EINSTEIN–PODOLSKY–ROSEN PARADOX, BELL’S INEQUALITY, AND THE PROJECTION POSTULATE*

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Abstract

Our aim is to understand the role of implicit assumptions which has been used by Einstein, Podolsky, and Rosen (EPR) in their famous article [*Phys. Rev.*, **47**, 777 (1935)] devoted to the so-called EPR paradox. We found that the projection postulate plays a crucial role in the EPR argument. It seems that EPR made a mistake in this paper — the projection postulate was applied not in its original form (as it has been formulated in von Neumann’s book [*Mathematical Foundations of Quantum mechanics*, Princeton University Press (1955)] but in the form which was later formalized as Lüders’ postulate [*Ann. Phys.*, Lpz. **8**, 322 (1951)]. Von Neumann’s postulate was crucially modified by extending it to observables with degenerate spectra. This modification is the real source of “quantum nonlocality.” The use of the original von Neumann postulate eliminates this problem — instead of (an action at a distance)-nonlocality, we obtain a classical measurement nonlocality, which is related to the synchronization of two measurements (produced on the two parts of a composite system). If one uses correctly von Neumann’s projection postulate, no “elements of reality” can be assigned to entangled systems.

Keywords: Einstein–Podolsky–Rosen paradox, Bell’s inequalities, von Neumann’s projection postulate, Lüders’ postulate, nonlocality.

1. Introduction

Unfortunately, the famous EPR paper [1] has been written on a level of rigorousness which is far from acceptable in theoretical physics. It is well known that Einstein himself was not happy with the remaining complications of this quantum treatment and was searching for simpler formulations. We point out that, besides the well-known conditions of (non)locality and (non)realism, the EPR consideration [1] contains a number of implicit assumptions which have never been discussed in debates on the EPR paradox. However, such assumptions play an important role within this framework. In this article, we are looking for a rigorous treatment of the EPR paradox.

In particular, we found recently that the projection postulate plays a crucial role in the EPR argument. It seems that EPR made a mistake in [1] – the projection postulate was applied not in its original form (as it was formulated in von Neumann’s book [2]), but in the form which was later formalized as Lüders’ postulate [3].

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*Talk presented at the oral issue of J. Russ. Laser Res. dedicated to the memory of Professor Vladimir A. Isakov, Professor Alexander S. Shumovsky, and Professor Andrei V. Vinogradov held in Moscow February 21–22, 2008.

We show that the main source of debate in the EPR argument [1], as well as in the discussion on the EPR–Bohm and Bell’s inequalities [4, 5], is the misuse of von Neumann’s projection postulate [2].

The projection postulate (PP) plays indeed a crucial role in the EPR argument [1]. We consider a composite system $s = (s_1, s_2)$. Assigning an element of reality to s_2 on the basis of a measurement performed on s_1 is fundamentally based on the PP. Although von Neumann had presented strong physical arguments stressing that the PP should be applied only to observables with nondegenerate spectra [2], it was, nevertheless, applied by Einstein, Podolsky, and Rosen [1] in the case of observables with degenerate spectra. This misapplication of von Neumann’s projection postulate [2] was later on formalized as a custom by Lüders [3].

Hence, we show that the EPR argument is, in fact, based on an improper extension of von Neumann’s postulate, and that as such it should not be considered as a valid attack against the Copenhagen interpretation (which is itself based solely on von Neumann’s axiomatic [2]). This means that quantum mechanics (QM) can (but, in principle, need not) be interpreted as a complete and local theory.

Our analysis shows that measurements on entangled systems, as predicted by the QM formalism, are nonlocal. However, such a measurement nonlocality is essentially a classical synchronization nonlocality and has nothing to do with the action-at-a-distance. Thus, instead of the “state- nonlocality,” everything is reduced to nonlocality of the (classical) design of the EPR experiment.

The presence of a measurement nonlocality in the EPR–Bohm experiment implies the violation of the conditions of Bell’s theorem and, hence, provides a possibility to violate Bell’s inequality (cf. works of Hess and Philipp on the role of the time parameter within the EPR–Bell framework [6–8]).

We remind the reader that the projection postulate is nowadays formulated in the following form:

PP: *Let a be a physical observable described by a self-adjoint operator $\hat{a}$ having purely discrete spectrum. Any measurement of the observable a on the pure quantum state ψ induces a transition from the state ψ into one of the eigenvectors e_a^k of the operator $\hat{a}$.*

It is in this form that the projection postulate was used by EPR in [1] as well as in numerous discussions on “quantum nonlocality.”

For reader’s convenience, we included in this paper the Appendix containing original considerations of von Neumann and Lüders on the projection postulate. We also note that the projection postulate is basic in defining quantum conditional probabilities which are widely used in quantum information theory (see Appendix). It is interesting that quantum conditional probabilities are defined on the basis of Lüders’ postulate (and not von Neumann’s postulate).

2. Quantum Description of Measurements on Composite Systems

Let H_1 and H_2 be two complex finite-dimensional Hilbert spaces, $\dim H_i \geq 2$. Let $\hat{a}_1 : H_1 \rightarrow H_1$ and $\hat{a}_2 : H_2 \rightarrow H_2$ be two self-adjoint operators. The Hilbert space H_i represents (quantum) states of the system s_i ($i = 1, 2$). The operator $\hat{a}_i$ represents an observable a_i corresponding to measurements on s_i ($i = 1, 2$). The composite system $s = (s_1, s_2)$ is described by the tensor product space $H = H_1 \otimes H_2$. The operators $\hat{A}_1 = \hat{a}_1 \otimes I$ and $\hat{A}_2 = I \otimes \hat{a}_2$ represent partial measurements A_1 and A_2 on s — a_1 on s_1 and a_2 on s_2 , respectively. To simplify considerations, we assume that both operators $\hat{a}_1$ and $\hat{a}_2$ have

purely discrete nondegenerate¹ spectra. We consider eigenvectors e_1^α and e_2^β of these operators

$$\hat{a}_1 e_1^\alpha = \lambda_1^\alpha e_1^\alpha, \quad \alpha = 1, \dots, N_1 = \dim H_1 \quad \text{and} \quad \hat{a}_2 e_2^\beta = \lambda_2^\beta e_2^\beta, \quad \beta = 1, \dots, N_2 = \dim H_2.$$

By taking two arbitrary (independent) eigenvectors for each operator, we construct an “entangled state”

$$\psi = c_1 e_1^i \otimes e_2^j + c_2 e_1^j \otimes e_2^i, \quad |c_1|^2 + |c_2|^2 = 1. \quad (1)$$

Suppose a measurement of A_1 was performed on the composite system $s = (s_1, s_2)$, i.e., a_1 was performed on s_1 . Suppose that the result

$$A_1 = \lambda_1^i$$

was obtained. This measurement is represented by the operator $\hat{A}_1 = \hat{a}_1 \otimes I$.

By the projection postulate (PP), the state ψ is projected onto

$$\psi_1^{ij} = e_1^i \otimes e_2^j.$$

Thus instantaneously (and, in principle, without any interaction of the A_1 -measurement device with the system s_2) the state of s_2 is changed. It became sharply determined: $\hat{A}_2 \psi_1^{ij} = \lambda_2^j \psi_1^{ij}$, and hence $A_2 = \lambda_2^j$.

This is nothing else than the so-called “quantum nonlocality.” To obtain it, one does not need to appeal to Bell’s inequality [4, 5] (and hidden variables at all).² If one takes this into account, Bell’s considerations would play a subsidiary role. The main problem is to explain “quantum nonlocality” as it follows from the quantum formalism.

3. Abuse of von Neumann’s Projection Postulate

Reading von Neumann’s book [2], we found that the modern formulation PP of the projection postulate (see Introduction) is not the original von Neumann’s formulation at all. One extremely important condition is omitted. It is the condition of *nondegeneration of spectrum* of the quantum observable. The formulation PP is, in fact, Lüders’ postulate [3] and not at all von Neumann’s [2].

In contrast to Lüders, von Neumann sharply separated the cases of nondegenerate and degenerate spectra. The PP could be applied only in the first case. In the second case, the result $a = \lambda$ does not determine any definite state. To obtain a definite state, one should perform a refinement d of the a measurement, such that $a = f(d)$ and d are represented by an operator $\hat{d}$ having nondegenerate spectrum.

Let us go back to the situation described in Sec. 2. Here the operators $\hat{A}_1$ and $\hat{A}_2$ (corresponding to measurements on s_1 and s_2) always have degenerate spectra:

If, e.g., $\hat{a}_1 e = \lambda e$, $e \in H_1$, then $\hat{A}_1 \psi = \lambda \psi$ for any $\psi = e \otimes \phi \in H$, $\phi \in H_2$. (Recall that $\dim H_2 \geq 2$.)

By von Neumann, the result $A_1 = \lambda_1^j$ does not induce projection onto a definite pure state. The state of the system $s = (s_1, s_2)$ is not determined after such a *partial* measurement!

Thus, if one follows really the Copenhagen interpretation, no trace of “quantum nonlocality” would be found.

¹The latter condition is redundant. We would like just to emphasize that degeneration of spectra in H_i does not play any role. Even operators with nondegenerate spectra in H_i induce operators with degenerate spectra in H .

²We note that precisely in this way the EPR–Bohm experiment was presented by Alain Aspect in his talk at the Växjö conference [9] — by measuring polarization of s_1 , one projects the state of the composite system and makes the state of s_2 determined (see also [10]).

4. Von Neumann's Postulate and "Classical Nonlocality"

What would von Neumann recommend to do for obtaining the definite post measurement state? He would recommend to perform a refinement A of the A_1 measurement which would be represented by an operator, say $\hat{A}$, having nondegenerate spectrum.

The crucial point is that one could not construct such a refinement $\hat{A}$ by operating only in H_1 , i.e., by using operators of the form

$$\hat{A} = \hat{C} \otimes I, \hat{C} : H_1 \rightarrow H_1.$$

One should consider "nonlocal measurements" which are represented by operators acting in the complete tensor product $H = H_1 \otimes H_2$.

In particular, one cannot create a nondegenerate refinement of the spin- s_1 measurement via modification of the spin operator in $H_1 = \mathbf{C}^2$. The corresponding nongenerate operator is nontrivial in the whole $H = \mathbf{C}^4$. The simplest refinement $\hat{A}$ can be constructed as $\hat{A}e_1^\alpha \otimes e_2^\beta = \gamma_{\alpha\beta}e_1^\alpha \otimes e_2^\beta$, where $\gamma_{\alpha\beta} \neq \gamma_{\alpha'\beta'}$, if $\alpha \neq \alpha'$ or $\beta \neq \beta'$.

Example. Spin refinement.

Let $\hat{a}_1 = \sum_{i=1}^3 x_i \sigma_i$ and $\hat{a}_2 = \sum_{i=1}^3 y_i \sigma_i$, where σ_i are Pauli matrices. Here $H_1 = H_2 = \mathbf{C}^2$, hence, $H = \mathbf{C}^4$. Consider eigenvectors $\hat{a}_i e_i^\pm = \pm e_i^\pm$, $i = 1, 2$. We consider the following encoding: $-- = 00 = 0$, $+- = 10 = 1$, $-+ = 01 = 2$, $++ = 11 = 3$. Now we set $\hat{A}e_1^\alpha \otimes e_2^\beta = \alpha\beta e_1^\alpha \otimes e_2^\beta$. This operator has nondegenerate spectrum $\lambda = 0, 1, 2, 3$. We have $\hat{A}_1 = f_1(\hat{A})$, where $f_1(\alpha\beta) = \alpha$. In the same way, $\hat{A}_2 = f_2(\hat{A})$, where $f_2(\alpha\beta) = \beta$. By von Neumann, only measurement of the observable A represented by $\hat{A}$ induces projection of the entangled state ψ onto the pure state $e_1^\alpha \otimes e_2^\beta$ (in the case of the result $\lambda = \alpha\beta$). Thus, instead of mysterious "quantum nonlocality" (or state nonlocality), we have measurement nonlocality which is purely classical nonlocality.

A measurement performed on a composite system $s = (s_1, s_2)$ consisting of two spatially separated parts is, of course, nonlocal. It is not surprising that it is represented by a "nonlocal operator" $\hat{A}$ (cf. [11]). Nothing nonlocal happens with the state of s . Only (classical) design of the experiment is nonlocal.

5. Simultaneous Measurement of Two Compatible Observables

We expect that our previous arguments on the classical measurement nonlocality in experiments with entangled systems would be criticized in the following way:

"There is nothing about the measurement nonlocality, since one can perform, instead of measurement of a nonlocal observable A , simultaneous measurements of two compatible observables A_1 and A_2 . The crucial point is that the operators $\hat{A}_1$ and $\hat{A}_2$ commute."

Such an argument would be based on another von Neumann's postulate, namely, about measurement of compatible observables (see [2], pp. 200–201):

(PC) *The probability, that in the state ψ the quantities with (commuting)³ operators $\hat{R}_1, \dots, \hat{R}_n$ take the values from corresponding intervals $\Delta_1, \dots, \Delta_n$, is*

$$||E_1(\Delta_1) \dots E_n(\Delta_n)\psi||^2,$$

where $E_1(\lambda), \dots, E_n(\lambda)$ are resolutions of the identity belonging to $\hat{R}_1, \dots, \hat{R}_n$, respectively.

³"Commuting" was absent in the original postulate. Von Neumann formulated (PC) first for arbitrary self-adjoint operators but then, after analyzing it, he pointed out that they should commute.

One may say that in measurements for entangled states one can consider simultaneous measurement of two compatible observables represented by the operators $\hat{A}_1 = \hat{a}_1 \otimes I$ and $A_2 = I \otimes \hat{a}_2$, $[\hat{A}_1, \hat{A}_2] = 0$. Hence the story is about simultaneous measurement of compatible observables and not about measurement of “nonlocal” observable A represented by $\hat{A}$.

Again by reading von Neumann (see [2], pp. 201–206), we understand that the above argument does not take into account the crucial fact that *simultaneous measurement of two compatible observables is not reduced to separate measurement of each of them*. By the conventional quantum formalism, the simultaneous measurement of a and b given by $\hat{a}$ and $\hat{b}$, $([\hat{a}, \hat{b}] = 0)$ is performed in the following way.

One should construct an observable d (having nondegenerate spectrum) represented by $\hat{d}$ such that $\hat{a} = f_1(\hat{d})$ and $\hat{b} = f_2(\hat{d})$. Then simultaneous measurement of a and b is performed in two steps:

- a) measurement of d ;
- b) a and b are obtained as $a = f_1(d)$ and $b = f_2(d)$.

Therefore, in the case of measurements on composite systems $s = (s_1, s_2)$, one could not (!) proceed without (classically) nonlocal measurement of a refinement A given by $\hat{A}$.

What does it mean physically?

It means that one should make synchronized measurement on both systems. Then the result $\lambda = \alpha\beta$ should be decoded into $A_1 = \alpha$ and $A_2 = \beta$. After this, one can calculate, e.g., the correlation $\langle A_1 A_2 \rangle$ between A_1 and A_2 . Nonlocality appears here via synchronization. An observer, to whom such a synchronization is not available, would not be able to find the correct matching between the results of A_1 and A_2 measurements.

Thus the EPR–Bohm experiment is really nonlocal, but it is measurement nonlocally which is completely classical time synchronization nonlocality.

6. The Original EPR Experiment

It is evident that Alain Aspect simply borrowed “quantum nonlocality” argument [10] from EPR. Thus the root of misunderstanding was in the original paper [1].

Let now $H_1 = H_2 = L_2(R^3, dx)$. Let a_1 and a_2 be observables represented by operators $\hat{a}_1$ and $\hat{a}_2$ with purely discrete nongenerate spectra

$$\hat{a}_i e_i^\alpha = \lambda_i^\alpha e_i^\alpha, \quad i = 1, 2.$$

Any state $\psi \in H = H_1 \otimes H_2$ can be represented as

$$\psi = \sum_{\alpha, \beta} c_{\alpha\beta} e_1^\alpha \otimes e_2^\beta,$$

where $\sum_{\alpha, \beta} |c_{\alpha\beta}|^2 = 1$. Einstein, Podolsky, and Rosen claimed that the measurement of A_1 given by $\hat{A}_1 = \hat{a}_1 \otimes I$ induces the projection of ψ onto one of states $e_1^\alpha \otimes u$, $u \in H_2$. In particular, for a state of the form $\psi = \sum_\gamma c_\gamma e_1^\gamma \otimes e_2^\gamma$, one of the states $e_1^\gamma \otimes e_2^\gamma$ would be realized. Thus by performing measurement on the s_1 with the result λ_1^γ , the “element of reality $a_2 = \lambda_2^\gamma$ ” is assigned to s_2 .

However, the EPR considerations did not match von Neumann’s projection postulate, because the spectrum of $\hat{A}_1$ is degenerate. Finally (after consideration of operators with discrete spectra), Einstein, Podolsky, and Rosen considered operators of position and momentum having continuous spectra. According to von Neumann [2], one should proceed by approximating operators with continuous spectra by operators with discrete spectra.

Thus to get “an element of reality” by von Neumann, one should perform measurement of “nonlocal observable” A given by a nonlocal refinement of, e.g., $\hat{A}_1 = \hat{q}_1 \otimes I$ and $\hat{A}_2 = I \otimes \hat{p}_2$.

We point out that von Neumann’s viewpoint coincides with Bohr’s viewpoint in his reply to Einstein [13]. Unfortunately, Bohr did not use the mathematical arguments of von Neumann [2] to explain to Einstein why, to produce “an element of reality” for the subsystem s_2 , one should measure A on $s = (s_1, s_2)$ and not just, e.g., A_1 on s_1 .

7. Bell’s Inequality

Bell’s inequality and interpretations of its violation are still the source of intense debates (see, e.g., [6, 8, 10–12, 14–38]).

What are the consequences of our analysis of the EPR–Bohm and the EPR experiments for Bell’s considerations?

7.1. Measurement Nonlocality as Opposed to Action at the Distance

The main consequence is that classical random variables (if they exist) for A_1 and A_2 measurements should have the form $A_1(\lambda) = f_1(d(\lambda))$, $A_2(\lambda) = f_2(d(\lambda))$, where $d(\lambda)$ is the variable corresponding to synchronized measurement on both systems. Since $d \equiv d_{A_1, A_2}$, Bell’s condition of locality is violated. Therefore, it is not surprising that Bell’s inequality can be violated.

The crucial point is that such *measurement nonlocality* has nothing to do with action at the distance. It is classical time synchronization nonlocality; cf. works of Hess and Philipp on the role of the time parameter within the EPR–Bell framework [6–8]. The authors of these papers correctly stressed the role of analysis of time synchronization in measurements on pairs of entangled photons.

7.2. Time Synchronization and Time Window

The time-synchronization viewpoint in the EPR–Bohm experiment means that experimenters should couple in time two clicks of detectors corresponding to measurements on the first and second subsystems, respectively. Of course, such a time coupling is nonlocal. Roughly speaking, if Alice and Bob count the first and second types of clicks, respectively, they should call each other to couple their clicks in time. This procedure has nothing to do with QM. It is a classical nonlocal design of the experiment.

8. Did EPR Make a Mistake?

Recently Karl Hess wrote in his email-comment on the arXiv-preprint of this paper: “Did EPR make a mistake? I was sure that they proved the incompleteness of quantum mechanics.”

The answer is not so simple, since the whole EPR story is very complicated. In fact, this story is not about QM itself, but about its interpretation. EPR wanted to show that QM endowed with the Copenhagen interpretation is not complete. It is crucial that they did not claim that QM endowed with any interpretation is incomplete. The main problem is that EPR did not formulate precisely the interpretation under attack! It was more or less clear that it was the interpretation of Bohr and Heisenberg.

As we have already pointed out, EPR's argument was heavily based on the projection postulate. They definitely assigned it to the criticized interpretation. The possibility to apply the projection postulate for operators with degenerate spectra played the fundamental role in the EPR considerations.

However, one might be curious: "Would fathers of the Copenhagen interpretation accept such a (mis?) use of the projection postulate?" Briefly the question is: "Was EPR's argument against the real Bohr–Heisenberg interpretation? Maybe it was simply based on EPR's misinterpretation of views of Bohr and Heisenberg?"

We remark that within this framework it would be better to speak about views of concrete persons, since the "Copenhagen interpretation" is an extremely diffuse collection of viewpoints (see Arcady Plotnitsky [39] for details). I have not yet studied well the viewpoints of Heisenberg. But I read Bohr's reply [13] to Einstein. I have the impression that Bohr wrote to Einstein (unfortunately, in nonmathematical terms) that no "element of reality" can be "created" by measurement on a single subsystem. To create an "element of reality," one should perform measurement on the second subsystem. This is nothing else than von Neumann's refinement measurement!

I would say that in 1935 EPR misinterpreted the Copenhagen interpretation (at least the Bohr–von Neumann one). They attacked a sort of "perverse Copenhagen interpretation" based on abuse of von Neumann's projection postulate.⁴ They demonstrated that QM endowed with this EPR version of the Copenhagen interpretation was incomplete.⁵ However, the EPR argument did not imply that QM with the real (Bohr–von Neumann) Copenhagen interpretation was incomplete.

Unfortunately, the situation was not clarified at the very beginning. Bohr's reply was not sufficiently clear and it was not connected with von Neumann's book. And later on the EPR story was developed in a really unexpected way. The pseudo-Copenhagen interpretation of EPR became the orthodox Copenhagen interpretation. In particular, Bohr's reply and hence his views were ignored. Nowadays, Lüders' postulate is widely used instead of von Neumann's projection postulate. Thus the EPR arguments became fundamentally important and they stimulated Bell toward his inequality argument. He definitely believed that QM is not complete (as it was "proved" by EPR). And incompleteness was understood in the EPR manner.

One may ask: "Is it the end of the hidden variable story?" The answer is: Not at all. It is just a new beginning. Instead of the EPR incompleteness and assigning to a hidden variable the λ values of quantum observables $\lambda \rightarrow a(\lambda)$, we can analyze possibilities of more complicated couplings between prequantum and quantum worlds (see, e.g., [38]).

Conclusion. *The EPR paradox is a consequence of misinterpretation of the Copenhagen interpretation based on the vague application of von Neumann's projection postulate.*

9. Appendix: Projection Postulate

After discussions with a few hundred scientists working in quantum foundations and quantum information (who visited Växjö University during a series of conferences on quantum foundations), it became completely clear that the majority has even no idea about the difference between von Neumann's and Lüders' projection postulates (PPs). Those who knew the story about PP expressed their strong opinion that von Neumann's book is old fashioned and at the very beginning many things were not sufficiently

⁴Von Neumann's book was on the book shelf in Einstein's office. Did Einstein read it?

⁵The so-called quantum nonlocality was considered as totally absurd at that time.

clear. Later a lot in quantum foundations was clarified. In particular, Lüders clarified PP, which was formulated by von Neumann, in a very complicated form. I strongly disagree with such a common opinion. The story is not about simplification of von Neumann's arguments. It is about misunderstanding of the basis postulate of QM. As was already pointed out, Lüders' misunderstanding was rooted in a similar misunderstanding of PP by Einstein, Podolsky, and Rosen in their fundamental article [1]. Therefore, I find it important to present this Appendix containing a practically forgotten comparison of von Neumann's PP and Lüders' PP. Finally, we recall that this comparison is extremely important not only for quantum foundations, EPR paradox, and Bell's inequality. The difference between two forms of PP should be pointed out even without any relation to the mentioned questions. We recall that the basic definition of quantum conditional probability which is widely used in quantum information theory is based on Lüders PP and not on von Neumann's!

It should be emphasized that comparison of the two PPs is not of only theoretical value. In principle, it can be tested experimentally — either the post-measurement state is pure (as Lüders and quantum majority claimed) or it is really not well defined. Of course, one should understand better the meaning of “not well defined state.” However, we prefer to come back to this problem in a future article.

9.1. Lüder's Projection Postulate

It is very important for our further considerations to remark that if the spectrum of an operator $\hat{a}$ is *degenerate*, then (according to von Neumann [2]) the observation of the result $a = \alpha_k$ *does not induce the transition of the initial pure state ψ into a new pure state*. In such a case, the resulting state is not determined. It can be determined only through a subsequent measurement of an observable d refining the original observable a . After such a refining measurement, we obtain not a pure state but a statistical mixture.

However, in the contemporary formulations of “von Neumann's projection postulate” the cases of nondegenerate and degenerate spectra are not distinguished! In fact, this was not the original von Neumann invention, but it was a new postulate proposed by G. Lüders in 1951 [3]:

Lüders' Projection Postulate. *For any operator $\hat{a}$ with purely discrete spectrum, a measurement of the corresponding observable a giving the result $a = \alpha$, where $\alpha \in \text{Spec}(\hat{a})$, always produces the projection of the initial state ψ onto the state*

$$\psi_\alpha = \frac{P_\alpha \psi}{\|P_\alpha \psi\|}, \quad (2)$$

where as always P_α is the orthogonal projector onto the subspace corresponding to the eigenvalue α .

The crucial point is that by the Lüders' PP the post-measurement state is also a pure state independently of degeneration of spectrum.

However, von Neumann emphasized several times in [2] (p. 218) that “if the eigenvalue α is multiple, then the state ϕ after the measurement is not uniquely determined by knowledge of the result of the measurement.”

The post-measurement state ϕ is not determined. What does it mean? J. von Neumann pointed out that, to determine ϕ , one should determine a subsequent measurement procedure which corresponds to the choice of a concrete orthonormal basis in the subspace $\mathcal{H}_\alpha = P_\alpha^a \mathcal{H}$. This ambiguity in the determination of the post-measurement state is an important difficulty in the foundations of quantum mechanics.⁶

⁶Let us take this problem seriously. It might be interpreted as a sign of *incompleteness of quantum mechanics*: the quantum formalism does not determine the post-measurement state in the case of observables represented by operators with

9.2. Von Neumann's and Lüders' Postulates for Mixed States

Let ψ be a pure state and let P be a projector. By Lüders' postulate, after measurement of the observable represented by P that gives the result 1, the initial pure state ψ is transformed again into a pure state, namely,

$$\psi' = \frac{P\psi}{\|P\psi\|}.$$

Thus for the corresponding density operators we have

$$\rho_{\psi'} = \psi' \otimes \psi' = \frac{P\psi \otimes P\psi}{\|P\psi\|^2}.$$

We remark that $(P\psi \otimes P\psi)\phi = \langle \phi, P\psi \rangle P\psi$ and $(P\rho_\psi P)\phi = \langle P\phi, \psi \rangle P\psi$. Since $P^* = P$, we obtain

$$\rho_{\psi'} = \frac{P\rho_\psi P}{\|P\psi\|^2}. \quad (3)$$

Finally, we see that $\|P\psi\|^2 = \langle P\psi, P\psi \rangle = \langle P^2\psi, \psi \rangle = \langle P\psi, \psi \rangle = \text{Tr } \rho_\psi P$. Thus

$$\rho_{\psi'} = \frac{P\rho_\psi P}{\text{Tr } \rho_\psi P}. \quad (4)$$

In this way, Lüders' postulate is represented within the framework of density operators (still in the case of pure states). Lüders generalized this formula (without any doubt) to an arbitrary state ρ . If we measure the observable represented by a projector P for the ensemble of systems described by ρ and then select a new ensemble corresponding to the result 1, we get the state

$$\rho' = \frac{P\rho P}{\text{Tr } \rho P}. \quad (5)$$

Let now consider an arbitrary self-adjoint operator with purely discrete spectrum

$$\hat{a} = \sum_m \alpha_m P_m, \quad \alpha_m \in \mathbf{R}, \quad P_m P_l = \delta_{ml} P_l. \quad (6)$$

Lüders pointed out [3] that in a measurement of $\hat{a}$ the initial state ρ is transformed into

$$\rho' = \sum_m P_m \rho P_m. \quad (7)$$

It is easy to see that such a ρ' is again positive and self-adjoint and its trace equals one. This is Lüders' postulate for the transformation of a state ρ through a measurement of an observable represented by $\hat{a}$. The crucial Lüders' assumption is that, *for a pure state after a measurement of $\hat{a}$ and selection with respect to the value α_m , we always obtain again a pure state.*

Von Neumann had a completely different viewpoint on such a transformation [2]. As was already pointed out, even for a pure state ψ the result will not again be a pure state (if the operator has a degenerated spectrum).

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degenerate spectra. However, von Neumann did not interpret this problem in such a way. It may be that the success of Lüders' modification of von Neumann's projection postulate is due to the common desire to escape this problem.

Let $\hat{a}$ have nondegenerate spectrum. Thus all P_m are one-dimensional projectors (onto eigenvectors $\{e_m\}$ of $\hat{a}$). Then by the von Neumann projection postulate a measurement of a giving the result $a = \alpha_m$ really induces the projection of the original pure state ψ onto e_m^a . The transformation of the density operator is given by

$$\rho_{\psi'} = \sum_m P_m \rho_{\psi} P_m \quad (8)$$

(so, in the nondegenerate case, Lüders' approach coincides with von Neumann's). Starting with an arbitrary initial state ρ we obtain the state

$$\rho' = \sum_m P_m \rho P_m. \quad (9)$$

We remark that, since all projectors are one-dimensional, we have

$$(P_m \rho P_m) \phi = P_m (\langle \phi, e_m \rangle \rho e_m) = \langle \rho e_m, e_m \rangle \langle \phi, e_m \rangle e_m = \langle \rho e_m, e_m \rangle P_m \phi.$$

Thus we can rewrite (8) as von Neumann did

$$\rho' = \sum_m \langle \rho e_m, e_m \rangle P_m. \quad (10)$$

Let us start with a pure state ψ . If $\hat{a}$ has degenerate (discrete) spectrum then, according to von Neumann [2], a measurement of a giving the value $a = \alpha_m$ *does not induce a projection of ψ* . The result will not be a pure state (in particular, not $\psi_m = P_m \psi$). Moreover, the resulting state is not determined. Only a subsequent measurement of an observable d such that $a = f(d)$ and d are represented by the operator $\hat{d}$ with nondegenerate spectrum will determine the final state.

Let $\hat{a} = P$ be an orthogonal projector onto a subspace $\mathcal{H}_0$ of the state space $\mathcal{H}$. Let us choose in $\mathcal{H}_0$ an orthonormal basis $\{\phi_n\}$. The basis $\{\phi_n\}$ can be completed to an orthonormal basis in $\mathcal{H} : \{\phi_n, \phi'_l\}$. Let us take two sequences of real numbers $\{\gamma_n\}, \{\gamma'_n\}$ such that all numbers are distinct. We define the corresponding self-adjoint operator $\hat{d}$ having eigenvectors $\{\phi_n, \phi'_l\}$ and eigenvalues $\{\gamma_n, \gamma'_n\}$

$$\hat{d} = \sum_n \gamma_n P_{\phi_n} + \sum_n \gamma'_n P_{\phi'_n}.$$

Its domain of definition is given by

$$D(\hat{d}) = \left\{ \psi \in \mathcal{H} : \psi = \sum_n \gamma_n^2 |\langle \psi, \phi_n \rangle|^2 + \sum_n (\gamma'_n)^2 |\langle \psi, \phi'_n \rangle|^2 < \infty \right\}.$$

J. von Neumann postulated [2] that one can construct a physical observable d described by the operator $\hat{d}$. Measurement of d can be considered as measurement of the observable a , because $a = f(d)$, where f is some function such that $f(\gamma_k) = 1$ and $f(\gamma'_k) = 0$. But the d measurement (without post-measurement selection with respect to eigenvalues) produces the statistical mixture

$$\bar{\rho} = \sum_n |\langle \psi, \phi_n \rangle|^2 P_{\phi_n} + \sum_n |\langle \psi, \phi'_n \rangle|^2 P_{\phi'_n}.$$

Since we can choose $\{\phi_n\}$ and $\{\phi'_n\}$ in many ways, by obtaining the result $a = \alpha_k$ we cannot determine the post-measurement state.

If we start with an arbitrary state ρ and an arbitrary self-adjoint operator $\hat{a}$ with purely discrete spectrum, then we can determine the post-measurement state only with the aid of the subsequent measurement of an observable d , $a = f(d)$ described by the operator $\hat{d}$ with nondegenerate spectrum. We denote by $\{\phi_{km}\}$ bases in subspaces $\mathcal{H}_m = P_m \mathcal{H}$. Then by von Neumann

$$\rho' = \sum_m \sum_k \langle \rho \phi_{km}, \phi_{km} \rangle P_{\phi_{km}}. \quad (11)$$

It seems that experimental investigations to compare von Neumann's or Lüders' laws of transformation of states have never been performed. It is amazing! It is not so complicated to check whether after measurement of an observable a (represented by a self-adjoint operator with degenerate purely discrete spectrum) the post measurement state is a pure state (Lüders' postulate) or some statistical mixture (von Neumann's viewpoint).

9.3. Conditional Probability

As in the classical Kolmogorov and von Mises probabilistic models, in QM Born's postulate about the probabilistic interpretation of quantum states should be completed by a definition of conditional probability. We present the contemporary definition which is conventional in quantum logic and quantum information theory:⁷

Definition. Let physical observables a and b be represented by self-adjoint operators with purely discrete (may be degenerate) spectra

$$\hat{a} = \sum_m \alpha_m P_m^a, \quad \hat{b} = \sum_m \beta_m P_m^b, \quad (12)$$

where P_m^a and P_m^b are projectors on subspaces corresponding to eigenvalues α_m and β_m .

Let ψ be a pure state and let $P_k^a \psi \neq 0$. Then the probability to get the value $b = \beta_m$ under the condition that the value $a = \alpha_k$ was observed in the preceding measurement of the observable a on the state ψ is given by

$$\mathbf{P}_\psi(b = \beta_m | a = \alpha_k) \equiv \frac{\|P_m^b P_k^a \psi\|^2}{\|P_k^a \psi\|^2}. \quad (13)$$

Sometimes the symbol $\mathbf{P}_\psi(P_m^b | P_k^a)$ is used. Set

$$\psi_k^a = \frac{P_k^a \psi}{\|P_k^a \psi\|}.$$

Then

$$\mathbf{P}_\psi(b = \beta_m | a = \alpha_k) = \|P_m^b \psi_k^a\|^2 = \mathbf{P}_{\psi_k^a}(b = \beta_m).$$

Let $\hat{a}$ have nondegenerate spectrum. We can write

$$\mathbf{P}_\psi(b = \beta_m | a = \alpha_k) = \|P_m^b e_k^a\|^2$$

⁷This definition is based on Lüders' postulate. Von Neumann's assumption that observable should have nondegenerate spectrum was totally ignored.

(here $\hat{a}e_k^a = \alpha_k e_k^a$). Thus the conditional probability in this case does not depend on the original state ψ . We can say that the memory about the original state was destroyed.

If also the operator $\hat{b}$ has nondegenerate spectrum, then we have

$$\mathbf{P}_\psi(b = \beta_m | a = \alpha_k) = |\langle e_m^b, e_k^a \rangle|^2$$

and

$$\mathbf{P}_\psi(a = \alpha_k | b = \beta_m) = |\langle e_k^a, e_m^b \rangle|^2.$$

By using symmetry of the scalar product, we obtain:

Let both operators $\hat{a}$ and $\hat{b}$ have purely discrete nondegenerate spectra and let $P_k^a \psi \neq 0$ and $P_m^b \psi \neq 0$. Then the conditional probability is symmetric and it does not depend on the original state ψ

$$\mathbf{P}_\psi(b = \beta_m | a = \alpha_k) = \mathbf{P}_\psi(a = \alpha_k | b = \beta_m) = |\langle e_m^b, e_k^a \rangle|^2.$$

Now we invent the notion of conditional probability for a quantum statistical state given by a density operator ρ . Let two observables be represented by operators (12). Then the probability to get the value $b = \beta_m$ under the condition that the value $a = \alpha_k$ has been observed in the preceding measurement of the observable a on the state ρ is given by

$$\mathbf{P}_\rho(b = \beta_m | a = \alpha_k) = \text{Tr } \rho_k^a P_m^b, \quad \text{where } \rho_k^a = \frac{P_k^a \rho P_k^a}{\text{Tr } \rho P_k^a}. \quad (14)$$

Here (according to Lüders) the density operator ρ_k^a describes the quantum state after the result $a = \alpha_k$ was obtained. We also use the notation $\mathbf{P}_\rho(P_m^b | P_k^a)$ for $\mathbf{P}_\rho(b = \beta_m | a = \alpha_k)$.

We remark that the validity of the conventional (Lüders' type) definition of conditional probability in QM can be, in principle, tested experimentally. Unfortunately, such experimental tests have never been performed.

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