lambda)$ reads

$$S(\lambda)f(x,\xi) = \int_0^\infty \exp\bigl[-\bigl(\nu(\xi)-\lambda\bigr)\tau\bigr] f(x+\xi\tau,\xi)d\tau, \tag{5.3}$$

and in the above formula we can put $f_n = K\phi_n$. With a proper choice of the vectors t_n we can assure that $x = 0 \in A_n$ for all n and we observe that as $\lim_{\xi\to\infty} |S(\lambda)K\phi| = 0$, for any n there exists such $\xi_n \in A_n$ that $\lim_{n\to\infty} |\xi_n| < \infty$. This observation is obvious for such Boltzmann operators for which $\lim_{\xi\to\infty} \nu(\xi) = \infty$ but for other interactions, as for example for the Maxwell model, we must use the more subtle estimates on the operator K given in [10] for classical hard interactions and in [5] for the relativistic case. The behaviour of these sets A_n is described in the lemma,

Lemma 5.1 *There exist sequences ϕ_n and w_n fulfilling relation* (5.1) *and such that there exists bounded set $A < R^6$ with $\mu(A) > 0$ and a constant $C > 0$ that $|\phi_n| > C$ a.e. in A.*

Proof Assume that we have chosen such a sequence ϕ_n that $x = 0$ belongs to A_n and without loss of generality we can assume that $\lim_{n\to\infty} \xi_n = 0$. Now we take a ball $\Omega(R) \subset R^6$ with a centre in point $(0,0)$ and such that for $n > N$ $\mu(\Omega \cap A_n) > 0$. We introduce then a projector operator P_Ω corresponding to the characteristic function of the set Ω. Acting with this operator on Eq. (5.2) we obtain

$$P_\Omega\phi_n = P_\Omega S(\lambda)K\phi_n + P_\Omega S(\lambda)w_n. \tag{5.4}$$

Expressing ϕ_n as $P_{\Omega'}\phi_n + Q_{\Omega'}\phi_n$, where $Q_\Omega = 1 - P_\Omega$, and I will specify the set Ω' later, we can write Eq. (5.4) as

$$P_\Omega\phi_n = P_\Omega S(\lambda)KP_{\Omega'}\phi_n + P_\Omega S(\lambda)w_n + P_\Omega S(\lambda)KQ_{\Omega'}\phi_n. \tag{5.5}$$

Now I consider the term $P_\Omega S(\lambda)KQ_{\Omega'}\phi_n$. According to Eq. (5.3), we see that splitting integral over $d\tau$ on two parts as follows

$$\begin{aligned} S(\lambda)f(x,\xi) &= \int_0^{\tau_0} \exp\bigl[-\bigl(\nu(\xi)-\lambda\bigr)\tau\bigr] f(x+\xi\tau,\xi)d\tau \\ &\quad + \int_{\tau_0}^\infty \exp\bigl[-\bigl(\nu(\xi)-\lambda\bigr)\tau\bigr] f(x+\xi\tau,\xi)d\tau \end{aligned} \tag{5.6}$$

and choosing in this integral sufficiently large τ_0, we see that the second part of it is less than any prescribed ϵ. In addition, if we assume that $\xi > \xi_0$ then with sufficiently large ξ_0 the $|S(\lambda)K\phi| < \epsilon$. Now it is clear that if we choose Ω' such that $\forall(x,\xi) \in \Omega\ (x+\zeta,\xi) \in \Omega'$ where $|\zeta| < \xi_0\tau_0$ we find $\|P_\Omega S(\lambda KQ_{\Omega'}\phi_n)\|_{L^\infty} < \epsilon$. Equation (5.5) can be written in the following form

$$P_\Omega\phi_n = P_\Omega S(\lambda)KP_{\Omega'}\phi_n + r_{n'} \tag{5.7}$$

where

$$\|r_n\|_{L^\infty} < \epsilon. \tag{5.8}$$

We can reiterate now Eq. (5.7) with the same conditions on sets Ω'', Ω''' and Q'''' and we obtain

$$P_\Omega \phi_n = P_\Omega S(\lambda) K P_{\Omega''} S(\lambda) K P_{\Omega'''} S(\lambda) K P_{\Omega''''} S(\lambda) K P_{\Omega'''''} \phi_n + z_n, \tag{5.9}$$

where $\|z_n\|_{L^\infty} < \epsilon$ and Ω''''' can be chosen in the form of the box in the x space. According to Lemma A.3 the operator $(S(\lambda)K)^4$ in the space $L^\infty_{x,\xi}(\Omega''''')$ with periodic boundary conditions is a compact operator, thus also the operator $P_\Omega(S(\lambda)K)^4$ is a compact operator in the space $L^\infty(\Omega'''''; d_3x \otimes d_3\xi)$. It is now easy to see that if we modify ϕ_n on the small vicinity of the boundary of Ω''''' in order to make ϕ_n periodic on the space boundary of this set, that the operator $P_\Omega S(\lambda) K P_{\Omega''} S(\lambda) K P_{\Omega'''} S(\lambda) K P_{\Omega''''} S(\lambda) K P_{\Omega'''''} \phi_n$ differs from the compact operator on the space $L^\infty_{x,\xi}(\Omega)$ on the operator with norm less than ϵ. We see then that, taking a subsequence if necessary, there exists such $\psi \in L^\infty_{x,\xi}(\Omega)$ that for sufficiently large N the following inequality holds

$$\|P_\Omega \phi_n - \psi\|_{L^\infty(\Omega; d_3x \otimes d_3\xi)} < \epsilon \tag{5.10}$$

First of all we see that $\|\psi\|_{L^\infty} \neq 0$ as for all n $\|P_\Omega \phi_n\|_{L^\infty_{x,\xi}(\Omega)} > 1 - \epsilon$. Thus there exists such set $A \subseteq \Omega$ and $C > 0$ that $|\psi| > C$ for $(x,\xi) \in A$ but it follows then from Eq. (5.10) that on this set $A |\phi_n| > C - \epsilon$ a.e. for $n > N$. Taking ϵ sufficiently small, such that $C - \epsilon > 0$ we conclude the proof of the lemma. The above construction shows that if there exist sequences ϕ'_n and w_n such that $(B^\infty - \lambda)\phi'_n = r_n$ with $\|r_n\|_{L^\infty} \to 0$, we can construct ϕ_n and w_n defined on the ball Ω centred at (0,0) such that $\phi_n = 0$ and $w_n = 0$ for x and ξ outside Ω and with $A \subset \Omega$, $\mu(A) > 0$ such that $|\phi_n| > C$ a.e. in A. It means that $|\phi_n|_{L^p_{x,\xi}} > C \cdot \mu(A)$ for all $1 \le p < \infty$. Moreover, $(B - \lambda)\phi_n = w_n$ with $\lim_{n\to\infty} \|w_n\|_{L^\infty} = 0$.

With a sequence ϕ_n we can define a sequence ψ_n as follows

$$\psi_n(x,\xi) = \left(1 + \left(x/C(n)\right)^2\right)^{-1} \phi_n(x,\xi), \tag{5.11}$$

where we have chosen $C(n)$ in such a way that

$$\lim_{n\to\infty} C^2(n) \|r_n\|_{L^\infty} = 0, \tag{5.12a}$$

$$\lim_{n\to\infty} C(n) = \infty. \tag{5.12b}$$

Acting on these ψ_n with an operator $B - \lambda$ we obtain the following equation

$$\begin{aligned}(B - \lambda)\psi_n(x,\xi) &= \left(1 + \left(x/C(n)\right)^2\right)^{-1} r_n(x,\xi) \\ &\quad + C(n)^{-2} \xi x \left(1 + \left(x/C(n)\right)^2\right)^{-2} \psi_n(x,\xi).\end{aligned} \tag{5.13}$$

I use now the space $L^2_{x,\xi}$. I calculate now $\|(B - \lambda)\psi_n\|_{L^2_{x,\xi}}$. Taking into account that from the definition of ψ_n and r_n we have $|(1 + |\xi|^k)\phi_n| \le (1 + |\xi_{max}|)^k |\phi_n|$ and $|(1 + |\xi|^k) w_n| \le (1 + |\xi_{max}|)^k |w_n|$, and using the following estimations:

$$\int \left(1 + \left(x/C(n)\right)^2\right)^{-2} \left(1 + |\xi|\right)^{-8} \left|r_n(x,\xi)\right|^2 d_3x d_3\xi$$

$$\leq \int C(n)^3\big[1+|y|^2\big]^{-2}\big[1+|\xi|\big]^{-8}\big|r_n\big(C(n)y,\xi\big)\big|^2 d_3x d_3\xi$$
$$\leq C_1 C(n)^3\|r_n\|^2_{L^\infty}, \tag{5.14}$$

$$\int C(n)^{-4}|\xi x|^2\big(1+\big(x/C(n)\big)^2\big)^{-2}\big|\psi_n(x,\xi)\big|^2 d_2x d_3\xi$$
$$\leq C(n)^{-2}\bigg[\int C(n)^{-4}|\xi x|^4\big(1+\big(x/C(n)\big)^2\big)^{-8}\big(1+|\xi|\big)^{-8}\big|\phi_n(x,\xi)\big|^2\bigg]^{1/2}, \tag{5.15}$$

we obtain:

$$\big\|(B-\lambda)\psi_n\big\|_{L^2_{x,\xi}} \leq C(n)^{-1/2}C_2\|\psi_n\|_{L^2_{x,\xi}}. \tag{5.16}$$

From these relations and Eq. (5.12b) we see that

$$\lim_{n\to\infty}\big[\|\psi_n\|_{L^2_{x,\xi}}\big]^{-1}\big\|(B-\lambda)\psi_n\big\|_{L^2_{x,\xi}} = 0, \tag{5.17}$$

Thus it follows from Eq. (5.16) that $\lambda \in \sigma(B_k^2)$ but I have shown in Sect. 4 that the spectrum of the operator B_k in the spaces L^p_ξ is independent of p for $1 \leq p \leq \infty$. We see then the spectrum of the operator B in $L^2_{x,\xi}$ is equal to $\bigcup_k \sigma(B_k)$. We conclude then that $\lambda \in \sigma(B^2)$, and it follows that for $\operatorname{Re}\lambda > -\nu_0$ $\sigma(B^\infty) \subseteq \sigma(B^2)$. For $\operatorname{Re}\lambda \leq -\nu_0$, we have shown that $\sigma(B^2) \subseteq \sigma(B^\infty)$ and as $\sigma(B^2)$ contains the whole half-space, we see that we have in general that $\sigma(B^2) = \sigma(B^\infty)$. □

In order to extend these results to arbitrary $1 < p < \infty$ we make use of the following lemma:

Lemma 5.2 *The resolvent sets $\rho(B^p)$ of the operators B^p are the same for all $1 \leq p \leq \infty$.*

Proof We denote as $\sum^p$ the spectral set of the operator B^p. For arbitrary $\lambda \in C/\sum^p$ operators $(B^1-\lambda)^{-1}$ and $(B^\infty-\lambda)^{-1}$ are bounded operators in $L^1_{x,\xi}$ and $L^\infty_{x,\xi}$ respectively. Let M^1_λ and M^∞_λ be their norms in these spaces. The Riesz-Thorin interpolation theorem [29] shows that operator $(B-\lambda)^{-1}$ is bounded in arbitrary $L^p_{x,\xi}$; $1 < p < \infty$ with a norm $M^p_\lambda = (M^1_\lambda)^{1-\theta}(M^\infty_\lambda)^\theta$ where $\theta = p^{-1}$. Thus λ belongs to the residual spectrum of the operator B^p or to the resolvent set. This first possibility is excluded because if λ is an element of the residual spectrum of the operator B^p, then it is also eigenvalue of the operator $(B^p)'$. This in fact means that there exists solution of the equation

$$B^q_{-k}\phi_{-k}(\xi) = \lambda\phi_{-k}(\xi), \tag{5.18}$$

with $\phi_k(\xi) \in L^q_\xi$ and constant λ. As the spectrum of the operators B^q_k is independent of q we see that in such case λ is also an eigenvalue of the operator B^2 but we have shown that the spectrum of the operator B in $L^2_{x,\xi}$ has no discrete part. We see that $\sum^p \subset \sum^1$. Applying again the interpolation theorem of Riesz-Thorin we see that $\sum^2 \subset \sum^p$ and as these two set are equal $\sum^2 \subset \sum^1$ we see that $\sum^p \subset \sum$ independent on p thus also resolvent sets $C/\sum^p = C/\sum$ and are independent of p.

I have shown that both spectral sets and resolvent sets of the operators B are independent of p for $1 \le p \le \infty$. Now I will describe details of the spectrum of these operators. First of all, simple argument shows that residual part of the spectrum of these operators $\sigma_{res}(B^p) = \oslash$ for $1 \le p \le \infty$. Consider first operator B^1. If we assume that there exists such $\lambda = \lambda_i(k)$ for some k, then we see that for the operator (B^∞) in $L^\infty_{x,\xi}$ functions $\phi^i(x,\xi) = \exp[-ikx]\phi^i_k(\xi) \in L^\infty_{x,\xi}$ are eigenfunctions of this operators with eigenvalues $\lambda_i(k)$. These eigenfunctions cannot be eigenfunctions of the operator B^1 because $\phi^i(x,\xi) \notin L^1_{x,\xi}$ but if we use a sequence of functions $\psi_n = n^{-3}\exp[-x^2/n^2]\phi_n$, we see that $\|\psi_n\| = C$ and $(B - \lambda_i(k))\psi_n$ converges to zero in L^1 norm. Thus we see that $\lambda(k)$ belongs simultaneously to the discrete or continuous and residual spectrum of the operator B^1 which is impossible. As for the operator B^∞ it is easy to check that functions $\phi^i(x,\xi) = \exp[-ikx]\phi^i_k(\xi) \in L^\infty_{x,\xi}$ are eigenfunctions belonging to the eigenvalue $\lambda_i(k)$. For $p > 1$ the argument on the absence of residual spectrum was provided in Lemma 5.1.

In Sect. 5, I have shown with a help of the theory of relatively bounded perturbation that for $\operatorname{Re}\lambda > -\nu_0$ operator B^2 has only continuous part of the spectrum. It then follows that the same is true for the operators B^p with $1 \le p \le \infty$. To see this, it is enough to observe that if ϕ is eigenfunction belonging to the eigenvalue λ of the operator B^p, then $\bar{\phi}$ is the eigenfunction of the operator $(B^p)' = B^q$ where $p^{-1} + q^{-1} = 1$ and thus $\phi \in L^2_{x,\xi}$ and λ is an eigenvalue of the operator B^2 and this is impossible for $\operatorname{Re}\lambda > -\nu_0$. This argument still allows for the existence of the eigenvalues with $\operatorname{Re}\lambda < -\nu_0$. We can give partial answer to this problem showing that the half-plane $\operatorname{Re}\lambda < -\nu_0$ belong to the continuous part of the spectrum. To this end we observe that for any λ belonging to this half-plane there exists such sequence $\phi^n_k(\xi)$ that functions defined as $\psi_n = \exp[ikx]\phi^n_k$ fulfill the following relation $\lim_{n\to\infty} \|(B^\infty - \lambda)\psi_n\|_{L^\infty} = 0$. Introducing now a new sequence as $\rho_n = n^{-3}\exp[-x^2/n^2]\psi_n$ we see that this sequence fulfills the following relation $\lim_{n\to\infty} \|(B^1 - \lambda)\rho_n\|_{L^1} = 0$ but ρ_n clearly does not converge in L^1. Similar considerations are valid for other $1 < p < \infty$.

Branches $\lambda = \lambda_i(k)$ with $\operatorname{Re}\lambda_i(k) > -\nu_0$ change as k is changing thus they form the continuous part of the spectrum of the operator B^p for $1 \le p < \infty$ but they also form the point spectrum of the operator B^∞. It is obvious that this spectral set is uncountable thus we see that operator B^∞ is not spectral. □

The fact that operators B^p have continuous spectrum different from the spectrum of the operators A^p for $1 \le p < \infty$ shows according to the Voigt Theorem that the operator K cannot be even relatively power compact with respect to the operator $S(\lambda)$ in $L^p(R^3 \times R^3)$.

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Appendix

I define now the streaming operators:

$$A^p\psi(x,\xi) = -\eta(\xi)\nabla\phi(x,\xi) - \nu(\xi)\phi(x,\xi). \tag{A.1}$$

$$A'^p\psi(x,\xi)=\eta(\xi)\nabla\psi(x,\xi)-\nu(\xi)\psi(x,\xi). \tag{A.2}$$

For $\lambda\in C$ such that $\operatorname{Re}\lambda>-\nu_0=\inf_{\xi\in R^3}\nu(\xi)$ we define two more operators:

$$S^p(\lambda)\phi(x,\xi)=\int_0^\infty \phi\big(x+\eta(\xi)\tau,\xi\big)\exp\big[-\big(\lambda+\nu(\xi)\big)\tau\big]d\tau, \tag{A.3}$$

$$S'^p(\lambda)\phi(x,\xi)=\int_0^\infty \psi\big(x-\eta(\xi)\tau,\xi\big)\exp\big[-\big(\lambda+\nu(\xi)\big)\tau\big]d\tau. \tag{A.4}$$

Properties of the operators A^p, A'^p, S^p and S'^p were investigated in [11]. For the sake of completeness we introduce several lemmas (without proofs) that we will need later:

Lemma A.1 *For* $\operatorname{Re}\lambda>-\nu_0$ *operators* $S^p(\lambda)$ *and* $S'^p(\lambda)$ *are closed, bounded operators in* $L^p_{x,\xi}$. *Moreover* $S^p(\lambda)=(A^p-\lambda I)^{-1}$ *and* $S'^p(\lambda)=(A'^p-\lambda I)^{-1}$.

Lemma A.2 *If* $S^p(\lambda)$ *is an operator in* $L^p_{x,\xi}$ $1\le p<\infty$ *then* $S'^q(\lambda)$ *in* $L^q_{x,\xi}$ *with* $p^{-1}+q^{-1}=1$ *is its adjoint operator. If* $S'^q(\lambda)$ *is an operator in* $L^q_{x,\xi}$ $1\le p<\infty$ *then* $S^p(\lambda)$ *in* $L^p_{x,\xi}$ *is its adjoint operator with* $q^{-1}+p^{-1}=1$.

From these definitions and lemmas we obtain finally for $p^{-1}+q^{-1}=1$:

Corollary A.1 A'^q *in* $L^q_{x,\xi}$ *is the adjoint operator of* A^p *in* $L^p_{x,\xi}$ $1\le p<\infty$; A^p *in* $L^p_{x,\xi}$ *is the adjoint operator of* A'^q *in* $L^q_{x,\xi}$ $1\le q<\infty$.

These relations are not fully symmetric due to the fact that domains of the operators A and A' are not dense in $L^\infty_{x,\xi}$.

In the following part of the paper we will need a version of compactness theorem for $S(\lambda)K$ valid for bounded spatial region with periodic boundary condition [11]. To this end we define linearized Boltzmann operator acting in a bounded spatial domain. We assume periodic boundary condition. The operator is given by expression (2.8) with $x\in\Omega\subset R^3$ and f a periodic function of $x\in\Omega$. We can define an operator $S^p(\lambda)$ and $S'^p(\lambda)$ in the space $L^p_{x,\xi}(\Omega)$ as:

$$S^p(\lambda)\phi=\int_0^\infty \tilde\phi\big(x-\eta(\xi)\mu,\xi\big)\exp\big[-\big(\lambda+\nu(\xi)\big)\mu\big]d\mu \tag{A.5}$$

$$S'^p(\lambda)\psi=\int_0^\infty \tilde\psi\big(x+\eta(\xi)\mu,\xi\big)\exp\big[-\big(\lambda+\nu(\xi)\big)\mu\big]d\mu \tag{A.6}$$

$\tilde\psi$ is an extension of ϕ by periodicity.

With this definition we have:

Lemma A.3 [11] *The operators* $KS^p(\lambda)$, $S^p(\lambda)K$, $S'^p(\lambda)K$ *and* $KS'^p(\lambda)$ *are compact in the spaces* $L^p_{x,\xi}(\Omega)$, $1<p<\infty$. *Fourth powers of these operators are compact in* $L^1_{x,\xi}(\Omega)$ *and* $L^\infty_{x,\xi}(\Omega)$.

Proof is given in [11].

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